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ON THE ASYMPTOTIC CONVERGENCE OF THE POLYNOMIAL COLLOCATION METHOD FOR SINGULAR INTEGRAL EQUATIONS AND PERIODIC PSEUDODIFFERENTIAL EQUATIONS

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Abstract. We prove the convergence of polynomial collocation method for periodic singular integral, pseudodifferential and the systems of pseudodifferential equations in Sobolev spaces H^s via the equivalence between the collocation and modified Galerkin methods. The boundness of the Lagrange interpolation operator in this spaces when $s > 1/2$ allows to obtain the optimal error estimate for the approximate solution i.e. it has the same rate as the best approximation of the exact solution by the polynomials.

Introduction

Arnold and Wendland in [2] proposed the original technique of justification the spline collocation methods for pseudodifferential equations in Sobolev spaces. The justification is based on the equivalence between spline collocation and modified Galerkin-Petrov methods and justification of the last one by reducing it to the standard Galerkin method. In [3] - [7] this approach was used for justification of spline collocation methods for the various classes of singular integral and pseudodifferential equations. It was shown that strong ellipticity of the equation is sufficient and in some cases (see [20] - [19]) necessary condition for the convergence of spline collocation method. Note that earlier in [11] Golberg used the analogy between collocation and Galerkin methods as a criterion of optimal nodes choice.

Here the approach of the article [2] is used for justification of the polynomial collocation method for the singular integral, periodic pseudodifferential and the systems of the pseudodifferential equations in Sobolev spaces. The results show that polynomial collocation method converges for wider class of equations than

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spline collocation method. Namely it converges for all elliptic i.e. uniquely solvable equations while spline collocation method converges only for strong elliptic ones. Moreover, the rate of convergence of polynomial collocation method grows up with the growing of the smoothness of exact solution infinitely while for spline collocation method this rate is bounded by the order of used splines. Finally, the calculation schemes of polynomial collocation methods are easier than of spline collocation methods for the wide range of equations including singular integral and integro-differential equations, because the singular integrals of polynomials could be calculated explicitly.

1. Collocation with trigonometric polynomials

Consider the linear equation

$$Au = f\,,\ A : H^{s+\beta} \to H^s\,, \qquad s, \beta \in \mathbf{R}\,, \tag{1}$$

where f is known and u is desired unknown 2π-periodic, complex-valued functions, H^s denotes Sobolev space of order s i.e. the closure of all smooth 2π-periodic complex-valued functions of a real variable t with respect to the norm

$$||f||_s = ||f||_{H^s} = \{|\widehat{f}(0)|^2 + \sum_{0 \neq k \in \mathbf{Z}} |\widehat{f}(k)|^2 |k|^{2s}\}^{1/2},$$

where

$$\widehat{f}(k) = (2\pi)^{-1} \int_{-\pi}^{\pi} f(t)\bar{e}_k(t)\,dt\,, \qquad k \in \mathbf{Z},$$

are the complex Fourier coefficients of the function $f \in H^s$,

$$e_k(t) = \exp(ikt)\,, \qquad k \in \mathbf{Z}\,,$$

are trigonometric monomials and $\beta \in \mathbf{R}$ is the order of the operator A. For the following we'll assume that $s > 1/2$ providing the embedding of H^s in C.

Denote the space of trigonometric polynomials of degree n by

$$\mathcal{T}_n = \mathrm{span}\{e_k(t)\ :\ |k| \le n\}\,, \qquad n \in \mathbf{N}_0\,.$$

Here $\mathbf{N}_0 = \{0, 1, ...\}$ is a set of natural numbers including zero. When zero is excluded we write $\mathbf{N} = \{1, 2, ...\}$. Since $\dim \mathcal{T}_n = 2n+1$, define the equally-spaced collocation points by

$$t_k = 2\pi k/(2n+1)\,, \qquad |k| \le n\,, \tag{2}$$

and seek $u_n^* \in \mathcal{T}_n$ satisfying

$$(Au_n^*)(t_k) = f(t_k)\,, \qquad |k| \le n\,. \tag{3}$$

When it exists, the function u_n^* is said to be a trigonometric collocation solution of (1).

