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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF SECOND–ORDER DIFFERENCE EQUATIONS

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Abstract. Using the method of variation of constants, discrete inequalities and Tychonoff's fixed-point theorem we study problem asymptotic equivalence of second order difference equations.

1. Introduction

Some asymptotic relationships between the solutions of the second order difference equations

$$(1) \qquad \Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n = 0$$

and

$$(2) \qquad \Delta(p_{n-1}\Delta y_{n-1}) + q_n y_n = f(n, y_n, \Delta y_{n-1})$$

are studied.

The purpose of this paper is to extend some of the results from [2] and [6] on differences equations.

Analogous problem for differential equations has been considered in paper [11] by J. Kuben.

We suppose that $n \in N(n_0+1) = \{n_0+1, n_0+2, \dots\}$, ($n_0$ is a fixed nonnegative integer), Δ is the forward difference operator; i.e., $\Delta u_n = u_{n+1} - u_n$ for any function u: $N(n_0) \to R$ (R is a real line), p: $N(n_0) \to (0,\infty)$, q: $N(n_0) \to R$, f: $N(n_0+1) \times R \times R \to R$ is for any $n \in N(n_0+1)$ continuous as a function of $(y,z) \in R \times R$. Hereafter, the term "solution" of (1) or (2) is always used as such real sequence $\{u_n\}$ satisfying (1) or (2) for each $n \in N(n_0+1)$. Such a solution we denote by u_n.

Notation 1. Let M_1 be the set of all solutions of the equation (1) and M_2 the set of all solutions of the equation (2) that exist for all $n \in N(n_0+1)$.

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Let μ: $N(n_0) \to R$. The symbols O and o have the usual meaning: $z_n = O(\mu_n)$ denotes that there exists $c_1 > 0$ such that $|z_n| \leq c_1|\mu_n|$ for large n, and $z_n = o(\mu_n)$ denotes that there exists h_n such that $z_n = \mu_n h_n$ and $\lim_{n\to\infty} h_n = 0$.

Definition 1. We shall say that the equations (1) and (2) are μ^0-asymptotically equivalent if for each $x \in M_1$ there exists $y \in M_2$ such that

$$x_n - y_n = o(\mu_n^0)\,, \tag{3}$$

and conversely.

Definition 2. We shall say that the equations (1) and (2) are weakly μ^1-asymptotically equivalent if for each $x \in M_1$ there exists $y \in M_2$ such that

$$\Delta x_n - \Delta y_n = o(\mu_n^1)\,, \tag{3'}$$

and conversely.

Definition 3. The equations (1) and (2) will be called strongly (μ^0, μ^1)-asymptotically equivalent if for appropriate x_n and y_n, (3) and (3′) holds.

The asymptotic equivalence was studied by many authors e.g. [1]–[10]. Our method is similar to that of [9] but is applied to the difference equation.

2. Equivalence of nonhomogeneous linear difference equations

Let in equation (2) $f(n,u,v) \equiv a_n$, where a: $N(n_0+1) \to R$. Then the equation (2) has the form

$$\Delta(p_{n-1}\Delta y_{n-1}) + q_n y_n = a_n\,. \tag{4}$$

The method of variation of constants formula gives for each solution y of the equation (4) the relation

$$y_n = c_1 u_n + c_2 v_n - c^{-1} u_n \sum_{s=n_0+1}^{n} v_s a_s + c^{-1} v_n \sum_{s=n_0+1}^{n} u_s a_s\,, \tag{5}$$

where c_1, c_2 are arbitrary constants, u_n, v_n are lineary independent solutions of the equation (1),

$$c = p_n[u_n v_{n+1} - v_n u_{n+1}]\,.$$

Notation 2. If u_n, v_n are lineary independent solutions of (1) then

$$y_n^0 = -cu \sum_{s=n_0+1}^{n} v_s a_s - c v_n \sum_{s=n+1}^{\infty} u_s a_s\,,$$

where $c^{-1} = p_n W[u_n, v_n], W[\cdot,\cdot]$-the Casorati matrix is a particular solution of (4).

