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A NOTE ON THE CAUCHY PROBLEM FOR FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

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ABSTRACT. Conditions for the existence and uniqueness of a solution of the Cauchy problem

$$u'(t) = p(t)u(\tau(t)) + q(t)\,, \qquad u(a) = c\,,$$

established in [2], are formulated more precisely and refined for the special case, where the function τ maps the interval $]a,b[$ into some subinterval $[\tau_0,\tau_1] \subseteq [a,b]$, which can be degenerated to a point.

INTRODUCTION

The following notation is used throughout.

R is the set of all real numbers,

$$[x]_+ = \frac{|x|+x}{2}, \qquad [x]_- = \frac{|x|-x}{2}\,.$$

$\widetilde{C}([a,b];R)$ is the set of absolutely continuous functions $u : [a,b] \to R$.

$L(]a,b[;R)$ is the space of Lebesgue integrable functions $p :]a,b[\to R$ with the norm

$$\|p\|_L = \int_a^b |p(s)|\,ds\,.$$

By a solution of the equation

$$u'(t) = p(t)u(\tau(t)) + q(t)\,, \tag{1}$$

where $p,q \in L(]a,b[;R)$, $\tau :]a,b[\to [a,b]$ is a measurable function, we understand a function $u \in \widetilde{C}([a,b];R)$ satisfying the equation (1) almost everywhere in $]a,b[$.

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Consider the problem of the existence and uniqueness of a solution of (1) satisfying the initial condition

$$u(a) = c\,, \tag{2}$$

where $c \in R$. According to Theorem 1.1 in [4] (for more general version of this result see [1]), the problem (1), (2) has a unique solution if and only if the corresponding homogeneous problem

$$u'(t) = p(t)u(\tau(t))\,, \qquad u(a) = 0 \tag{3}$$

has only the trivial solution. In [2] and [3] there were established optimal in some sense sufficient conditions for the existence and uniqueness of a solution of the problem (1), (2). In the present paper, those conditions are formulated more precisely and refined for the case, where the function τ maps the segment $[a,b]$ into some subsegment $[\tau_0, \tau_1] \subseteq [a,b]$, which can be eventually degenerated to a point. Precisely, we will suppose that there exist $\tau_0, \tau_1 \in [a,b]$, $\tau_0 \leq \tau_1$ such that $\tau(t) \in [\tau_0, \tau_1]$ for almost all $t \in [a,b]$. Thus, in the sequel it will be assumed that

$$\tau_0 = \text{ess inf}\{\tau(t) : t \in \,]a,b[\}\,, \qquad \tau_1 = \text{ess sup}\{\tau(t) : t \in \,]a,b[\}\,.$$

1. Main Results

Theorem 1. *Let there exist a function $\gamma \in \widetilde{C}([a,\tau_1];\,]0;+\infty[)$ such that*

$$\gamma'(t) \geq [p(t)]_+\gamma(\tau(t)) + [p(t)]_- \tag{4}$$

and either

$$\gamma(\tau_1) - \gamma(a) < 3 \tag{5}$$

or

$$\gamma(\tau_0) - \gamma(a) > 1 \quad \text{and} \quad \gamma(\tau_1) - \gamma(\tau_0) < 1 + \frac{1}{\gamma(\tau_0) - \gamma(a)}\,. \tag{6}$$

Then the problem (1), (2) has a unique solution.

Remark 1. In general, the strict inequality in (5) cannot be replaced by the nonstrict one. However, according to the condition (6), in the case $\gamma(\tau_0)-\gamma(a) > 1$, the constant 3 in (5) can be improved.