2. Some preliminaries

It's known that being equipped with the inner product

$$\langle f,g\rangle_s = \widehat{f}(0)\cdot\bar{\widehat{g}}(0) + \sum_{0\neq k\in\mathbf{Z}} \widehat{f}(k)\cdot\bar{\widehat{g}}(k)|k|^{2s}\,,$$

H^s becomes the Hilbert space and $H^0 = L_2$.

Denote the polynomial of the best approximation to $x \in H^s$ of degree $\leq n$ and the corresponding best approximation

$$(\mathcal{S}_n x)(t) = \sum_{|k|\leq n} \widehat{x}(k)e_k(t)\,, \quad E_n(x)_s = \inf_{x_n\in\mathcal{T}_n} \|x - x_n\|_s = \|x - \mathcal{S}_n x\|_s\,,$$

where $(\mathcal{S}_n x)(t)$ is the n-th partial sum of the Fourier series of x.

Lemma 1. *If $x \in H^p$ and $s \leq p$, then*[1]*,*

$$E_n(x)_s \leq cn^{s-p}E_n(x)_p\,.$$

Proof. Indeed

$$\begin{aligned} E_n(x)_s = \|x - \mathcal{S}_n x\|_s &= \{\sum_{|k|>n} |\widehat{x}(k)|^2|k|^{2s}\}^{1/2} \\ &= \{\sum_{|k|>n} |\widehat{x}(k)|^2|k|^{2p}|k|^{2(s-p)}\}^{1/2} \\ &\leq (n+1)^{s-p}E_n(x)_p \leq cn^{s-p}E_n(x)_p\,. \end{aligned}$$

□

Let $\mathcal{L}_n$ be the usual polynomial Lagrangian interpolation operator that assigns to every function $x \in H^s$ the polynomial

$$(\mathcal{L}_n x)(t) = (2n+1)^{-1}\sum_{|k|\leq n} x(t_k)\frac{\sin((2n+1)(t-t_k)/2)}{\sin((t-t_k)/2)}$$

coinciding with x in the nodes (2).

Lemma 2. *The operator $\mathcal{L}_n$ is bounded and the following estimate is valid*

$$\|\mathcal{L}_n\|_{H^s\to H^s} \leq \sqrt{1+\zeta(2s)}\,, \qquad n = 1,2,...,$$

where $\zeta(t)$ - is the Riemann's ζ-function bounded and decreasing for $t > 1$.

Proof. Let's rewrite Lagrange polynomial $\mathcal{L}_n x$ in a form

$$(\mathcal{L}_n x)(t) = \sum_{|k|\leq n} \widehat{x}(k)^{(n)}e_k(t)\,,$$

where

$$\widehat{x}(k)^{(n)} = (2n+1)^{-1}\sum_{|l|\leq n} x(t_l)\bar{e}_k(t_l)\,, \qquad |k| \leq n\,,$$

[1] Here and in c denotes generic constant not depending on n and having different values at different places.

are Fourier-Lagrange coefficients of the function x with respect to the nodes (2). Substituting x by it's Fourier series expansion and omitting summands equal to zero we will obtain

$$\widehat{x}(k)^{(n)} = (2n+1)^{-1} \sum_{|l|\le n} \bar{e}_k(t_l) \sum_{m\in\mathbf{Z}} \widehat{x}(m) e_m(t_l) = \sum_{\mu\in\mathbf{Z}} \widehat{x}(k+\mu(2n+1)) .$$

Now, denoting for the convenience $k' = k + \delta_{0k}$ (δ_{ij} is the Kronecker symbol) estimate with the help of Hölder inequality the norm of the polynomial $\mathcal{L}_n x$

$$\begin{aligned}
\|\mathcal{L}_n x\|_s^2 &= \sum_{|k|\le n} |\widehat{x}(k)^{(n)}|^2 |k'|^{2s} = \sum_{|k|\le n} |\sum_{\mu\in\mathbf{Z}} \widehat{x}(k+\mu(2n+1))|^2 |k'|^{2s} \\
&= \sum_{|k|\le n} |\sum_{\mu\in\mathbf{Z}} |(k+\mu(2n+1))'|^{-s} \widehat{x}(k+\mu(2n+1)) |(k+\mu(2n+1))'|^{s}|^2 \, |k'|^{2s} \\
&\le \sum_{|k|\le n} (\sum_{\mu\in\mathbf{Z}} |\widehat{x}(k+\mu(2n+1))|^2 |(k+\mu(2n+1))'|^{2s} \sum_{\mu\in\mathbf{Z}} |k'/(k+\mu(2n+1))'|^{2s}) \\
&\le \max_{|k|\le n} \{ \sum_{\mu\in\mathbf{Z}} |k'/(k+\mu(2n+1))'|^{2s} \} \|x\|_s^2 .
\end{aligned}$$

