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RICCI CURVATURE OF REAL HYPERSURFACES IN COMPLEX HYPERBOLIC SPACE

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Abstract. First we prove a general algebraic lemma. By applying the algebraic lemma we establish a general inequality involving the Ricci curvature of an arbitrary real hypersurface in a complex hyperbolic space. We also classify real hypersurfaces with constant principal curvatures which satisfy the equality case of the inequality.

1. Statement of main result

Let M^n be a Riemannian n-manifold. For each 2-plane section $\pi \subset T_pM^n$, $p \in M^n$. We denote by $K(\pi)$ the sectional curvature of π. Let X be a unit vector in T_pM^n. If we choose an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM^n such that $e_1 = X$, then the Ricci curvature $Ric(X)$ at X is given by

$$Ric(X) = K_{12} + \cdots + K_{1n}\,, \tag{1.1}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j. The maximal Ricci curvature is defined by

$$\max Ric(p) = \max\{Ric(X) : X \in T_pM^n, |X| = 1\}, \quad p \in M^n\,. \tag{1.2}$$

The scalar curvature τ of M^n is defined by $\tau = \sum_{1\le i<j\le n} K_{ij}$.

Let $CH^m(-4)$ denote the complex hyperbolic m-space with constant holomorphic sectional curvature -4 and J be the almost complex structure on $CH^m(-4)$. Assume that M is a real hypersurface in $CH^m(-4)$. We denote by $\langle\ ,\ \rangle$ the inner product for M as well as for $CH^m(-4)$. Denote by ∇ and $\tilde\nabla$ the Levi-Civita connections of M and $CH^m(-4)$, respectively.

For any vector X tangent to M we put

$$JX = PX + FX\,, \tag{1.3}$$

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where PX and FX are the tangential and the normal components of JX, respectively. P is a well-defined endomorphism of the tangent bundle TM of M.

The Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\,, \tag{1.4}$$

$$\tilde{\nabla}_X \eta = -A_\eta X + D_X \eta \tag{1.5}$$

for vector fields X, Y tangent to M and vector field η normal to M, where h is the second fundamental form, D the normal connection, and A the shape operator of the real hypersurface. Let $||h||^2$ denote the square norm of the second fundamental form h.

The mean curvature vector $\overrightarrow{H}$ of M is given by

$$\overrightarrow{H} = \frac{1}{2m-1} \sum_{i=1}^{2m-1} h(e_i, e_i) \tag{1.6}$$

where $\{e_1, \ldots, e_{2m-1}\}$ is a local orthonormal frame of the tangent bundle TM. The length of $\overrightarrow{H}$ is called the *mean curvature* of M. The squared mean curvature function is $H^2 = \langle \overrightarrow{H}, \overrightarrow{H} \rangle$.

A real hypersurface M of $CH^m(-4)$ is called a *Hopf hypersurface* if the shape operator of M satisfies $A_\xi J\xi = \alpha J\xi$ for some function α, where ξ is a unit normal vector field of M in $CH^m(-4)$. The tangent vector field $J\xi$ on M is known as the *Hopf vector field*.

For real hypersurfaces in a complex hyperbolic space, we have the following general result.

Theorem 1. *Let $m \geq 2$ and M be a real hypersurface of the complex hyperbolic space $CH^m(-4)$ of constant holomorphic sectional curvature -4. Then the maximal Ricci curvature of M satisfies*

$$\max Ric \leq \frac{(2m-1)^2}{4} H^2 - 2(m-1)\,. \tag{1.7}$$

The equality sign of (1.7) *holds identically if and only if M is a Hopf hypersurface with constant mean curvature given by $2\alpha/(2m-1)$, where α is the principal curvature associated with the Hopf vector field $J\xi$, i.e., $A_\xi J\xi = \alpha J\xi$.*

Moreover, if M has constant principal curvatures, then M satisfies the equality case of inequality (1.7) *identically if and only if M is an open portion of one of the following real hypersurfaces:*

(i) *The horosphere in $CH^2(-4)$.*

(ii) *$m \geq 3$ and M is the tubular hypersurface over a totally geodesic complex hypersurface $CH^{m-1}(-4)$ in $CH^m(-4)$ with radius $r = \tanh^{-1}(1/\sqrt{2m-3})$.*

2. An algebraic lemma and its geometric interpretation

Let n be a natural number ≥ 2 and $n_1, \dots, n_k$ be k natural numbers. Then $(n_1, \dots, n_k)$ is called a *partition of* n if $n_1 + \cdots + n_k = n$.

