ARCHIVUM MATHEMATICUM (BRNO)
Tomus 37 (2001), 273 – 278

THE DUAL NOTION OF PRIME SUBMODULES

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Abstract. In this paper the concept of the second submodule (the dual notion of prime submodule) is introduced.

0. Introduction

Throughout this note the ring R is commutative with a non-zero identity. A proper submodule N of a module M over a ring R is said to be *prime submodule* if for each $a \in R$ the homothety $M/N \xrightarrow{a\cdot} M/N$ is either injective or zero. This implies that $\mathrm{Ann}(M/N) = \mathfrak{p}$ is a prime ideal of R, and N is said to be $\mathfrak{p}$-prime submodule. In other words, N is a $\mathfrak{p}$-prime submodule if it is a $\mathfrak{p}$-primary submodule whose radical is identical with $\mathfrak{p} = \mathrm{Ann}(M/N)$, c.f. [1], [2], [4] and [6].

In [3] MacDonald introduced the notion of secondary module, which is (in a certain sense) a dual to the notion of primary module. An R-module $M \neq 0$ is called *secondary* if for each $a \in R$ the homothety $M \xrightarrow{a\cdot} M$ is either surjective or nilpotent. Then $\mathrm{nilrad}(M) = \mathfrak{p}$ is a prime ideal and M is called $\mathfrak{p}$-secondary.

The purpose of this paper is to introduce second submodules (the dual notion of prime submodules). The non-zero submodule S of M is said to be *second submodule* of M if for each $a \in R$ the homothety $S \xrightarrow{a\cdot} S$ is either surjective or zero. This implies that $\mathrm{Ann}(S) = \mathfrak{p}$ is a prime ideal of R, and S is said to be $\mathfrak{p}$-*second*. We show that for a finitely cogenerated (the dual notion of finitely generated) module, every non-zero submodule contains a second submodule. Let $\mathfrak{p}$ be a prime ideal of R and let N be a submodule of the R-module M. We prove that N is a minimal $\mathfrak{p}$-secondary if and only if N is a minimal $\mathfrak{p}$-second.

Let the zero submodule of M be a prime submodule. Then it is shown that M is a second submodule of itself if and only if M is an injective $R/\mathrm{Ann}(M)$-module. Also the dual of this result is proved.

We show that, if M is a finitely generated R-module and if M is a second submodule of itself then the zero submodule of M is a prime submodule. Also the dual of this result (in some sense) is shown.

2000 *Mathematics Subject Classification*: 13C13, 13C11.
Key words and phrases: prime submodule, second submodule, injective and flat module.
Received September 14, 1998.

In the last section we bring some functorial results for the class of all modules M such that the zero submodule is a prime submodule (or $M = 0$), and the class of all modules that are second as a submodule of itself.

1. Second Submodules

First we bring some definitions and notations of [10] and [7] that we use in this paper.

Let M be an R-module. The dual notion of $\mathrm{Z}(M)$, the set of zero divisors of M, is denoted by $\mathrm{W}(M)$ and it is defined by

$$\mathrm{W}(M) = \{a \in R | M \xrightarrow{a\cdot} M \quad \text{is not surjective}\}.$$

The submodule L of M is said to be *cocyclic* if it is a submodule of $\mathrm{E}(R/\mathfrak{m})$, injective envelope of $R/\mathfrak{m}$, for some $\mathfrak{m} \in \mathrm{Max}(R)$.

A prime ideal $\mathfrak{p}$ of R is said to be a *weakly coassociated* prime of M if there exists a cocyclic homomorphic image L of M such that $\mathfrak{p}$ is a minimal element in $\mathrm{V}(\mathrm{Ann}(L)) = \{\mathfrak{q} \in \mathrm{Spec}(R) | \mathrm{Ann}(L) \subseteq \mathfrak{q}\}$. The set of weakly coassociated prime ideals of M is denoted by $\widetilde{\mathrm{Coass}}(M)$.

The R-module M is said to be *divisible* if for each $x \in M$ and for each non-zero divisor $r \in R$ there exists $y \in M$ such that $ry = x$.

The module M over a domain R is said to be *torsion-free* if $rx = 0$ implies $r = 0$ or $x = 0$ (where $r \in R$ and $x \in M$).

Definition 1.1. The non-zero submodule S of M is said to be *second submodule* of M if for each $a \in R$ the homothety $S \xrightarrow{a\cdot} S$ is either surjective or zero. This implies that $\mathrm{Ann}(S) = \mathfrak{p}$ is a prime ideal of R, and S is said to be $\mathfrak{p}$-*second.*

Lemma 1.2. *Let $\mathfrak{p} \in \mathrm{Spec}(R)$. The following are equivalent for any $0 \neq S \subseteq M$;*

(i) *S is a $\mathfrak{p}$-second submodule of M.*
(ii) $\mathrm{Ann}(S) = \mathrm{W}(S) = \mathfrak{p}$.