The chain of the inequalities

$$\begin{aligned}
\max_{|k|\le n} \{ \sum_{\mu\in\mathbf{Z}} |k'/(k+\mu(2n+1))'|^{2s} \} &\le 1 + \sum_{\mu\in\mathbf{N}} ((2\mu+1)^{-2s} + (2\mu-1)^{-2s}) \\
&\le 1 + \sum_{\mu\in\mathbf{N}} \mu^{-2s} = 1 + \zeta(2s)
\end{aligned}$$

concludes the proof. $\square$

Let

$$Jx = (2\pi)^{-1} \int_{-\pi}^{\pi} x(t)dt \quad \text{and} \quad J_n x = (2n+1)^{-1} \sum_{|k|\le n} x(t_k)$$

be the Riemann's integral of the function $x \in H^s$ and quadrature rule for its approximate calculation correspondently.

Lemma 3. *Let $x \in H^s$. Then*

$$x(t_l) = 0 , \qquad |l| \le n , \tag{4}$$

if and only if

$$\langle x - e_k J(x\bar{e}_k) + e_k J_n(x\bar{e}_k), e_k \rangle_s = 0 , \qquad |k| \le n . \tag{5}$$

Proof. Denote $r_k = J(x\bar{e}_k) - J_n(x\bar{e}_k)$, $|k| \le n$ the residuals of the quadrature sums and form a polynomial

$$x_n(t) = \sum_{|k|\le n} r_k e_k(t) . \tag{6}$$

We can rewrite (5) simply as $\langle x - x_n, e_k \rangle_s = 0$, $|k| \le n$. Assume that (4) are valid, then

$$\langle x - x_n, e_k \rangle_s = \langle x, e_k \rangle_s - \langle x_n, e_k \rangle_s = J_n(x\bar{e}_k)|k'|^{2s} = 0 , \qquad |k| \le n .$$

Now assume that (5) are valid, i.e

$$(2n+1)^{-1}|k'|^{2s}\sum_{|l|\le n} x(t_l)\bar{e}_k(t_l)=0\,,\qquad |k|\le n\,.$$

The coefficients $\{\bar{e}_k(t_l)\ :\ |k|\le n,\ |l|\le n\}$ form the kind of Vandermond's determinant. So the homogeneous system of equations (5) has the unique solution $x(t_l)=0,\ |l|\le n$. □

Denote by R_n the operator

$$R_n(x)=\sum_{|k|\le n} r_k e_k(t)\,,\quad R_n\ :\ H^s\to H^s\,,$$

which assigns the polynomial (6) to every function $x\in H^s$. The estimation of its norm gives the following

Lemma 4. *For every $x\in H^s$*

$$||R_n x||_s\le cE_n(x)_s\,,\qquad n\in\mathbf{N}_0\,.$$

Proof. The coefficients of the polynomial $R_n x$

$$r_k=J(x\bar{e}_k)-J_n(x\bar{e}_k)=J(x\bar{e}_k)-J(\bar{e}_k\mathcal{L}_n x)=J((x-\mathcal{L}_n x)\bar{e}_k)\,,\qquad |k|\le n\,,$$

are the first $2n+1$ Fourier coefficients of the function $(x-\mathcal{L}_n x)(t)$, so with the help of Lemma 2 we'll obtain

$$||R_n x||_s=||\sum_{|k|\le n} r_k e_k||_s=||\mathcal{S}_n(x-\mathcal{L}_n x)||_s=||\mathcal{L}_n(\mathcal{S}_n x-x)||_s\le\sqrt{1+\zeta(2s)}E_n(x)_s\,.$$

□

Lemmas 3 and 4 play a crucial role in the following account. The first one allows to prove the equivalence of the collocation and modified Galerkin methods. The second allows to estimate the rate of convergence of the last one.

