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GENERALIZED PICONE'S FORMULA AND FORCED OSCILLATIONS IN QUASILINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract. In the paper a comparison theory of Sturm-Picone type is developed for the pair of nonlinear second-order ordinary differential equations first of which is the quasilinear differential equation with an oscillatory forcing term and the second is the so-called half-linear differential equation. Use is made of a new nonlinear version of the Picone's formula.

1. Introduction

In this paper we are concerned with the forced quasilinear ordinary differential equation

$$(P(t)|y'|^{\alpha}\operatorname{sgn} y')' + Q(t)|y|^{\beta}\operatorname{sgn} y = f(t)\,, \quad t \ge t_0\,, \tag{A}$$

where $0 < \alpha \le \beta$ are constants and $P, Q, f : [t_0, \infty) \to R$ are continuous real-valued functions with $P(t) > 0$ for $t \ge t_0$.

By a solution of (A) on an interval $I \subset [t_0, \infty)$ we understand a function $y : I \to R$ which is continuously differentiable on I together with $P|y'|^{\alpha}\operatorname{sgn} y'$ and satisfies (A) at every point of I. Such a solution is called *oscillatory* if it is defined on an interval of the form $[t_x, \infty), t_x \ge t_0$, and has arbitrarily large zeros in this interval.

In [5], the present authors obtained Sturm-Picone type theorems for the special case of equation (A) with $\alpha = 1$ and $\beta > 1$ by comparing the forced superlinear equation (A) to an unforced linear equation employing a modified version of the well-known Picone's identity (see [3] and [4]). The main results improved and extended the corresponding results in [7]. The purpose of this paper is to extend comparison theorems from [5] to the pair of nonlinear equations first of which is

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the forced "super-half-linear" equation (A) and the second is the unforced half-linear equation (B) given below. Use is made of a half-linear version of Picone-type formula introduced in [3].

2. Sturmian theorems for the forced super-half-linear equation (A)

Define $\varphi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, $\alpha > 0$, and consider the nonlinear second-order differential equation

$$L_{\alpha\beta}[y] \equiv (P(t)\varphi_\alpha(y'))' + Q(t)\varphi_\beta(y) = f(t), \tag{A}$$

where $0 < \alpha \leq \beta$ are constants and P, Q and f are continuous real-valued functions on a given interval $I \subset [t_0, \infty)$ with $P(t) > 0$ for all $t \in I$. Denote by $\mathcal{D}_{L_{\alpha\beta}}(I)$ the domain of the operator $L_{\alpha\beta}$, i.e. the set of all continuous real-valued functions y defined on I such that y and $P\varphi_\alpha(y')$ are continuously differentiable on I.

The following lemma which is the modified nonlinear version of an identity introduced in [3] will be needed in order to prove our main results. The proof is straightforward and it is omitted.

Lemma 1. *If $y \in D_{L\alpha\beta}(I_0)$ for some non-degenerate subinterval $I_0 \subset I$ and $y(t) \neq 0$ in I_0, then for any $x \in C^1(I_0)$ the following identity holds:*

$$\begin{aligned} \frac{d}{dt}\left[\frac{|x|^{\alpha+1}}{\varphi_\alpha(y)}P(t)\varphi_\alpha(y')\right] &= P(t)|x'|^{\alpha+1} - \left[Q(t)|y|^{\beta-\alpha} - \frac{f(t)}{\varphi_\alpha(y)}\right]|x|^{\alpha+1} \\ &\quad - P(t)\Phi_\alpha(x', xy'/y) + \frac{|x|^{\alpha+1}}{\varphi_\alpha(y)}\{L_{\alpha\beta}[y] - f(t)\} \end{aligned} \tag{1}$$

where Φ_α denotes the form defined by

$$\Phi_\alpha(u, v) := |u|^{\alpha+1} + \alpha|v|^{\alpha+1} - (\alpha+1)u\varphi_\alpha(v)$$

which satisfies $\Phi_\alpha(u, v) \geq 0$ for all $u, v \in R$ with the equality holding if and only if $u = v$.