Corollary 1. *Let $(t - \tau(t))[p(t)]_+ \geq 0$ for almost all $t \in \,]a,b[$,*

$$\int_a^{\tau_0} \exp\left(\int_s^{\tau_0} [p(\xi)]_+ d\xi\right) [p(s)]_- ds > 1\,,$$

and

$$\int_a^{\tau_1} \exp\left(\int_s^{\tau_1} [p(\xi)]_+ d\xi\right) [p(s)]_- ds - \int_a^{\tau_0} \exp\left(\int_s^{\tau_0} [p(\xi)]_+ d\xi\right) [p(s)]_- ds$$
$$< 1 + \left[\int_a^{\tau_0} \exp\left(\int_s^{\tau_0} [p(\xi)]_+ d\xi\right) [p(s)]_- ds\right]^{-1}.$$

Then the problem (1), (2) *has a unique solution.*

Theorem 2. *Let*

$$\int_a^{\tau_1} [p(s)]_+ ds < 1 \tag{7}$$

and either

$$\int_a^{\tau_1} [p(s)]_- ds < 1 + 2\sqrt{1 - \int_a^{\tau_1} [p(s)]_+ ds} \tag{8}$$

or

$$\int_{\tau_0}^{\tau_1} [p(s)]_- ds < 1 + \frac{1 - \int_a^{\tau_1} [p(s)]_+ ds}{\int_a^{\tau_0} [p(s)]_- ds}\,, \qquad \int_a^{\tau_0} [p(s)]_- ds > \sqrt{1 - \int_a^{\tau_1} [p(s)]_+ ds}\,. \tag{9}$$

Then the problem (1), (2) *has a unique solution.*

Remark 2. For $\tau_0 = a$ and $\tau_1 = b$, from Theorems 1 and 2 we obtain Theorems 1.2 and 1.3 in [2].

The following theorem can be understand as a supplement of the previous theorem for the case, where the norm of the positive part of the coefficient p is greater than one.

Theorem 3. *Let*

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds < 1\,, \qquad \int_{\tau_0}^{\tau_1} [p(s)]_- ds < 1\,, \tag{10}$$

and

$$\int_a^{\tau_0} [p(s)]_+ ds > 1 + \frac{1}{1-T} \int_a^{\tau_1} [p(s)]_- ds\,, \tag{11}$$

where

$$T = \max\left\{ \int_{\tau_0}^{\tau_1} [p(s)]_+ ds\,,\ \int_{\tau_0}^{\tau_1} [p(s)]_- ds \right\}\,. \tag{12}$$

Then the problem (1), (2) *has a unique solution.*

Remark 3. All of the above theorems are optimal in the sense that the strict inequalities in the conditions (5)–(11) cannot be replaced by the nonstrict ones.

2. Proofs

According to Theorem 1.1 in [4] (see also [1]), to prove the theorems it is sufficient to show that the homogeneous problem (3) has only the trivial solution.

Proof of Theorem 1. Assume the contrary that there exists a nontrivial solution u of (3). According to Theorem 1.1 in [2] and condition (4), u changes its sign in $[\tau_0, \tau_1]$. Put

$$M = \max\{u(t) : t \in [\tau_0, \tau_1]\}, \qquad m = \max\{-u(t) : t \in [\tau_0, \tau_1]\},$$

and choose $c_1, c_2 \in [\tau_0, \tau_1]$ such that $u(c_1) = M$, $u(c_2) = -m$. Without loss of generality we can assume that $c_1 < c_2$.

In view of the condition (4) and Theorem 1.1 in [2], the problems

$$\alpha'(t) = [p(t)]_+\alpha(\tau(t)) + \frac{1}{M}[p(t)]_-[u(\tau(t))]_+, \qquad \alpha(a) = 0, \tag{13}$$

$$\beta'(t) = [p(t)]_+\beta(\tau(t)) + \frac{1}{m}[p(t)]_-[u(\tau(t))]_-, \qquad \beta(a) = 0 \tag{14}$$

are uniquely solvable on $[a, \tau_1]$. Let α, resp. β, be a solution of the problem (13), resp. (14). Then, according to (4) and Theorem 1.1 in [2], we have $\alpha(t) \geq 0$, $\beta(t) \geq 0$ for $t \in [a, \tau_1]$, and so due to (4) and (13), resp. (14), $\gamma'(t) \geq \alpha'(t)$, resp. $\gamma'(t) \geq \beta'(t)$ for almost all $t \in]a, \tau_1[$, i.e.,

$$\begin{aligned} \gamma(x) - \gamma(y) &\geq \alpha(x) - \alpha(y) \\ \gamma(x) - \gamma(y) &\geq \beta(x) - \beta(y) \end{aligned} \qquad \text{for } x, y \in [a, \tau_1], \quad x \geq y. \tag{15}$$