First we give the following general algebraic lemma for later use.

Lemma 2. *Suppose that $a_1, \dots, a_n$ are n real numbers, k is an integer satisfying $2 \leq k \leq n-1$. Then, for any partition $(n_1, \dots, n_k)$ of n, we have*

$$\begin{aligned}\sum_{1\leq i_1<j_1\leq n_1} a_{i_1}a_{j_1} + \sum_{n_1+1\leq i_2<j_2\leq n_1+n_2} a_{i_2}a_{j_2} + \cdots + \sum_{n_1\cdots+n_{k-1}+1\leq i_k<j_k\leq n} a_{i_k}a_{j_k}\\ \geq \frac{1}{2k}\Big\{(a_1+\cdots+a_n)^2 - k(a_1^2+\cdots+a_n^2)\Big\},\end{aligned} \tag{2.1}$$

with the equality holding if and only if

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_n\,. \tag{2.2}$$

Proof. Let $a_1, \dots, a_n$ be n real numbers, k be an integer satisfying $2 \leq k \leq n-1$, and $(n_1, \dots, n_k)$ be a partition of n. Then we have

$$\begin{aligned}
&2k\Bigg\{\sum_{1\leq i_1<j_1\leq n_1} a_{i_1}a_{j_1} + \sum_{n_1+1\leq i_1<j_1\leq n_1+n_2} a_{i_2}a_{j_2} + \cdots + \sum_{n_1+\cdots+n_{k-1}+1\leq i_k<j_k\leq n} a_{i_k}a_{j_k}\Bigg\}\\
&\quad - \Big(\sum_{\alpha=1}^{n} a_\alpha\Big)^2 + k\sum_{\alpha=1}^{n} a_\alpha^2\\
&= 2k\Bigg\{\sum_{1\leq i_1<j_1\leq n_1} a_{i_1}a_{j_1} + \sum_{n_1+1\leq i_1<j_1\leq n_1+n_2} a_{i_2}a_{j_2} + \cdots + \sum_{n_1+\cdots+n_{k-1}+1\leq i_k<j_k\leq n} a_{i_k}a_{j_k}\Bigg\}\\
&\quad + (k-1)\sum_{\alpha=1}^{n} a_\alpha^2 - 2\sum_{1\leq\alpha<\beta\leq n} a_\alpha a_\beta\\
&= \Bigg\{\sum_{1\leq a_{i_1}\leq n_1} a_{i_1} - \sum_{n_1+1\leq a_{i_2}\leq n_1+n_2} a_{i_2}\Bigg\}^2 + \Bigg\{\sum_{1\leq a_{i_1}\leq n_1} a_{i_1} - \sum_{n_1+n_2+1\leq a_{i_3}\leq n_1+n_2+n_3} a_{i_3}\Bigg\}^2\\
&\quad + \cdots + \Bigg\{\sum_{n_1+\cdots+n_{k-2}+1\leq i_{k-1}\leq n_1+\cdots+n_{k-1}} a_{i_{k-1}} - \sum_{n_1+\cdots+n_{k-1}+1\leq i_k\leq n_1+\cdots+n_k} a_{i_k}\Bigg\}^2\\
&\geq 0\,,
\end{aligned}$$

with equality holding if and only if (2.2) holds. □

Remark 2.1. Lemma 3.1 of [3] is a special case of Lemma 2. In fact, if we put $(n_1, \dots, n_{n-1}) = (2, 1, \dots, 1)$, then Lemma 1 reduces to Lemma 3.1 of [3].

Remark 2.2. *Geometric Interpretation of Inequality* (2.1).

Let M be a Riemannian n-manifold and L be a subspace of T_pM of dimension $r \geq 2$. Suppose that $\{e_1, \dots, e_r\}$ is an orthonormal basis of L. Then the scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta)\,. \tag{2.3}$$

The scalar curvature $\tau(p)$ of M at p is nothing but the scalar curvature of the tangent space of M at p. And if L is a 2-plane section, $\tau(L)$ is nothing but the sectional curvature $K(L)$ of L. In general, $\tau(L)$ is nothing but the scalar curvature of the image $\exp_p(L)$ of L at p under the exponential map at p. If L is 1-dimensional subspace of $T_pM, p \in M$, we simply put $\tau(L) = 0$.