Applying the operator Δ to both sides of relation (5) we obtain

$$\Delta y_n = c_1 \Delta u_n + c_2 \Delta v_n - c^{-1} \Delta u_n \sum_{s=n_0+1}^{n} v_s a_s + c^{-1} \Delta v_n \sum_{s=n_0+1}^{n} u_s a_s\,. \tag{5'}$$

Theorem 1. *The equations* (1) *and* (4) *are* μ^0*-asymptotically equivalent (weakly* μ^1*-asymptotically equivalent, strongly* (μ^0, μ^1)*-asymptotically equivalent) if there exists a solution* y_n^0 *of the equation* (4) *such that*

$$y_n^0 = o(\mu_n^0),\ (\Delta y_n^0 = o(\mu_n^1),\ \Delta^i y_n^0 = o(\mu_n^i),\ i = 0,1) \quad \textit{where} \quad \Delta^0 y_n = y_n\,.$$

Proof. Each solution of the equation (4) can be expressed in the form

$$y_n = x_n + y_n^0$$

where x_n is an arbitrary solution of the equation (1). This implies the assertion of the theorem. □

Theorem 2. *Assume that*

$$u_n \sum_{s=n_0+1}^{n} v_s a_s + v_n \sum_{s=n+1}^{\infty} u_s a_s = o(\mu_n^0) \tag{6}$$

or

$$\Delta u_n \sum_{s=n_0+1}^{n} v_s a_s + \Delta v_n \sum_{s=n+1}^{\infty} u_s a_s = o(\mu_n^1) \tag{6'}$$

or both (6) *and* (6′) *hold. Then the equation* (4) *has a solution* y^0 *with property*

$$y_n^0 = o(\mu_n^0) \quad \textit{or} \quad \Delta y_n^0 = o(\mu_n^1) \quad \textit{or} \quad \Delta^i y_n^0 = o(\mu_n^i)\,; \qquad i = 0,1\,.$$

Proof. The assertion is an immediate consequence of the relations

$$y_n = c_1 u_n + c_2 v_n - c^{-1} u_n \sum_{s=n_0+1}^{n} v_s a_s - c^{-1} v_n \sum_{s=n+1}^{\infty} u_s a_s\,,$$

$$\Delta y_n = c_1 \Delta u_n + c_2 \Delta v_n - c^{-1} \Delta u_n \sum_{s=n_0+1}^{n} v_s a_s - c^{-1} \Delta v_n \sum_{s=n+1}^{\infty} u_s a_s\,.$$ □

Theorem 2'. *Assume that*

$$u_n \sum_{s=n}^{\infty} v_s a_s - v_n \sum_{s=n}^{\infty} u_s a_s = o(\mu_n^0) \tag{6''}$$

or

$$\Delta u_n \sum_{s=n}^{\infty} v_s a_s - \Delta v_n \sum_{s=n}^{\infty} u_s a_s = o(\mu_n^1) \tag{6'''}$$

or both (6″) *and* (6‴) *hold. Then the equation* (4) *has a solution* y^0 *with property*

$$y_n^0 = o(\mu_n^0) \quad \textit{or} \quad \Delta y_n^0 = o(\mu_n^1) \quad \textit{or} \quad \Delta^i y_n^0 = o(\mu_n^i)\,; \qquad i = 0,1\,.$$

Corollary 1. *If the hypotheses of Theorem 2 (or Theorem 2') holds, then the equations* (1) *and* (4) *are* μ^0*-asymptotically equivalent, weakly* μ^1*-asymptotically equivalent or strongly* (μ^0, μ_1) *asymptotically equivalent respectively.*

3. Equivalence of nonlinear difference equations

In this chapter we shall give sufficient conditions for the types of asymptotic equivalence defined above. We suppose that the following hypotheses hold:

(i) $f\colon N(n_0+1)\times R\times R\to R$

(ii) there exists a nonnegative function

$$F\colon N(n_0+1)\times R_+\times R_+\to R_+$$

which is continuous and nondecreasing with respect two last arguments for each fixed $n\in N(n_0+1)$ such that

$$|f(n,u,v)|\le F(n,|u|,|v|)\,. \tag{7}$$

Here R_+ is the set of all nonnegative real numbers.