Moreover, from (3), (13) and (14) it follows that

$$M\alpha'(t) \geq -u'(t), \qquad m\beta'(t) \geq u'(t) \qquad \text{for } t \in]a, \tau_1[.$$

Integration of the latter inequalities from c_1 to c_2, resp. from a to c_1, results in

$$M\big(\alpha(c_2) - \alpha(c_1)\big) \geq m + M, \qquad \text{resp.} \qquad m\big(\beta(c_1) - \beta(a)\big) \geq M.$$

Consequently, in view of the monotonicity of γ and (15),

$$\gamma(\tau_1) - \gamma(c_1) \geq \frac{m}{M} + 1, \qquad \gamma(c_1) - \gamma(a) \geq \frac{M}{m},$$

i.e.,

$$1 + \frac{1}{\gamma(c_1) - \gamma(a)} \leq \gamma(\tau_1) - \gamma(c_1). \tag{16}$$

Assume that the condition (5) holds. Then from (16) we have

$$3 \leq 1 + \frac{1}{\gamma(c_1) - \gamma(a)} + \gamma(c_1) - \gamma(a) \leq \gamma(\tau_1) - \gamma(a),$$

which contradicts (5).

Assume now that the inequality (6) holds. Then from (16) we have

$$\gamma(\tau_1) - \gamma(\tau_0) \geq 1 + \frac{1}{\gamma(c_1) - \gamma(a)} + \gamma(c_1) - \gamma(a) - \big(\gamma(\tau_0) - \gamma(a)\big).$$

Hence, taking into account $\gamma(c_1) \geq \gamma(\tau_0)$, $\gamma(\tau_0) - \gamma(a) > 1$ and the fact that the mapping $t \mapsto t + \frac{1}{t}$ is increasing for $t > 1$, we get

$$\gamma(\tau_1) - \gamma(\tau_0) \geq 1 + \frac{1}{\gamma(\tau_0) - \gamma(a)}\,,$$

which contradicts the condition (6). □

Proof of Corollary 1. Choose $\varepsilon > 0$ such that

$$\varepsilon\left(\exp\left(\int_a^{\tau_1}[p(\xi)]_+d\xi\right) - \exp\left(\int_a^{\tau_0}[p(\xi)]_+d\xi\right)\right)$$
$$+\int_a^{\tau_1}\exp\left(\int_s^{\tau_1}[p(\xi)]_+d\xi\right)[p(s)]_-ds - \int_a^{\tau_0}\exp\left(\int_s^{\tau_0}[p(\xi)]_+d\xi\right)[p(s)]_-ds$$
$$< 1 + \left[\int_a^{\tau_0}\exp\left(\int_s^{\tau_0}[p(\xi)]_+d\xi\right)[p(s)]_-ds + \varepsilon\left(\exp\left(\int_a^{\tau_0}[p(\xi)]_+d\xi\right) - 1\right)\right]^{-1}.$$

Put

$$\gamma(t) = \varepsilon\exp\left(\int_a^{t}[p(\xi)]_+d\xi\right) + \int_a^{t}\exp\left(\int_s^{t}[p(\xi)]_+d\xi\right)[p(s)]_-ds\,.$$

Then the inequalities (4) and (6) in Theorem 1 are fulfilled. □

Proof of Theorem 2. Assume the contrary that there exists a nontrivial solution u of the problem (3).

First suppose that u does not change its sign in $[\tau_0, \tau_1]$. Without loss of generality we can assume that $u(t) \geq 0$ for $t \in [\tau_0, \tau_1]$. Put

$$\overline{M} = \max\{u(t)\ :\ t \in [\tau_0, \tau_1]\}$$

and choose $t_0 \in [\tau_0, \tau_1]$ such that $u(t_0) = \overline{M}$. It is clear that $\overline{M} > 0$, since otherwise from (3) it would follow $u'(t) = 0$, $u(a) = 0$, and we would obtain $u(t) = 0$ for $t \in [a, b]$.