The inequality (2.1) is equivalent to the following geometric result according to the equation of Gauss.

Proposition 3. *Let M be a hypersurface of Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ and k be an integer in $\{2, \dots, n-1\}$. Then, for any partition $(n_1, \dots, n_k)$ of n, we have*

$$\tau(L_1) + \cdots + \tau(L_k) \geq \frac{1}{2k}(n^2H^2 - k\|h\|^2)\,, \tag{2.4}$$

where $L_1, \dots, L_k$ are the mutually orthogonal subspaces spanned by the principal vectors $\{e_1, \dots, e_{n_1}\}, \dots, \{e_{n_1+\cdots+n_{k-1}+1}, \dots, e_n\}$, respectively.

Proof. We only need to assume that $a_1, \dots, a_n$ are the principal curvatures of M in $\mathbb{E}^{n+1}$ associated with principal vectors $e_1, \dots, e_n$, respectively. □

Remark 2.3. Similar to the results given in [3-7], inequality (2.4) provides us a simple relationship between intrinsic and extrinsic invariants of submanifolds.

3. Proof of Theorem 1

Let M be a real hypersurface of the complex hyperbolic space $CH^m(-4)$. Let $e_1, \dots, e_{2m-1}$ be a local orthonormal frame of the tangent bundle TM. We put

$$a_j = h_{jj}\,, \quad h_{ij} = h(e_i, e_j)\,, \quad i, j = 1, \dots, 2m-1\,. \tag{3.1}$$

Since $(n_1, n_2) = (2m-2, 1)$ is a partition of $2m-1$, we may apply Lemma 2 to obtain the following inequality:

$$\sum_{1 \leq i < j \leq 2m-2} a_i a_j \geq \frac{1}{4}\Big\{(a_1 + \cdots + a_{2m-1})^2 - 2(a_1^2 + \cdots + a_{2m-1}^2)\Big\}\,, \tag{3.2}$$

with the equality holding if and only if

$$a_1 + \cdots + a_{2m-2} = a_{2m-1}\,. \tag{3.3}$$

On the other hand, the equation of Gauss together with the curvature expression of complex hyperbolic space imply that the Riemannian curvature tensor of M satisfies (cf. [2,4])

$$\begin{aligned} \langle R(X,Y)Z,W\rangle &= \langle h(X,W),h(Y,Z)\rangle - \langle h(X,Z),h(X,Z)\rangle \\ &\quad - \{\, \langle X,W\rangle \langle Y,Z\rangle - \langle X,Z\rangle \langle Y,W\rangle + \langle JY,Z\rangle \langle JX,W\rangle \\ &\quad - \langle JX,Z\rangle \langle JY,W\rangle + 2\langle X,JY\rangle \langle JZ,W\rangle \,\}. \end{aligned} \tag{3.4}$$

Hence, the sectional curvature K_{ij} of the 2-plane section section spanned by e_i and e_j is given by

$$K_{ij} = a_i a_j - h_{ij}^2 - 1 - 3\langle Pe_i, e_j\rangle^2 . \tag{3.5}$$

Substituting (3.5) into (3.2) gives

$$\begin{aligned} \sum_{1\le i<j\le 2m-2} K_{ij} &\ge \frac{1}{4}\Big\{(2m-1)^2H^2 - 2\|h\|^2 + 2\sum_{1\le i\ne j\le 2m-1} h_{ij}^2\Big\} \\ &\quad - (m-1)(2m-3) - \sum_{1\le i<j\le 2m-2} h_{ij}^2 - \frac{3}{2}\sum_{i,j=1}^{2m-2}\langle Pe_i, e_j\rangle^2 . \end{aligned} \tag{3.6}$$

From (3.4) we also know that the scalar curvature and the mean curvature of M satisfy

$$\|h\|^2 = (2m-1)^2H^2 - 2\tau - 2(m-1)(2m-1) - 3\|P\|^2, \tag{3.7}$$

where

$$\|P\|^2 = \sum_{\beta,\gamma=1}^{2m-1}\langle e_\beta, Pe_\gamma\rangle^2 \tag{3.8}$$

is the squared norm of the endomorphism P on TM.