Notation 3. Let $r^i\colon N(n_0)\to(0,\infty)$, $(i=0,1)$ be a positive function such that

$$\Delta^i u_n=O(r_n^i),\quad \Delta^i v_n=O(r_n^i)\,,\qquad (i=0,1)\,. \tag{8}$$

For example, we can take

$$r^i=|\Delta^i u_n|+|\Delta^i v_n|\,;\qquad (i=0,1)\,.$$

Theorem 3. *Suppose that* (7) *holds and let for any* $\alpha\ge 0$

$$\sum_{s=n_0}^{\infty}|u_s|F(s,\alpha r_s^0,\alpha r_s^1)<\infty$$

and

$$|\Delta^i u_n|\sum_{s=n_0+1}^{n}|v_s|F(s,\alpha r_s^0,\alpha r_s^1)=o(r_n^i)\,,\qquad (i=0,1)\,. \tag{9^i}$$

Let for each solution $y\in M_2$,

$$\Delta^i y_n=O(r_n^i)\,,\qquad (i=0,1) \tag{10^i}$$

and there exist finite limits for $\{\Delta^i u_n\}$, $\{\Delta^i u_v\}$, $i=0,1$.

The the equation (1) *and* (2) *are strongly* (μ^0,μ^1)*-asymptotically equivalent for each pair of functions* μ^0,μ^1*, such that for any* $\alpha\ge 0$

(11^i)

$$|\Delta^i u_n|\sum_{s=n_0+1}^{n}|v_s|F(s,\alpha r_s^0,\alpha r_s^1)+|\Delta^i v_n|\sum_{s=n+1}^{\infty}|u_s|F(s,\alpha r_s^0,\alpha r_s^1)=o(\mu_n^i)\,,$$
$$(i=0,1)\,.$$

Proof I. Let $y \in M_2$. Consider a nonhomogeneous linear difference equation

$$\Delta(p_{n-1}\Delta z_{n-1}) + q_n z_n = f(n, y_n, \Delta y_{n-1})$$

that possesses the solution y_n. From assumption of the theorem for appropriate $\alpha > 0$ we have

$$\begin{aligned} &\Big|\Delta^i u_n \sum_{s=n_0+1}^{n} v_s f(s, y_s, \Delta y_{s-1}) + \Delta^i v_n \sum_{s=n+1}^{\infty} u_s f(s, y_s, \Delta y_{s-1})\Big| \\ &\le |\Delta^i u_n| \sum_{s=n_0+1}^{n} |v_s| F(s, \alpha r_s^0, \alpha r_s^1) + |\Delta^i v_n| \sum_{s=n+1}^{\infty} |u_s| F(s, \alpha r_s^0, \alpha r_s^1) = o(\mu_n^i)\,, \\ &\qquad (i = 0, 1)\,. \end{aligned}$$

Theorem 2 guarantees the existence of a solution z such that $\Delta^i z_n = o(\mu_n^i)$, $(i = 0, 1)$. Then $x_n = y_n - z_n$ is the desired solution of the equation (1) that satisfies the order relations (3) and (3′).

II. Let $x \in M_1$ and consider eqautions

$$\begin{aligned} y_n &= x_n - c u_n \sum_{s=n_1+1}^{n} v_s f(s, y_s, \Delta y_{s-1}) \\ &\quad - c v_n \sum_{s=n+1}^{\infty} u_s f(s, y_s, \Delta y_{s-1}) \\ \Delta y_n &= \Delta x_n - c\Delta u_n \sum_{s=n_1+1}^{n} v_s f(s, y_s, \Delta y_{s-1}) \\ &\quad - c\Delta v_n \sum_{s=n+1}^{\infty} u_s f(s, y_s, \Delta y_{s-1}) \end{aligned} \tag{12}$$

for $n \ge n_1$ where $n_1 \ge n_0$ will be choosen later.

We denote by $\Phi = \Phi(N_{n_1}, R^2)$ the set all pairs functions defined on $N(n_1)$. For $g \in \Phi$, let $p_m(g) = \sup\{\|g_n\|\colon n \in N_m(n_1) = \{n_1, n_1+1, \dots, n_1+m\}\}$, $m = 0, 1, \dots$, here $\|\cdot\|$ is some convenient norm in R^2. Then p_m is a pseudo-norm and Φ with the topology induced by the family of pseudo-norms $\{p_m\}_{m=1}^{\infty}$ is a Frechet space.