Integration of (3) from a to t_0 yields

$$\overline{M} = \int_a^{t_0}[p(s)]_+u(\tau(s))ds - \int_a^{t_0}[p(s)]_-u(\tau(s))ds \leq \overline{M}\int_a^{\tau_1}[p(s)]_+ds\,,$$

which together with (7) results in the contradiction $\overline{M} < \overline{M}$.

Now assume that u changes its sign in $[\tau_0, \tau_1]$. Put

$$M = \max\{u(t)\ :\ t \in [\tau_0, \tau_1]\}\,, \qquad m = \max\{-u(t)\ :\ t \in [\tau_0, \tau_1]\}\,. \tag{17}$$

It is clear that $M > 0$, $m > 0$. Choose $\alpha, \beta \in [\tau_0, \tau_1]$ such that

$$u(\alpha) = M, \qquad u(\beta) = -m\,.$$

Without loss of generality we can assume that $\alpha < \beta$. Set

$$A_1 = \int_a^\alpha [p(s)]_+ ds\,, \qquad A_2 = \int_\alpha^{\tau_1} [p(s)]_+ ds\,, \qquad A = \int_a^{\tau_1} [p(s)]_+ ds\,, \tag{18}$$

$$\begin{aligned} B_1 &= \int_a^{\tau_0} [p(s)]_- ds\,, \qquad & B_2 &= \int_{\tau_0}^{\alpha} [p(s)]_- ds\,, \\ B_3 &= \int_\alpha^{\tau_1} [p(s)]_- ds\,, \qquad & B &= \int_a^{\tau_1} [p(s)]_- ds\,, \end{aligned} \tag{19}$$

$$f(t) = \frac{1-A}{B_1+t} + t \qquad \text{for } t > -B_1\,. \tag{20}$$

Integrating (3) from a to α, resp. from α to β and taking into account (17)–(19), we obtain

$$\begin{aligned} M &\le MA_1 + m(B_1+B_2)\,, \\ M+m &\le mA_2 + MB_3\,. \end{aligned}$$

On the other hand, due to (6) we have $A_1 < 1$, $A_2 < 1$. Thus from the last two inequalities we get $B_1 + B_2 > 0$, $B_3 > 1$, and

$$B_3 \ge 1 + \frac{m}{M}(1-A_2), \qquad \frac{m}{M} \ge \frac{1-A_1}{B_1+B_2}\,. \tag{21}$$

On account of $(1-A_1)(1-A_2) \ge 1-(A_1+A_2) = 1-A$, from (21) we find

$$B_3 \ge 1 + \frac{1-A}{B_1+B_2}\,. \tag{22}$$

Suppose that the condition (8) is satisfied. From (22) we have

$$1 - A \le (B_1+B_2)(B_3-1)\,.$$

This, according to $A < 1$ and $(B_1+B_2)(B_3-1) \le \frac{1}{4}(B_1+B_2+B_3-1)^2 = \frac{1}{4}(B-1)^2$, implies

$$2\sqrt{1-A} \le B - 1\,,$$

which in view of (18) and (19) contradicts (8).

Now suppose that the condition (9) is satisfied. It is easy to verify that the function f defined by (20) is increasing in the interval $]\sqrt{1-A} - B_1, +\infty[$, and so, on account of $B_1 \ge \sqrt{1-A}$, it is increasing in the interval $]0, +\infty[$. Therefore from (22) we obtain

$$\int_{\tau_0}^{\tau_1} [p(s)]_- ds = B_3 + B_2 \ge 1 + \frac{1-A}{B_1+B_2} + B_2 = 1 + f(B_2) \ge 1 + f(0) = 1 + \frac{1-A}{B_1}\,,$$

which contradicts (9). $\square$

Proof of Theorem 3. Assume the contrary that there exists a nontrivial solution u of (3). First suppose that u has a zero in $[\tau_0, \tau_1]$. Put