3. Singular integral equations

Suppose that (1) is the singular integral equation

$$A\equiv aPu+bQu+Ku=f\,,\quad A:H^s\to H^s\,,\tag{7}$$

where P and Q are the complementary projection operators defined by

$$(Pu)(t)=\sum_{l\ge 0}\widehat{u}(l)e_l(t)\,,\quad (Qu)(t)=\sum_{l\le -1}\widehat{u}(l)e_l(t)\,,$$

and where $K\ :\ H^s\to H^s$ is a compact linear operator. For simplicity, assume that the coefficients $a,\ b\in C^\infty$ are 2π-periodic complex-valued functions of a real variable $t\in\mathbf{R}$, then

$$A:H^s\to H^s$$

is a bounded linear operator. If the coefficients satisfy

$$|a(t)|^2+|b(t)|^2\ne 0\quad\text{for all}\quad t\in\mathbf{R}\,,$$

then A is said to be *elliptic* (or non-degenerate).

The function $t \to a(t)$ parametrizes a smooth, closed curve in the complex plane, whose winding number (about the origin) is denoted by

$$W(a) = (1/2\pi)[\arg\, a(t)]_{t=-\pi}^{\pi}\,.$$

Also, the kernel, image, cokernel and index of $A : H^s \to H^s$ are denoted by

$$\begin{aligned}\ker(A) &= \{u \in H^s : Au = 0\}\,,\\ \operatorname{im}(A) &= \{f \in H^s : f = Au \quad \text{for some} \quad u \in H^s\}\,,\\ \operatorname{coker}(A) &= H^s/\overline{\operatorname{im}\,(A)}\,,\\ \operatorname{ind}(A) &= \dim\ker(A) - \dim\ \operatorname{coker}\ (A)\,.\end{aligned}$$

Theorem 1. *Suppose that for the equation* (7) *the following is held:*

A1 *the singular integral operator A is elliptic with $W(a) = W(b)$ and* $\ker(A) = \{0\}$,

A2 *there is an $\varepsilon > 0$ such that $K : H^s \to H^{s+\varepsilon}$ is bounded,*

A3 $s > 1/2$.

Then for all n sufficiently large, there exists a unique collocation solution u_n^* which converges to the exact solution u^* and

$$\|u^* - u_n^*\|_s \le cE_n(u^*)_s\,.$$

Corollary 1. *If in the Theorem 1 $u^* \in H^{s+p}$, then*

$$\|u^* - u_n^*\|_s \le cn^{-p}E_n(u^*)_{s+p}\,. \tag{8}$$

Proof of the Theorem 1. The index Theorem for singular integral operators (see [17]) states that A is a Fredholm operator if and only if A is elliptic, in which case

$$\operatorname{ind}\ (A) = W(a) - W(b)$$

and A is invertible.

Define the pair of spaces

$$\begin{aligned}C_+^\infty &= \{f \in C^\infty : \widehat{f}(l) = 0 \quad \text{for all} \quad l \le 0\}\,,\\ C_-^\infty &= \{f \in C^\infty : \widehat{f}(l) = 0 \quad \text{for all} \quad l \ge 0\}\,.\end{aligned}$$

It is not difficult to verify that the following are equivalent:

1. $f \in C_+^\infty$.
2. For all $N > 0$, $\widehat{f}(l) = O(l^{-N})$ as $l \to \infty$.
3. The function $f \in C_+^\infty$ admits an analytic continuation into the upper half plane, which is bounded and 2π-periodic (i.e. $f(z + 2\pi) = f(z)$ for $Imz \ge 0$).

There are analogous characterization for C_-^∞, and as a consequence one has the well known factorization property [9, p. 191], [10, p. 78].