Denote

$$B_\rho(n_1+1) = \{\varphi = [\varphi^0, \varphi^1] \in \Phi\colon |\varphi_n^i| \le \rho r_n^i\,, i = 0, 1\}\,, \qquad n_1 \ge n_0\,.$$

There exists $\alpha > 0$ such that

$$\lfloor \Delta^0 x, \Delta x\rfloor\,, \lfloor \Delta^0 u, \Delta u\rfloor\,, \lfloor \Delta^0 v, \Delta v\rfloor \in B_\alpha(n_0+1)\,.$$

Let $\rho \geq 2\alpha$ and choose n_1 so that

$$\sum_{s=n_1+1}^{\infty} |u_s| F(s, \rho r_s^0, \rho r_s^1) \leq \frac{|c|^{-1}}{2}$$

and

$$|\Delta^i u_n| \sum_{s=n_1+1}^{n} |v_s| F(s, \rho r_s^0, \rho r_s^1) \leq \frac{1}{2} \alpha r_n^i |c|^{-1}$$

for $n_1 \geq n_0$, $(i = 0, 1)$.

Let T: $B_\rho(n_1+1) \to B_\rho(n_1+1)$ be an operator. $T\varphi = [T_0\varphi, T_1\varphi]$, $\varphi = \lfloor \varphi^0, \varphi^1 \rfloor$ where

$$(T_i\varphi)(n) = \Delta^i x_n - c\Delta^i u_n \sum_{s=n_1+1}^{n} v_s f(s, \varphi_s^0, \varphi_s^1) - c\Delta^i v_n \sum_{s=n+1}^{\infty} u_s f(s, \varphi_s^0, \varphi_s^1)\,,$$
$$i = 0, 1\,.$$

The convergence in Φ is the uniform convergence on every compact subinterval on $\langle n_1 + 1, \infty)$.

Let $\varphi \in B_\rho(n_1 + 1)$, then

$$|(T_i\varphi)(n)| \leq \alpha r_n^i + \frac{1}{2}|c| \cdot \alpha \cdot |c|^{-1} r_n^i + \frac{1}{2}|c| \cdot \alpha \cdot |c|^{-1} r_n^i = 2\alpha r_n^i \leq \rho r_n^i$$

for $n \geq n_1 + 1$, $i = 0, 1$. Therefore $TB_\rho(n_1 + 1) \subset B_\rho(n_1 + 1)$.

Next, we will verify that the transformation T is continuous.

Let $\{\varphi_{ni}\}_{i=1}^{\infty}$ be a sequence of element $B_\rho(n_1+1)$ such that $\varphi_{ni} \underset{i\to\infty}{\longrightarrow} \varphi_{n0}$ in the Frechet space Φ.

Let $n_2 > n_1 + 1$ and $\varepsilon > 0$. Denote $d = \max r_n^0$ for $n \in \langle n_1 + 1, n_2 + 1\rangle$. Choose $n_3 > n_2 + 1$ such that

$$\sum_{s=n_3}^{\infty} |u_s| F(s, \rho r_s^0, \rho r_s^1) < \frac{|c|^{-1}\varepsilon}{8d}\,.$$

Put

$$\Theta = \min\left\{ \frac{\varepsilon|c|^{-1}}{2d \sum\limits_{s=n_1+1}^{n_2} |v_s|}\,, \frac{\varepsilon|c|^{-1}}{4d \sum\limits_{s=n_1+1}^{n_3} |u_s|} \right\}\,.$$

Since f is continuous and $\varphi_{ni} \to \varphi_{n0}$ convergent uniformly on $\langle n_1 + 1, n_3\rangle$, there exists a positive constant N_0 such that if $i \geq N_0$, then

$$|f(n, \varphi_{ni}^0, \varphi_{ni}^1) - f(n, \varphi_{n0}^0, \varphi_{n0}^1)| < \Theta$$