$$\overline{M} = \max\{u(t) \,:\, t \in [\tau_0, \tau_1]\}, \qquad \overline{m} = \max\{-u(t) \,:\, t \in [\tau_0, \tau_1]\},$$

and choose $t_1, t_2 \in [\tau_0, \tau_1]$ such that

$$u(t_1) = -\overline{m}, \qquad u(t_2) = \overline{M}.$$

We have $\overline{M} + \overline{m} > 0$. Without loss of generality we can assume that $t_1 < t_2$. Integration of (3) from t_1 to t_2 yields

$$\overline{M} + \overline{m} \le \overline{M} \int_{\tau_0}^{\tau_1} [p(s)]_+ ds + \overline{m} \int_{\tau_0}^{\tau_1} [p(s)]_- ds,$$

which together with (10) results in the contradiction $\overline{M} + \overline{m} < \overline{M} + \overline{m}$.

Now suppose that u is of constant sign in $[\tau_0, \tau_1]$. Without loss of generality we can assume that $u(t) > 0$ for $t \in [\tau_0, \tau_1]$. Put

$$M = \max\{u(t) \,:\, t \in [\tau_0, \tau_1]\}, \qquad m = \min\{u(t) \,:\, t \in [\tau_0, \tau_1]\},$$

and choose $\alpha, \beta \in [\tau_0, \tau_1]$ such that

$$u(\alpha) = M, \qquad u(\beta) = m.$$

First assume that $\alpha < \beta$. Then the integration of (3) from α to β yields

$$m - M \ge -M \int_{\alpha}^{\beta} [p(s)]_- ds,$$

and, consequently,

$$1 - \int_{\tau_0}^{\tau_1} [p(s)]_- ds \le \frac{m}{M}. \tag{23}$$

Now assume that $\beta \le \alpha$. Then the integration of (3) from β to α results in

$$M - m \le M \int_{\beta}^{\alpha} [p(s)]_+ ds,$$

and, consequently,

$$1 - \int_{\tau_0}^{\tau_1} [p(s)]_+ ds \le \frac{m}{M}. \tag{24}$$

From (23) and (24) we get

$$\frac{M}{m} \le \frac{1}{1 - T}, \tag{25}$$

where T is defined by (12).

Integrating (3) from a to β, we find

$$m = \int_a^\beta [p(s)]_+ u(\tau(s))ds - \int_a^\beta [p(s)]_- u(\tau(s))ds \geq m \int_a^{\tau_0} [p(s)]_+ ds - M \int_a^{\tau_1} [p(s)]_- ds\,.$$

Consequently,

$$\int_a^{\tau_0} [p(s)]_+ ds \leq 1 + \frac{M}{m} \int_a^{\tau_1} [p(s)]_- ds\,.$$

The last inequality together with (25) contradicts (11). □

3. On Remark 1.3

In the examples below we will construct functions p and τ such that the corresponding homogeneous problem (3) has a nontrivial solution. According to the Fredholm property of the Cauchy problem for a linear functional differential equation (see [1,4]), there exist $q \in L(]a,b[;R)$ and $c \in R$ such that the problem (1), (2) has no solution.

Example 1. Let $\varepsilon_0 \in [0,1[$, $k \in [0,+\infty[$,

$$p(t) = \begin{cases} -k & \text{for } \ t \in]0,1[\\ 1 & \text{for } \ t \in]1, 1+\varepsilon_0[\\ -\frac{1}{\sqrt{1-\varepsilon_0}} & \text{for } \ t \in]1+\varepsilon_0, 2[\\ -1 & \text{for } \ t \in]2, 3+\sqrt{1-\varepsilon_0}[\end{cases},$$

(26)

$$\tau(t) = \begin{cases} 1 & \text{for } \ t \in]0,1[\\ 2 & \text{for } \ t \in]1, 1+\varepsilon_0[\cup]2, 3+\sqrt{1-\varepsilon_0}[\\ 3+\sqrt{1-\varepsilon_0} & \text{for } \ t \in]1+\varepsilon_0, 2[\end{cases}.$$