Lemma 5. *Let $a \in C^\infty$ satisfy $a(t) \neq 0$ for all $t \in \mathbf{R}$. If $W(a) = \kappa$ then there exist functions $a_\pm \in C_\pm^\infty$ such that*

1. *$a(t) = a_+(t)e^{i\kappa t}a_-(t)$ for all $t \in \mathbf{R}$.*
2. *$1/a_\pm \in C_\pm^\infty$.*

We'll also need the following identities (see [10, p. 71]):

$$\begin{aligned} \mathcal{S}_n(a_+P + a_-Q)\mathcal{S}_n &= \mathcal{S}_n(a_+P + a_-Q)\,, \\ \mathcal{S}_n(Pa_- + Qa_+)\mathcal{S}_n &= (Pa_- + Qa_+)\mathcal{S}_n\,, \end{aligned} \tag{9}$$

for any $a_\pm \in C^\infty_\pm$.

According to the Theorem 1

$$W(b^{-1}a) = W(a) - W(b) = 0\,,$$

so due to Lemma 5 there exists a factorization

$$b^{-1}a = \rho_+\rho_-\,, \qquad \rho_\pm \in C^\infty_\pm\,.$$

Define the operators

$$M = b\rho_+\,, \quad N = P\rho_- + Q\rho_+^{-1}\,,$$

then, using the fact that $P + Q = I$, it is easy to see that

$$M^{-1} = \rho_+^{-1}b^{-1}\,, \quad N^{-1} = P\rho_-^{-1} + Q\rho_+\,.$$

Let $[\cdot,\cdot]$ be the usual commutator bracket, and define

$$T = M^{-1}K + [\rho_-, P] + [\rho_+^{-1}, Q]\,,$$

then elementary algebra gives

$$A = M(N+T)\,.$$

This representation in other notations appeared in the monographs of N. I. Muskhelishvili [16] and F. D. Gakhov [8]. For the justification of the approximate methods for singular integral equations it was used for the first time by V.V. Ivanov [12].

Consider the equation

$$Bu \equiv (N+T)u = y\,, \quad y = M^{-1}f\,, \quad B : H^s \to H^s\,, \tag{10}$$

which is equivalent to the equation (1). Equivalence here means that the equations (1) and (10) are either both solvable or not, and their solutions coincide. Moreover, McLean and Wendland [14, p. 367] have shown that the operator $T : H^s \to H^{s+\varepsilon}$ is bounded for the same s and ε as the operator K.

Analogously, consider the system of equations

$$(Bu_n)(t_k) = y(t_k)\,, \qquad |k| \le n\,,$$

which is equivalent to the system of equations (3). Lemma 3 allows us to rewrite the system of equations (3) as the system of equations of modified Galerkon-Petrov method

$$\langle Bu_n - y - R_n(Bu_n - y), e_l\rangle_s = 0\,, \qquad |l| \le n\,,$$

or in the form of operator equation of projection method

$$\mathcal{S}_nBu_n = \mathcal{S}_n(y + R_n(Bu_n - y))\,. \tag{11}$$

With the help of (9) and consequential identities

$$\mathcal{S}_nN\mathcal{S}_n = N\mathcal{S}_n\,, \quad \mathcal{S}_nNu_n = \mathcal{S}_nN\mathcal{S}_nu_n = N\mathcal{S}_nu_n = Nu_n\,, \tag{12}$$

left hand side of the equation (11) could be represented as

$$B_n u_n = \mathcal{S}_n B u_n = \mathcal{S}_n (N+T) u_n = (N + \mathcal{S}_n T) u_n \,.$$

Therefore from the invertibility of the operator $N+T : H^s \to H^s$ and the convergence

$$\|(T - \mathcal{S}_n T)u\|_z = E_n(Tu)_s \le cn^{-\varepsilon} E_n(Tu)_{s+\varepsilon} \le cn^{-\varepsilon}\|u\|_s$$

it follows that there is such a number $n_1 \in \mathbf{N}$ that for all $n \ge n_1$ the perturbed operator $B_n = N + \mathcal{S}_n T$ has an inverse satisfying the uniform bound

$$\|B_n^{-1}\|_{H^s \to H^s} \le c \,.$$

To estimate the rate of convergence of the approximate solutions $u_n^* \in \mathcal{T}_n$ to the exact solution $u^* \in H^s$ of the equation (1) we denote by $\bar{u}_n = \mathcal{S}_n u^* \in \mathcal{T}_n$ the polynomial of the best approximation of u^* and taking into account that $\bar{u}_n = B_n^{-1} \mathcal{S}_n B \bar{u}_n$ write