for $n \in \langle n_1+1, n_3\rangle$. Thus

$$\begin{aligned}
&|(T_0\varphi_i)(n) - (T_0\varphi_0)(n)| \\
&\quad \le |c|\,|u_n| \sum_{s=n_1+1}^{n} |v_s|\,|f(s,\varphi^0_{si},\varphi^1_{si}) - f(s,\varphi^0_{so},\varphi^1_{so})| \\
&\qquad + |c|\,|v_n| \sum_{s=n+1}^{\infty} |u_s|\,|f(s,\varphi^0_{si},\varphi^1_{si}) - f(s,\varphi^0_{so},\varphi^1_{so})| \\
&\quad \le |c|\,|u_n| \sum_{s=n_1+1}^{n} |v_s|\,|f(s,\varphi^0_{si},\varphi^1_{si}(-f(s,\varphi^0_{so},\varphi^1_{so})| \\
&\qquad + |c|\,|v_n| \sum_{s=n+1}^{n_3} |u_s|\,|f(s,\varphi^0_{si},\varphi^1_{si}(-f(s,\varphi^0_{so},\varphi^1_{so})| \\
&\qquad + |c|\,|v_n| \sum_{s=n_3+1}^{\infty} |u_s|\,|f(s,\varphi^0_{si},\varphi^1_{si}(-f(s,\varphi^0_{so},\varphi^1_{so})| \\
&\quad \le |c|d\Theta \sum_{s=n_1+1}^{n} |v_s| + |c|d\Theta \sum_{s=n+1}^{n_3} |u_s| + 2|c|d \sum_{s=n_3+1}^{\infty} |u_s| F(s,r^0_s,r^1_s) < \varepsilon
\end{aligned}$$

for $i \ge N_0$ and $n \in \langle n_1+1, n_2+1\rangle$.

Therefore, the mapping T_0 is continuous. The same is true for T_1. This implies that T is continuous. Since $TB_\rho(n_1+1) \subset B_\rho(n_1+1)$, then $TB_\rho(n_1+1)$ is uniformly bounded for each n.

It suffices to prove that elements of $TB_\rho(n_1+1)$ satisfy Cauchy's condition uniformly on $TB_\rho(n_1+1)$. In fact, let $\varphi \in B_\rho(n_1+1)$ and $n > m \in N(n_1+1)$. Then we have

$$\begin{aligned}
&|(T_0\varphi)(n) - (T_0\varphi)(m)| \\
&\quad \le |x_n - x_m| + |c|\,|u_n \sum_{s=n_1+1}^{n} v_s\, f(s,\varphi^0_s,\varphi^1_s - u_m \sum_{s=n_1+1}^{m} v_s f(s,\varphi^0_s,\varphi^1_s)| \\
&\qquad + |c|\,|v_n \sum_{s=n+1}^{\infty} u_s f(x,\varphi^0_s,\varphi^1_s) - v_m \sum_{s=m+1}^{\infty} u_s f(s,\varphi^0_s,\varphi^1_s)| \\
&\quad \le |c| \Big\{ |u_n| \sum_{s=m+1}^{n} |v_s| F(s,\rho r^0_s,\rho r^1_s) + |u_n| \sum_{s=n_1+1}^{n} |v_s| F(s,\rho r^0_s,\rho r^1_s) \\
&\qquad + |u_m| \sum_{s=n_1+1}^{m} |v_s| F(s,\rho r^0_s,\rho r^1_s) + |v_n| \sum_{s=m+1}^{\infty} |u_s| F(s,\rho r^0_s,\rho r^1_s) \Big\}.
\end{aligned}$$

By assumptions of Theorem for given $\varepsilon > 0$, there exists $n_4 \in N(n_1+1)$ such that

$$|(T_0\varphi)(n) - (T_0\varphi)(m)| < \varepsilon$$

for all $n, m \in N(n_4)$.

The same is true for T_1. By Ascoli's theorem $TB_\rho(n_1+1)$ is relatively compact in Φ. Therefore as $B_\rho(n_1+1)$ is convex and closed in Φ. T has a fixed point in $B_\rho(n_1+1)$. This assertion is due to Tychonoff's fixed theorem – see e.g. [3], p. 45. At the same time, we have proved that the system (12) has a solution. The relations (11^i) and (12) imply that (3) and (3′) hold.