To obtain our first result concerning forced super-half-linear equation (A), assume that $Q(t) \geq 0$ on some subinterval $[a, b] \subset I$. Let $U[a, b] = \{\eta \in C^1[a, b] : \eta(a) = \eta(b) = 0\}$ and define the functional $J_{\alpha\beta} : U[a, b] \to R$ by

$$J_{\alpha\beta}[\eta] \equiv \int_a^b \left[P(t)|\eta'|^{\alpha+1} - \alpha^{-\alpha/\beta}\beta(\beta-\alpha)^{\frac{\alpha-\beta}{\beta}}[Q(t)]^{\frac{\alpha}{\beta}}|f(t)|^{\frac{\beta-\alpha}{\beta}}|\eta|^{\alpha+1}\right] dt$$

with the convention that $0^0 = 1$.

Theorem 1. *If there exists an $\eta \in U$, $\eta \not\equiv 0$, such that*

$$J_{\alpha\beta}[\eta] \le 0\,, \tag{2}$$

then every solution y of (A) *defined on $[a,b]$ and satisfying*

$$y(t)f(t) \le 0 \tag{3}$$

in this interval must have a zero in $[a,b]$.

Proof. Assume to the contrary that (A) has a solution y satisfying (3) and $y(t) \neq 0$ on $[a,b]$. Then the identity (1) with $x(t)$ replaced by $\eta(t)$ reduces to

$$\begin{aligned}\left[\frac{|\eta|^{\alpha+1}}{\varphi_\alpha(y)}P(t)\varphi_\alpha(y')\right]' = P(t)|\eta'|^{\alpha+1} - \left[Q(t)|y|^{\beta-\alpha} - \frac{f(t)}{\varphi_\alpha(y)}\right]|\eta|^{\alpha+1}\\ - P(t)\Phi_\alpha(\eta', \frac{\eta y'}{y})\,.\end{aligned} \tag{4}$$

Denote by $F(y)$ the expression in the brackets on the right-hand side of (4) considered as the function of y and observe that

$$\min_{y\neq 0} F(y) = \min_{y \neq 0}\left[Q|y|^{\beta-\alpha} + \frac{|f|}{|y|^\alpha}\right] = \alpha^{-\frac{\alpha}{\beta}}\beta(\beta-\alpha)^{\frac{\alpha-\beta}{\beta}}Q^{\frac{\alpha}{\beta}}|f|^{\frac{\beta-\alpha}{\beta}} \tag{5}$$

if $\alpha < \beta$, and

$$F(y) \ge Q(t) \tag{6}$$

if $\alpha = \beta$. Thus, with the convention that $0^0 = 1$, in both cases (4) reduces to

$$\begin{aligned}\left[\frac{|\eta|^{\alpha+1}}{\varphi_\alpha(y)}P(t)\varphi_\alpha(y')\right]' \le P(t)|\eta'|^{\alpha+1} - \alpha^{-\frac{\alpha}{\beta}}\beta(\beta-\alpha)^{\frac{\alpha-\beta}{\beta}}[Q(t)]^{\frac{\alpha}{\beta}}|f|^{\frac{\beta-\alpha}{\beta}}|\eta|^{\alpha+1}\\ - P(t)\Phi_\alpha(\eta', \frac{\eta y'}{y})\,,\end{aligned} \tag{7}$$

and integrating the inequality (7) from a to b we obtain

$$0 \le J_{\alpha\beta}[\eta] - \int_a^b P(t)\Phi_\alpha(\eta', \frac{\eta y'}{y})\,dt\,, \tag{8}$$

which is a contradiction unless $J_{\alpha\beta}[\eta] \equiv 0$ and $\Phi_\alpha(\eta', \frac{\eta y'}{y}) \equiv 0$ in $[a,b]$. The last relation implies that y must be a constant multiple of η, and so we get, in particular, that $y(a) = y(b) = 0$. This completes the proof. $\square$

The following corollary is an immediate consequence of Theorem 1.

Corollary 1. *Let there exist two sequences of disjoint intervals* (a_n^-, b_n^-), (a_n^+, b_n^+), $t_0 \le a_n^- < b_n^- \le a_n^+ < b_n^+$, $a_n^- \to \infty$ *as* $n \to \infty$ *such that*

$$Q(t) \ge 0 \quad on \quad [a_n^-, b_n^-] \cup [a_n^+, b_n^+]\,, \tag{9}$$

$$f(t) \le 0 \quad on \quad [a_n^-, b_n^-]\,, \tag{10}$$

$$f(t) \ge 0 \quad on \quad [a_n^+, b_n^+]\,, \tag{11}$$

$n = 1, 2, \dots,$ *and two sequences of nontrivial continuously differentiable functions* $\eta_n^-(t)$ *and* $\eta_n^+(t)$ *defined on* $[a_n^-, b_n^-]$ *and* $[a_n^+, b_n^+]$, *respectively, such that*