On the segment $[0, 3+\sqrt{1-\varepsilon_0}]$ consider the problem

$$u'(t) = p(t)u(\tau(t))\,, \qquad u(0) = 0\,. \tag{27}$$

Then $a = 0$, $\tau_0 = 1$, $\tau_1 = 3+\sqrt{1-\varepsilon_0}$,

$$\int_a^{\tau_1} [p(s)]_+ ds = \varepsilon_0\,, \qquad \int_a^{\tau_1} [p(s)]_- ds = k + 1 + 2\sqrt{1-\varepsilon_0} \geq 1 + 2\sqrt{1 - \int_a^{\tau_1} [p(s)]_+ ds}\,.$$

On the other hand,

$$u(t) = \begin{cases} 0 & \text{for } \ t \in [0,1[\\ t-1 & \text{for } \ t \in [1,2[\\ 3-t & \text{for } \ t \in [2, 3+\sqrt{1-\varepsilon_0}] \end{cases} \tag{28}$$

is a nontrivial solution of (27).

Moreover, let $\varepsilon \geq 0$. Put $k = 0$ and choose $\varepsilon_0 \in [0,1[$ such that

$$\frac{\varepsilon_0}{1-\varepsilon_0} + \frac{1}{\sqrt{1-\varepsilon_0}} + \sqrt{1-\varepsilon_0} = 2 + \varepsilon \,.$$

Then the function

$$\gamma(t) = \begin{cases} 1 & \text{for } \ t \in [0,1[\\ \frac{1+\sqrt{1-\varepsilon_0}}{1-\varepsilon_0}(t-1)+1 & \text{for } \ t \in [1,1+\varepsilon_0[\\ \frac{t-1}{\sqrt{1-\varepsilon_0}} + \frac{1}{1-\varepsilon_0} & \text{for } \ t \in [1+\varepsilon_0, 2[\\ t-2+\frac{1}{\sqrt{1-\varepsilon_0}} + \frac{1}{1-\varepsilon_0} & \text{for } \ t \in [2, 3+\sqrt{1-\varepsilon_0}] \end{cases}$$

satisfies the inequality (4), where p and τ are defined by (26). Furthermore,

$$\gamma(\tau_1) - \gamma(a) = 1 + \frac{\varepsilon_0}{1-\varepsilon_0} + \frac{1}{\sqrt{1-\varepsilon_0}} + \sqrt{1-\varepsilon_0} = 3 + \varepsilon \,.$$

However, as we have shown, the problem (27) has the nontrivial solution (28).

This example verifies the optimality of the condition (5) in Theorem 1 as well as the optimality of the condition (8) in Theorem 2.

Example 2. Let $k > 1$,

$$p(t) = \begin{cases} -k & \text{for } \ t \in]0,1[\\ -1 & \text{for } \ t \in]1, 2+\frac{1}{k}[\end{cases}, \qquad \tau(t) = \begin{cases} 2+\frac{1}{k} & \text{for } \ t \in]0,1[\\ 1 & \text{for } \ t \in]1, 2+\frac{1}{k}[\end{cases},$$

and on the segment $[0, 2+\frac{1}{k}]$ consider the problem (27). Then $a = 0$, $\tau_0 = 1$, $\tau_1 = 2 + \frac{1}{k}$, and the function

$$\gamma(t) = \begin{cases} kt+1 & \text{for } \ t \in [0,1[\\ t+k & \text{for } \ t \in [1, 2+\frac{1}{k}] \end{cases}$$

satisfies the inequality (4). Moreover,

$$\gamma(\tau_0) - \gamma(a) = k > 1 \,, \qquad \gamma(\tau_1) - \gamma(\tau_0) = 1 + \frac{1}{k} = 1 + \frac{1}{\gamma(\tau_0) - \gamma(a)} \,.$$

On the other hand, the problem (27) has a nontrivial solution

$$u(t) = \begin{cases} t & \text{for } \ t \in [0,1[\\ 2-t & \text{for } \ t \in [1, 2+\frac{1}{k}] \end{cases}.$$

This example shows that the strict inequality in the condition (6) cannot be replaced by the nonstrict one.