By combining (3.6) and (3.7) we find

$$\begin{aligned} \sum_{1\le i<j\le 2m-2} K_{ij} &\ge \tau - \frac{(2m-1)^2}{4}H^2 + 2(m-1) \\ &\quad + \sum_{1\le j\le 2m-2} h_{j\,2m-1}^2 - \frac{3}{2}\sum_{i,j=1}^{2m-2}\langle Pe_i, e_j\rangle^2 + \frac{3}{2}\|P\|^2, \end{aligned} \tag{3.9}$$

which implies

$$\begin{aligned} &Ric(e_{2m-1}) \\ &\le \frac{(2m-1)^2}{4}H^2 - 2(m-1) - \sum_{1\le j\le 2m-2} h_{j\,2m-1}^2 - 3\sum_{i=1}^{2m-2}\langle Pe_{2m-1}, e_i\rangle^2 \\ &\le \frac{(2m-1)^2}{4}H^2 - 2(m-1). \end{aligned} \tag{3.10}$$

Since e_{2m-1} can be chosen to be any unit vector X tangent to M, (3.10) implies

$$\max Ric \le \frac{(2m-1)^2}{4} H^2 - 2(m-1). \tag{3.11}$$

Suppose that $\max Ric = Ric(e_{2m-1})$ and the equality sign of (3.11) holds. Then all of the inequalities in (3.2) and (3.10) become equalities. Thus, we have

$$h_{12m-1} = \cdots = h_{2m-22m-1} = 0\,, \tag{3.12}$$

$$\langle Pe_{2m-1}, e_j \rangle = 0, \quad j = 1, \dots, 2m-2\,, \tag{3.13}$$

$$a_1 + \cdots + a_{2m-2} = a_{2m-1}\,. \tag{3.14}$$

Condition (3.12) implies that e_{2m-1} is an eigenvector of A_ξ. And Condition (3.13) means that Je_{2m-1} is a normal vector of M. Thus, without loss of generality, we may assume $e_{2m-1} = J\xi$. Therefore, the Hopf vector field $J\xi$ is an eigenvector of A_ξ, *i.e.,* M is a Hopf's hypersurface. Hence, by applying a result of [1], we conclude that the principal curvature function $\alpha = a_{2m-1}$ corresponding to the Hopf vector field $J\xi$ is constant. Consequently, Condition (3.14) becomes that the trace of A_ξ is equal to 2α which is constant. Therefore, M is a Hopf hypersurface with constant mean curvature given by $2\alpha/(2m-1)$.

Conversely, it is easy to verify that every Hopf hypersurface with constant mean curvature $2\alpha/(2m-1)$ satisfies the equality case of (3.11).

Next, let us assume that the Hopf hypersurface M has constant principal curvatures. Then, by the classification theorem of Hopf hypersurfaces in $CH^m(-4)$ with constant principal curvatures given in [1], we know that M is orientable and it is an open portion of one of the following hypersurfaces:

(i) A tubular hypersurface with radius $r \in \mathbf{R}_+$ over a totally geodesic $CH^\ell(-4)$ for an integer $\ell \in \{0, \dots, m-1\}$;

(ii) A tubular hypersurface with radius $r \in \mathbf{R}_+$ over a totally geodesic $RH^m(-1)$;

(iii) A horosphere in $CH^m(-4)$.

For Case (i), M has principal curvatures $\{2\coth(2r), \tanh(r), \coth(r)\}$ of multiplicities $\{1, 2\ell, 2(m-\ell-1)\}$, respectively. For Case (ii), M has principal curvatures $\{2\tanh(2r), \tanh(r), \coth(r)\}$ of multiplicities $\{1, m-1, m-1\}$. And for Case (iii), M has principal curvatures $\{2, 1\}$ of multiplicities $\{1, 2m-2\}$. It is also known that the multiplicity of the principal curvature with respect to the Hopf vector field is one.