$$\eta_n^-(a_n^-) = \eta_n^-(b_n^-) = \eta_n^+(a_n^+) = \eta_n^+(b_n^+) = 0\,,$$

for $n = 1, 2, \dots,$ *and*

$$J_{\alpha\beta}[\eta_n^\pm] \equiv \int_{a_n^\pm}^{b_n^\pm} \left[P(t)|\eta_n^{\pm\prime}|^{\alpha+1} - \alpha^{-\frac{\alpha}{\beta}}\beta(\beta-\alpha)^{\frac{\alpha-\beta}{\beta}}[Q(t)]^{\frac{\alpha}{\beta}}|f(t)|^{\frac{\beta-\alpha}{\beta}}|\eta_n^\pm|^{\alpha+1}\right] dt \le 0 \tag{12}$$

for every $n \in N$. *Then all solutions of* (A) *are oscillatory.*

Our next results will be obtained by comparing the super-half-linear equation (A) with the unforced half-linear equation

$$l_\alpha[x] \equiv (p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0\,, \tag{B}$$

where $p, q : [t_0, \infty) \to R$ are continuous functions and $p(t) > 0$ for $t \ge t_0$. Analogously as in the case of the nonlinear differential operator $L_{\alpha\beta}$, by $D_{l_\alpha}(I)$ we denote the set of all real-valued functions which are defined and continuous on an interval $I \subset [t_0, \infty)$ and such that both x and $p\varphi_\alpha(x')$ are continuously differentiable on I.

Theorem 2 (Leighton-type comparison theorem)**.** *If there exists a nontrivial solution* $x \in D_{l_\alpha}([a,b])$ *of the half-linear equation* (B) *in* $[a,b]$ *such that* $x(a) = x(b) = 0$ *and*

$$V_{\alpha\beta}[x] \equiv \int_a^b \left[(p(t)-P(t))|x'|^{\alpha+1} + (\alpha^{-\frac{\alpha}{\beta}}\beta(\beta-\alpha)^{\frac{\alpha-\beta}{\beta}}[Q(t)]^{\frac{\alpha}{\beta}}|f(t)|^{\frac{\beta-\alpha}{\beta}} - q(t))|x|^{\alpha+1}\right] dt \ge 0\,, \tag{13}$$

then every solution y *of the forced super-half-linear equation* (A) *satisfying* $y(t)f(t) \le 0$ *in* (a,b) *has a zero in* $[a,b]$.

Proof. If $x \in \mathcal{D}_{l_\alpha}([a,b])$ is a nontrivial solution of (B) satisfying $x(a) = x(b) = 0$, then integration by parts yields

$$\int_a^b [p(t)|x'|^{\alpha+1} - q(t)|x|^{\alpha+1}]\, dt = 0\,. \tag{14}$$

Thus, combining (2) with (14) we obtain

$$V_{\alpha\beta}[x] = -J_{\alpha\beta}[x] \geq 0$$

and the conclusion follows from Theorem 1. □

Corollary 2 (Sturm-Picone type comparison theorem). *Let $Q(t) \geq 0$ in $[a,b]$. If*

$$p(t) \geq P(t) > 0\,, \tag{15}$$

$$\alpha^{-\frac{\alpha}{\beta}}\beta(\beta-\alpha)^{\frac{\alpha-\beta}{\beta}}[Q(t)]^{\frac{\alpha}{\beta}}|f(t)|^{\frac{\beta-\alpha}{\beta}} \geq q(t) \tag{16}$$

in $[a,b]$ and there exists a nontrivial solution $x \in \mathcal{D}_{l_\alpha}([a,b])$ of the half-linear equation (B) *such that $x(a) = x(b) = 0$, then any solution of* (A) *satisfying $y(t)f(t) \leq 0$ in (a,b) has a zero in $[a,b]$.*

As a consequence of Theorem 2, we have the following general comparison result which relates oscillation of the forced super-half-linear equation (A) to that of conjugacy of two sequences of associated "minorant" half-linear equations (B_n^-) and (B_n^+) below considered on the sequences of corresponding disjoint intervals $[a_n^-, b_n^-]$ and $[a_n^+, b_n^+]$, respectively.