$$\begin{aligned} u^* - u_n^* &= u^* - \bar{u}_n + B_n^{-1}\mathcal{S}_n B \bar{u}_n - B_n^{-1}\mathcal{S}_n (y + R_n(Bu_n^* - y)) \\ &= u^* - \bar{u}_n + B_n^{-1}\mathcal{S}_n B \bar{u}_n - B_n^{-1}\mathcal{S}_n B u^* - B_n^{-1}\mathcal{S}_n R_n (Bu_n^* - y) \\ &= (I - B_n^{-1}\mathcal{S}_n B)(u^* - \bar{u}_n) + B_n^{-1}\mathcal{S}_n R_n (Bu^* - Bu_n^*) \,. \end{aligned}$$

The boundness of the operators B_n^{-1}, $\mathcal{S}_n$, B and the obvious identity $\mathcal{S}_n R_n = R_n$ allow us to obtain the estimation

(13)
$$\begin{aligned} \|u^* - u_n^*\|_s &\le c\|u^* - \bar{u}_n\|_s + c\|R_n(Bu^* - Bu_n^*)\|_s \\ &\le cE_n(u^*)_s + c\|R_n(Nu^* - Nu_n^*)\|_s + c\|R_n(Tu^* - Tu_n^*)\|_s \end{aligned}$$

With the help of the Lemma 4 the third summand of the right hand side can be estimated as follows:

$$\begin{aligned} \|R_n(Tu^* - Tu_n^*)\|_s \le cE_n(T(u^* - u_n^*))_s &\le cn^{-\varepsilon} E_n(T(u^* - u_n^*))_{s+\varepsilon} \\ &\le cn^{-\varepsilon}\|u^* - u_n^*\|_s \,. \end{aligned}$$

To estimate the second summand we will use once more Lemma 4, and the identities (12)

$$\begin{aligned} \|R_n(Nu^* - Nu_n^*)\|_s &\le c\|N(u^* - u_n^*) - \mathcal{S}_n N(u^* - u_n^*)\|_s \\ &\le c\|N(u^* - u_n^*) - N(\bar{u}_n - u_n^*) + \mathcal{S}_n N(\bar{u}_n - u_n^*) - \mathcal{S}_n N(u^* - u_n^*)\|_s \\ &\le c\|N(u^* - \bar{u}_n)\|_s + c\|\mathcal{S}_n N(u^* - \bar{u}_n)\|_s \le cE(u^*)_s \,. \end{aligned}$$

Hence, the estimation (13) will take the form

$$\|u^* - u_n^*\|_s \le cE_n(u^*)_s + cn^{-\varepsilon}\|u^* - u_n^*\|_s \,.$$

Choosing $n_2 \ge n_1$ on condition that $cn^{-\varepsilon} < 1/2$ is valid for all $n \ge n_2$ and transposing the last summand of the estimation to the left hand side we obtain finally

$$\|u^* - u_n^*\|_s \le 2cE_n(u^*)_s \,. \tag{14}$$

□

Lemma 1 and the estimation (14) allow to obtain the estimation (8) of the Corollary 1.

Remark. As it was shown in [13, p. 135] when **A1** is not satisfied, it may nevertheless be possible to obtain approximate solutions of $Au = f$ by the following modification due to Mikhlin and Prössdorf [15, p. 442-443].

With $\kappa = W(a) - W(b) \neq 0$ introduce the new operator

$$D = e_{-\kappa} aP + bQ + K(e_{-\kappa} P + Q)\,,$$

whose coefficients have equal winding numbers. Now let $v_n = \sum_{|l|\leq n} \xi_l e_l$ denote the approximate solution of the equation $Dv = f$ obtained by applying the collocation method. Then the sequence

$$u_n = \sum_{l=0}^{n} \xi_l e_{l-\kappa} + \sum_{l=-n}^{-1} \xi_l e_l$$

converges to a solution u of the original equation $Au = f$ provided a solution u exists.