If Case (i) occurs, then $\alpha = 2\coth(2r)$. Thus, Condition (3.14) implies

$$2\coth(2r) = 2\ell\tanh(r) + 2(m-\ell-1)\coth(r)\,, \tag{3.15}$$

for some $\ell \in \{0 \dots, m-1\}$. If we put $x = \tanh(r)$, then Equation (3.15) becomes $1 + x^2 = 2\ell x^2 + 2(m-\ell-1)$. Thus, we obtain

$$x^2 = \frac{2\ell - 2m + 3}{2\ell - 1}\,. \tag{3.16}$$

Since $0 < \tanh^2(r) < 1$, Equation (3.16) implies

$$0 < \frac{2\ell - 2m + 3}{2\ell - 1} < 1\,. \tag{3.17}$$

Clearly, (3.17) cannot occur unless $\ell \geq 1$. Thus, we obtain from the second inequality of (3.17) that $m \geq 3$. On the other hand, the first inequality of (3.17) implies that $\ell > m - 3/2$. Thus, we must have $\ell = m - 1$. Substituting this into (3.16) yields $x^2 = 1/(2m-3)$. Hence, the radius r of the tubular hypersurface M is given by $r = \tanh^{-1}(1/\sqrt{2m-3})$.

If Case (ii) occurs, then Condition (3.14) implies

$$2\tanh(2r) = (m-1)\big(\tanh(r) + \coth(r)\big)\,. \tag{3.18}$$

Hence, we get $4x^2 = (m-1)(1+x^2)^2$, where $x = \tanh(r)$ as in Case (i). After solving this equation for x^2, we obtain

$$x^2 = \frac{3 - m \pm \sqrt{2-m}}{m-1}\,. \tag{3.19}$$

Since $x^2 = \tanh^2(r)$ is a real number, (3.19) implies that $m = 2$. Substituting this into (3.19) yields $x^2 = 1$ which is impossible since $-1 < \tanh(r) < 1$. Therefore, Case (ii) cannot occurs.

If Case (iii) occurs, then Condition (3.14) implies $m = 2$. Thus, M is an open portion of the horosphere in $CH^2(-4)$.

Conversely, it is easy to verify that the horosphere in $CH^2(-4)$ and the tubular hypersurface with radius $r = \tanh^{-1}(1/\sqrt{2m-3})$ over a totally geodesic complex hypersurface $CH^{m-1}(-4)$ in $CH^m(-4)$ with $m \geq 3$ have constant principal curvatures and constant mean curvature given by $2\alpha/(2m-1)$. $\square$

4. Real hypersurfaces in CH^2 satisfying the equality

When $m = 2$, the assumption of constant principal curvatures given in Theorem 1 holds automatically. In fact, we have the following.

Corollary 4. *Let M be a real hypersurface of the complex hyperbolic space $CH^2(-4)$. Then we have*

$$\max Ric \leq \frac{9}{4}H^2 - 2\,. \tag{4.1}$$

The equality sign of (4.1) *holds identically if and only if M is an open portion of the horosphere in $CH^2(-4)$.*

Proof. When $m = 2$, inequality (1.7) reduces to inequality (4.1).

Suppose that the equality case of (4.1) holds identically, then Theorem 1 implies that M is a Hopf hypersurface with constant mean curvature $2\alpha/3$, where α is the principal curvature associated with the Hopf vector field. Let $\mathcal{D}$ denote the distribution of rank 2 on M which is orthogonal to the Hopf vector field. Then $\mathcal{D}$ is a complex distribution. It is well-known that the shape operator of a Hopf hypersurface in $CH^2(-4)$ satisfies

$$2\langle X, PY\rangle = \alpha\langle AX, PY\rangle + 2\langle PAX, AY\rangle - \alpha\langle PX, AY\rangle \tag{4.2}$$

for X, Y tangent to M. From (4.2) it follows that the other two principal curvatures a_1, a_2 of M satisfy

$$2 + 2a_1a_2 = \alpha a_1 + \alpha a_2 \,. \tag{4.3}$$

Since $\alpha = a_1 + a_2$ is constant, (4.3) implies that both a_1, a_2 are constant too. Thus, we may apply Theorem 1 to conclude that M is an open portion of the horosphere in $CH^2(-4)$.

The converse have already been proved in Theorem 1. □

Remark 4.1. In views of Theorem 1 and Corollary 4, it is an interesting problem to determine whether there exist Hopf hypersurfaces in the complex hyperbolic space $CH^m(-4)$ with $m \geq 3$ which satisfy the equality case of (1.7), but the principal curvatures of the Hopf hypersurfaces are not all constant.

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