Theorem 4. *Suppose that* (7) *holds and let for any* $\alpha \geq 0$

$$\sum_{s=n_0+1}^{\infty} (|u_s| + |v_s|) F(s, \alpha r_s^0, \alpha r_s^1) < \infty .$$

Let for each $y \in M_2$ (10^i) *hold.*

If F *does not depend on* u *or* v*, the assumption* (10^i) *can be omitted. Then the equations* (1) *and* (2) *are strongly* (μ^0, μ^1)*-asymptotically equivalent for each pair of functions* μ^0, μ^1 *such that for any* $\alpha \geq 0$

$$\sum_{s=n+1}^{\infty} (|\Delta^i u_n \cdot v_s| + |u_s \cdot \Delta^i v_n|) F(s, \alpha r_s^0, \alpha r_s^1) = o(\mu_s^i) , \qquad i = 0, 1 .$$

Proof. In an aim to prove this theorem one should consider the equations

$$y_n = x_n + c^{-1} u_n \sum_{s=n}^{\infty} v_s f(s, y_s, \Delta y_{s-1}) - c^{-1} v_n \sum_{s=n}^{\infty} u_s f(s, y_s, \Delta y_{s-1})$$

and

$$\Delta y_n = \Delta x_n + c^{-1} \Delta u_n \sum_{s=n+1}^{\infty} v_s f(s, y_s, \Delta y_{s-1}) - c^{-1} \Delta v_n \sum_{s=n+1}^{\infty} u_s f(s, y_s, \Delta y_{s-1}) ,$$

and follow an analogous way as in the case Theorem 3. □

4. Special cases of perturbations

Suppose that

$$|f(n, u, v)| \leq h_n |u| \tag{13}$$

or

$$|f(n, u, v)| \leq g_n |v| \tag{13'}$$

where $h, g\ N(n_0) \to \langle 0, \infty)$ are nonnegative.

Lemma 1. *Let* (8), (13) *and* $\sup_n l_0(r_n^0)^2 h_n \leq \gamma < 1$ *hold, where* l_0 *is a positive constant, then each solution of the equation* (2) *exists on* $N(n_0)$ *and*

$$y_n = O\Big(r_n^0 \exp\Big(\sum_{s=n_0+1}^{n-1} \frac{l_0}{1-\gamma}(r_s^0)^2 h_s\Big)\Big).$$

Proof. From the relation (5), assumption of theorem and generalised Gronwall's inequality we obtain the needed estimate. □

Lemma 2. *Let* (8) *and* (13′) *hold, then each solution of the equation* (2) *exists on* $N(n_0)$ *and*

$$\Delta y_n = O\Big(r_n^1 \exp\Big(\sum_{s=n_0}^{n-1} \overline{l_0} g_{s+1} r_{s+1}^0 r_s^1\Big)\Big),$$

where $\overline{l_0}$ *is a positive constant.*

Proof. In an aim to prove this Lemma one sholud consider the equation

$$\Delta y_n = c_1 \Delta u_n + c_2 \Delta v_n - c^{-1} \Delta u_n \sum_{s=n_0+1}^{n} v_s f(s, y_s, \Delta y_{s-1}) +$$
$$+ c^{-1} \Delta v_n \sum_{s=n_0+1}^{n} u_s f(s, y_s, \Delta y_{s-1})$$

and follow an analogous way as in the case of Lemma 1. □

Lemma 3. *Assume that*

1° (7) *holds,*

2° *for any* $\lambda \geq 0$, $\sum_{n=n_0+1}^{\infty} r_n^0 F(n, \lambda r_n^0, \lambda r_n^1) < \infty$,

3° *there exists* $\lambda_0 > 0$ *such that*

$$\sup_{\lambda \in \langle \lambda_0, \infty)} \frac{1}{\lambda} \sum_{n=n_1+1}^{\infty} r_n^0 F(n, \lambda r_n^0, \lambda r_n^1) = S < |c| \tag{14}$$

for an appropriate $n_1 \geq n_0$.

Then each solution y *of the equation* (2) *exists for* $n \geq n_1 + 1$ *and* $\Delta^i y_n = O(r_n^i)$, $i = 0, 1$.