4. Periodic pseudodifferential equations

We consider equations with operators

$$Bv \equiv (aP + bQ + K)\Lambda^\beta v = f \tag{15}$$

where the operator Λ^β denotes the Bessel potential operator of order $\beta \in \mathbf{R}$, given by

$$\Lambda^\beta e_l = |l + \delta_{0l}|^\beta e_l\,, \qquad l \in \mathbf{Z}\,,$$

(δ_{ij} is the Kronecker symbol) and corresponding continuous extension. If $\beta = 0$, then B is just a singular integral operator.

From Agranovich's Theorem [1], [21] it is known that any one-dimensional (classical) pseudodifferential operator of order β acting on periodic functions can be written in the form (15). In fact suppose B_0 is the principal part of B, i.e.

$$(B_0 u)(t) = \sum_{k \in \mathbf{Z}} \sigma_0(t,k)\widehat{u}(k)e^{ikt}\,,$$

where $\sigma_0(t,k)$ - the principal symbol of B - is 2π-periodic in t, and positive homogeneous of degree β in $k \neq 0$, i.e.,

$$\sigma_0(t,k) = \begin{cases} |k|^\beta \sigma_0(t, k/|k|)\,, & k \neq 0\,, \\ 1\,, & k = 0\,. \end{cases}$$

A simple calculation shows that with

$$a(t) = \sigma_0(t,+1),\ \ b(t) = \sigma_0(t,-1)\,, \tag{16}$$

the principal part of B can be written as $B_0 = (aP + bQ)\Lambda^\beta$. The following mapping property of Λ^β allows the extension of the former results to (15), see [23, p. 149].

Lemma 6. *For arbitrary $\beta,\ s \in \mathbf{R}$ the mapping*

$$\Lambda^\beta : H^{s+\beta} \to H^s$$

is an isomorphism.

Lemma 7. *Operator Λ^β commutes with $\mathcal{S}_n$, with P and with Q.*

Just as for the case of a singular integral operator ($\beta = 0$), we say that B is *elliptic* if the functions (16) satisfy

$$a(t) \neq 0 \quad \text{and} \quad b(t) \neq 0 \quad \text{for all} \quad t \in \mathbf{R}\,.$$

Obviously, when B is elliptic, the mapping

$$B : H^{s+\beta} :\to H^s\,, \qquad \beta \in \mathbf{R}\,,$$

is an isomorphism if and only if $W(a) = W(b)$ and $\ker(B) = \{0\}$.

Let us write $A = aP + bQ + K$, then the equation

$$Bv \equiv A\Lambda^\beta v = f \tag{17}$$

is equivalent to

$$Au = f \quad \text{with} \quad u = \Lambda^\beta v\,,$$

and the system of collocation method

$$(Bv_n)(t_k) = f(t_k)\,, \qquad |k| \leq n\,, \tag{18}$$

is equivalent to finding the polynomial

$$u_n = \Lambda^\beta v_n \in \mathcal{T}_n \tag{19}$$

satisfying

$$(Au_n)(t_n) = f(t_n)\,, \qquad |k| \leq n\,. \tag{20}$$

Therefore, the solvability of the (18) is determined by the solvability of the (20) and (19). The solution of the last one consists in the calculation of the Fourier coefficients of the polynomial v_n according to the formulas

$$\widehat{v}_n(l) = |l|^\beta \widehat{u}_n(l)\,, \qquad |l| \leq n\,. \tag{21}$$

Theorem 2. *Suppose that for the equation* (17) *the following is held:*

- **B1** *the pseudodifferential operator B is elliptic with $W(a) = W(b)$ and* $\ker\,(B) = \{0\}$,
- **B2** *there is an $\varepsilon > 0$ such that $K : H^{s+\beta} \to H^{s+\beta+\varepsilon}$ is bounded for all $s \in \mathbf{R}$,*
- **B3** *$s > 1/2$.*

Then for all n sufficiently large, there exists a unique collocation solution v_n^* of (18)*, and*

$$\|v^* - v_n^*\|_{s+\beta} \leq cE_n(v^*)_{s+\beta}\,.$$

Corollary 2. *If in the Theorem 2 $v^* \in H^{s+\alpha}$, $\alpha > \beta$, then*

$$\|v^* - v_n^*\|_{s+\beta} \leq cn^{\beta-\alpha} E_n(v^*)_{s+\alpha}\,.$$

Proof of the Theorem 2. The solvability of the system of equations (18) follows immediately from the Theorem 1 and the identities (21). Theorem 1 also allows to obtain the error estimation

$$\|v^* - v_n^*\|_{s+\beta} = \|\Lambda^\beta v^* - \Lambda^\beta v_n^*\|_s \le cE_n(\Lambda^\beta v^*)_s = cE_n(v^*)_{s+\beta} \,. \qquad \square$$

The last estimation and the Lemma 1 yield the estimation of the Corollary 2.