Corollary 3. *Let there exist two sequences of disjoint intervals (a_n^-, b_n^-) and (a_n^+, b_n^+), $t_0 \leq a_n^- < b_n^- \leq a_n^+ < b_n^+$, $a_n^- \to \infty$ as $n \to \infty$ such that*

$$Q(t) \geq 0 \quad on \quad [a_n^-, b_n^-] \cup [a_n^+, b_n^+] \tag{17}$$

$$f(t) \leq 0 \quad on \quad [a_n^-, b_n^-]\,, \tag{18}$$

$$f(t) \geq 0 \quad on \quad [a_n^+, b_n^+]\,, \tag{19}$$

$n = 1, 2, \ldots$, and two sequences of half-linear equations

$$l_n^-[x] \equiv (p_n^-(t)\varphi_\alpha(x'))' + q_n^-(t)\varphi_\alpha(x) = 0\,, \tag{B_n^-}$$

$$l_n^+[x] \equiv (p_n^+(t)\varphi_\alpha(x'))' + q_n^+(t)\varphi_\alpha(x) = 0\,, \tag{B_n^+}$$

where $p_n^-, q_n^- : [a_n^-, b_n^-] \to R$ and $p_n^+, q_n^+ : [a_n^+, b_n^+] \to R$ are continuous functions with $p_n^-(t) > 0$ and $p_n^+(t) > 0$, with respective nontrivial solutions $x_n^- \in \mathcal{D}_{l_n^-}([a_n^-, b_n^-])$ and $x_n^+ \in \mathcal{D}_{l_n^+}([a_n^+, b_n^+])$ satisfying

$$x_n^-(a_n^-) = x_n^-(b_n^-) = x_n^+(a_n^+) = x_n^+(b_n^+) = 0\,, \tag{20}$$

$n = 1, 2, \dots,$ *and*

$$V_{\alpha\beta}[x_n^{\pm}] \equiv \int_{a_n^{\pm}}^{b_n^{\pm}} \{[p_n^{\pm}(t) - P(t)]|x_n^{\pm\prime}|^{\alpha+1} \tag{21}$$
$$+(\alpha^{-\frac{\alpha}{\beta}}\beta(\beta-\alpha)^{\frac{\alpha-\beta}{\beta}}[Q(t)]^{\frac{\alpha}{\beta}}|f(t)|^{\frac{\beta-\alpha}{\beta}} - q_n^{\pm}(t))|x_n^{\pm}|^{\alpha+1}\}\, dt \geq 0$$

for every $n \in N$. *Then all solutions of* (A) *are oscillatory.*

In our next result, by *consecutive sign change points* of the oscillatory forcing function f we understand points $t_1, t_2 \in [t_0, \infty), t_1 < t_2$, such that $f(t) \geq 0$ (resp. $f(t) \leq 0$) on $[t_1, t_2]$ and $f(t) < 0$ (resp. $f(t) > 0$) on $(t_1 - \epsilon, t_1) \cup (t_2, t_2 + \epsilon)$ for some $\epsilon > 0$ (see [2]).

Corollary 4. *Assume that* $Q(t) \geq 0$ *on* $[t_0, \infty)$,

$$p(t) \geq P(t)\,, \tag{22}$$

$$\alpha^{-\frac{\alpha}{\beta}}\beta(\beta-\alpha)^{\frac{\alpha-\beta}{\beta}}[Q(t)]^{\frac{\alpha}{\beta}}|f(t)|^{\frac{\beta-\alpha}{\beta}} \geq q(t)\,, \tag{23}$$

for $t \geq t_0$ *and either* (22) *or* (23) *do not become an identity on any open interval where* $f(t) \equiv 0$. *Moreover, suppose that the half-linear equation* (B) *is oscillatory and the distance between consecutive zeros of any solution of* (B) *is less that the distance between consecutive sign change points of the forcing function f. Then every nontrivial solution of the super-half-linear equation* (A) *is oscillatory, too.*

In our last Corollary, a solution of Eq. (B) is called *quickly oscillatory* if it is oscillatory and the sequence of its consecutive zeros $t_n, n = 1, 2, \dots,$ is such that $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$.

Corollary 5. *Let* $Q(t) \geq 0$ *for* $t \geq t_0$. *If* (22) *and* (23) *hold and every solution of* (B) *is quickly oscillatory, then every notrivial solution of the forced equation* (A) *is oscillatory, too, provided that the forcing function* $f(t)$ *changes sign on* $[T, \infty)$ *for each* $T \geq t_0$ *and the distance between consecutive sign change points of f is bounded from below.*

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