Proof. As

$$y_n = c_1 u_n + c_2 v_n - c^{-1} u_n \sum_{s=n_0+1}^{n} v_s(s, y_s, \Delta y_{s-1})$$
$$+ c^{-1} v_n \sum_{s=n_0+1}^{n} u_s f(s, y_s, \Delta y_{s-1})$$

and

$$\Delta y_n = c_1\Delta u_n + c_2\Delta v_n - c^{-1}\Delta u_n \sum_{s=n_0+1}^{n} v_s(s, y_s, \Delta y_{s-1})$$
$$+ c^{-1}\Delta v_n \sum_{s=n_0+1}^{n} u_s f(s, y_s, \Delta y_{s-1}),$$

then for $n \in \langle n_1+1, N^0), n_1 \ge n_0, N^0 < \infty$ we have

$$|\Delta^i y_n| \le K r_n^i + |c|^{-1} r_n^i \sum_{s=n_1+1}^{n} r_s^0 F(s, |y_s|, |\Delta y_{s-1}|), \qquad i = 0, 1$$

K is a positive constant.

Denote

$$z_m = K|c| + \sum_{s=n_1+1}^{m} r_s^0 F(s, |y_s|, |\Delta y_{s-1}|), \qquad i = 0, 1 \tag{15}$$

for $m \in \langle n_1+1, N^0)$.

Then

$$|\Delta^i y_n| \le |c|^{-1} r_n^i z_m \quad \text{for} \quad n \in \langle n_1+1, m\rangle, \qquad i = 0, 1. \tag{16}$$

If $z_m < |c|\lambda_0$ for each $m \in \langle n_1+1, N^0)$ then

$$|\Delta^i y_n| \le \lambda_0 r_n^i, \quad n \in \langle n_1+1, N^0), \qquad i = 0, 1. \tag{17}$$

If there exists $m_0 \in \langle n_1+1, N^0)$ such that $z_{m_0} \ge |c|\lambda_0$ then $z_m \ge |c|\lambda_0$ for $m \in \langle m_0, N^0)$. From relation (14) we obtain

$$\sup_{\lambda\in\rangle\lambda_0,\infty)} \frac{1}{\lambda} \sum_{s=n_1+1}^{m} r_n^0 F(n, \lambda r_s^0, \lambda r_s^1) = S_1 \le S < |c|.$$

Put $\lambda = |c|^{-1} z_m$ for $m \in \langle m_0, N^0)$, then

$$\sum_{s=n_1+1}^{m} r_n^0 F(s, |c|^{-1} z_m r_s^0, |c|^{-1} z_m r_s^1) \le |c|^{-1} S z_m.$$

Now from (15) and (16) we obtain

$$z_m \le K|c| + |c|^{-1} S z_m, \quad m \in \langle m_0, N^0).$$

Therefore

$$z_m \le \frac{K|c|}{1 - |c|^{-1} S},$$

since $|c|^{-1}S < 1$.

Relation (16) implies

(18)
$$|\Delta^i y_n| \le \frac{K}{1-|c|^{-1}S} r_n^i\,, \quad \text{for} \quad n \in \langle n_1+1, m)\,,\ m \in \langle m_0, N^0)\,, \qquad i = 0,1\,.$$

But this estimate does not depend on m, thus (18) holds for each $n \in \langle n_1+1, N^0)$.

As (17) or (18) holds, we get $\Delta^i y_n$ $(i = 0,1)$ are bounded on $\langle n_1+1, N^0)$. This is a contradiction and hence necessarily $N^0 = \infty$. At the same time we have obtained that
$$|\Delta^i y_n| = O(r_n^i)\,, \qquad i = 0,1\,.$$
□

Theorem 5. *Let the assumptions of Lemma 3 hold. Then the equations* (1) *and* (2) *are strongly* (r^0, r^1)*-asymptotically equivalent.*

Proof. The proof is a consequence of Theorem 4 and Lemma 3. □

Using Theorem 4 and Lemmas 1 and 2 we obtain

Theorem 6. *In addition to the assumptions of Lemma 1, suppose that*
$$\sum_{n=n_0+1}^{\infty} (r_n^0)^2 h_n < \infty\,.$$
Then the equations (1) *and* (2) *are* r^0*-asymptotically equivalent.*

Theorem 7. *In addition to the assumptions of Lemma 2, suppose that*
$$\sum_{n=n_0+1}^{\infty} r_n^0 r_n^1 g_n < \infty\,.$$
Then the equations (1) *and* (2) *are* r^1*-asymptotically equivalent.*

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