To verify the optimality of the condition (9) means to show that whenever ε_0, x_0, and y_0 are such that

$$\varepsilon_0 \in [0,1[\,, \qquad x_0 > \sqrt{1-\varepsilon}\,, \qquad y_0 \geq 1 + \frac{1-\varepsilon_0}{x_0}\,, \tag{29}$$

then there exist a function $p \in L(]a,b[;R)$ and a measurable function $\tau :]a,b[\to [a,b]$ such that

$$\int_a^{\tau_1} [p(s)]_+ ds = \varepsilon_0 \,, \qquad \int_a^{\tau_0} [p(s)]_- ds = x_0 \,, \qquad \int_{\tau_0}^{\tau_1} [p(s)]_- ds = y_0 \,, \tag{30}$$

and the problem (3) has a nontrivial solution.

Example 3. Let ε_0, x_0, and y_0 be such that the conditions (29) are fulfilled. Put

$$p(t) = \begin{cases} 1 & \text{for } \ t \in]0, \varepsilon_0[\\ -\frac{x_0}{1-\varepsilon_0} & \text{for } \ t \in]\varepsilon_0, 1[\\ -1 & \text{for } \ t \in]1,2[\cup]3, 3 + \frac{1-\varepsilon_0}{x_0}[\\ 1 + \frac{1-\varepsilon_0}{x_0} - y_0 & \text{for } \ t \in]2,3[\end{cases},$$

$$\tau(t) = \begin{cases} 1 & \text{for } \ t \in]0, \varepsilon_0[\cup]1,2[\cup]3, 3 + \frac{1-\varepsilon_0}{x_0}[\\ 3 + \frac{1-\varepsilon_0}{x_0} & \text{for } \ t \in]\varepsilon_0, 1[\\ 2 & \text{for } \ t \in]2,3[\end{cases}.$$

On the segment $[0, 3 + \frac{1-\varepsilon_0}{x_0}]$ consider the problem (27). Then $a = 0$, $\tau_0 = 1$, $\tau_1 = 3 + \frac{1-\varepsilon_0}{x_0}$, and the equalities (30) are fulfilled.

On the other hand,

$$u(t) = \begin{cases} t & \text{for } \ t \in [0,1[\\ 2-t & \text{for } \ t \in [1,2[\\ 0 & \text{for } \ t \in [2,3[\\ 3-t & \text{for } \ t \in [3, 3 + \frac{1-\varepsilon_0}{x_0}] \end{cases}$$

is a nontrivial solution of the problem (27).

Example 4. Let $k > 1$,

$$p(t) = \begin{cases} k & \text{for } \ t \in]0,1[\\ -\frac{k-1}{k} & \text{for } \ t \in]1,2[\end{cases}, \qquad \tau(t) = \begin{cases} 2 & \text{for } \ t \in]0,1[\\ 1 & \text{for } \ t \in]1,2[\end{cases},$$

and on the segment $[0,2]$ consider the problem (27). Then $a = 0$, $\tau_0 = 1$, $\tau_1 = 2$,

$$\int_{\tau_0}^{\tau_1} [p(s)]_+ ds = 0 \,, \qquad \int_{\tau_0}^{\tau_1} [p(s)]_- ds = \int_a^{\tau_1} [p(s)]_- ds = \frac{k-1}{k} \,, \qquad T = \frac{k-1}{k} \,,$$

and

$$\int_a^{\tau_0} [p(s)]_+ ds = k = 1 + \frac{1}{1-T} \int_a^{\tau_1} [p(s)]_- ds \,.$$

On the other hand,

$$u(t) = \begin{cases} t & \text{for } \; t \in [0,1[\\ -\frac{k-1}{k}\, t + \frac{2k-1}{k} & \text{for } \; t \in [1,2] \end{cases}$$

is a nontrivial solution of the problem (27).

This example shows that the strict inequality in the condition (11) cannot be replaced by the nonstrict one.

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