5. Systems of periodic pseudodifferential equations

As in [13] (see also the bibliography there) we will extend the analysis of the previous paragraphs to systems of singular integral equations and, further, to elliptic systems of periodic pseudodifferential equations. As indicated in [24], and using the results of [1], [21], every system of periodic pseudodifferential equations can be written in the form

$$Bv \equiv (\sigma_0(t,+1)P + \sigma_0(t,-1)Q + K)\Lambda^\beta v = f \,, \tag{22}$$

where σ_0 is the principal symbol of B, a matrix valued 2π-periodic by t function, and where $\beta = (\beta_1, \dots, \beta_L) \in \mathbf{R}^L$ is a suitable vector of orders. Λ^β is defined by the diagonal matrix of operators

$$\Lambda^\beta = (\delta_{ij}\Lambda^{\beta_i}) \,, \qquad i,j = 1, \dots, L \,,$$

(δ_{ij} is the Kronecker symbol), K satisfies assumption **B2**.

The trigonometric collocation method for (22) reads as to find the vector of polynomials $v_n \in \mathcal{T}_n^L$ satisfying

$$(Bv_n)(t_k) = f(t_k), \quad |k| \le n \,. \tag{23}$$

This system of equations is equivalent to the system

$$(Au_n)(t_k) = f(t_k), \quad |k| \le n \,,$$

with $u_n = \Lambda^\beta v_n$ and

$$A = \sigma_0(t,+1)P + \sigma_0(t,-1)Q + K \,.$$

Theorem 3. *Suppose that for the equation* (22) *the following is held:*

B1' $\det\sigma_0(t,\pm 1) \ne 0$ *for all* $t \in \mathbf{R}$ *and the left indices of* $\sigma_0(t,+1)$ *and the right indices of* $\sigma_0(t,-1)$ *and* $\sigma_0(t,-1)^{-1}\sigma_0(t,+1)$ *are all equal to zero,* $\ker(B) = \{0\}$,

B2' *there is an* $\varepsilon > 0$ *such that*

$$K : H_L^{s+\beta} \to H_L^{s+\beta+\varepsilon}, \quad H_L^{s+\beta} = \prod_{j=1}^{L} H^{s+\beta_j} \,, \quad \|\cdot\|_{s+\beta} = \sum_{j=1}^{L} \|\cdot\|_{s+\beta_j}$$

is bounded for all $s \in R$,

B3' $s > 1/2$.

Then for all n sufficiently large, there exists a unique collocation solution v_n^* *of* (23)*, and*

$$\|v^* - v_n^*\|_{s+\beta} \le cE_n(v^*)_{s+\beta}$$

Corollary 3. *If in the Theorem 3* $v^* \in H_L^{s+\alpha}$, $\alpha = (\alpha_1, \ldots, \alpha_L)$, $\alpha_j > \beta_j$, $j = 1, \ldots, L$, *then*

$$\|v^* - v_n^*\|_{s+\beta} \le cn^{\gamma} E_n(v^*)_{s+\alpha}\,, \quad \gamma = \max_{1\le j\le L} (\beta_j - \alpha_j)\,.$$

Proof of the Theorem 3. Assumption **B1'** guarantees the existence of the matrix factorizations

$$\sigma_0(t,+1) = a_+ a_-\,, \quad \sigma_0(t,-1) = b_- b_+\,,$$

with $a_\pm, b_\pm \in (C_\pm^\infty)^{L\times L}$, having all the desired properties needed in §3 and §4. The rest of the proof follows as in the proofs of the Theorems 1 and 2 and their Corollaries if the function factorizations are replaced by matrix factorizations. □

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