Proof. It is obvious. □

Theorem 1.3. *Let S be a non-zero submodule of M, with $\mathrm{Ann}(S) = \mathfrak{p}$. Then the following are equivalent;*

(i) *S is a $\mathfrak{p}$-second submodule of M*
(ii) *S is a divisible $R/\mathfrak{p}$-module;*
(iii) *$rS = S$ for all $r \in R \backslash \mathfrak{p}$;*
(iv) *$\mathfrak{a}S = S$ for all ideals $\mathfrak{a} \nsubseteq \mathfrak{p}$;*
(v) $\mathrm{W}(S) = \mathfrak{p}$;
(vi) $\widetilde{\mathrm{Coass}}S = \{\mathfrak{p}\}$.

Proof. The parts "(i)⇒(ii)", "(ii)⇒(iii)", "(iii)⇒(iv)" and "(iv)⇒(v)" are straightforward.

"(v)⇒(i)" Use (1.2).

"(vi)⇒(v)" Use [10; 2.14].

"(v)⇒(vi)" Set $\mathfrak{q} \in \widetilde{\mathrm{Coass}}S$. Then there exists a cocyclic homomorphic image L of S such that $\mathfrak{q}$ is a minimal element in $\mathrm{V}(\mathrm{Ann}(L))$. Since $\mathrm{Ann}(S) \subseteq \mathrm{Ann}(L)$

we have that $\mathfrak{p} \subseteq \mathfrak{q}$. Now if $\mathfrak{p} \neq \mathfrak{q}$ then $\mathfrak{q} \not\subseteq \mathrm{W}(S)$, which is a contradiction, cf. [10; 2.14]. $\square$

Proposition 1.4. *If S is a submodule of M and $\mathrm{Ann}(S) = \mathfrak{m} \in \mathrm{Max}(R)$ then S is a $\mathfrak{m}$-second submodule.*

Proof. Since $\mathrm{Ann}(S) = \mathfrak{m}$, we have that S is a vector space over $R/\mathfrak{m}$, and hence S is a divisible $R/\mathfrak{m}$-module. Thus S is a $\mathfrak{m}$-second by (1.3). $\square$

Definition 1.5. We call a non-zero submodule N of M, a minimal submodule if N is a simple module.

Proposition 1.6. *If S is a minimal submodule of M, then S is a second submodule.*

Proof. Since S is a minimal we have that S is a simple R-module and hence $\mathrm{Ann}(S)$ belongs to $\mathrm{Max}(R)$. Now the assertion follows from (1.4). $\square$

Definition 1.7. An R-module M is said to be *finitely cogenerated* (the dual notion of finitely generated) if $\mathrm{E}(M)$ (the injective envelope of M) is isomorphic to a direct sum of finitely many injective envelopes of simple modules. It is well-known that a module is finitely cogenerated if and only if its socle is a finitely generated and essential submodule.

In the Proposition (1.6) we show that any minimal submodule is second submodule. So it is good to know, which modules have minimal submodules. The next theorem gives these modules.

Theorem 1.8. *The R-module M has minimal submodules if and only if there exists non-zero finitely cogenerated submodule L of M.*

Proof. The proof is trivial. $\square$

Corollary 1.9. *If M is a finitely cogenerated module, then every non-zero submodule of M contains a simple, hence second, submodule.*

Proof. The proof is trivial. $\square$

In [3] MacDonald introduced the notion of a secondary module, which is (in a certain sense) a dual to the notion of primary module. An R-module $M \neq 0$ is called *secondary* if for each $a \in R$ the homothety $M \xrightarrow{a\cdot} M$ is either surjective or nilpotent. Then $\mathrm{nilrad}(M) = \mathfrak{p}$ is a prime ideal and M is called $\mathfrak{p}$-*secondary*.

Proposition 1.10. *The following hold;*

(a) *Let S be a secondary submodule of M. Then S is a second if and only if* $\mathrm{Ann}(S) \in \mathrm{Spec}(R)$.

(b) *Let K be a submodule of a $\mathfrak{p}$-second module M. Then K is a $\mathfrak{p}$-secondary submodule if and only if K is a $\mathfrak{p}$-second submodule.*

Proof. "(a)" This is obvious.

"(b)" Assume K is a $\mathfrak{p}$-secondary submodule of M. Then $\mathfrak{p} = \mathrm{Ann}(M) \subseteq \mathrm{Ann}(K) \subseteq \mathrm{nilrad}(K) = \mathfrak{p}$. $\square$

We shall call a submodule N of M a minimal $\mathfrak{p}$-secondary (resp. $\mathfrak{p}$-second) submodule of M if N is a $\mathfrak{p}$-secondary (resp. $\mathfrak{p}$-second) submodule which is not strictly contains any other $\mathfrak{p}$-secondary (resp. $\mathfrak{p}$-second) submodule of M.

Theorem 1.11. *The submodule N of M is minimal $\mathfrak{p}$-secondary if and only if N is a minimal $\mathfrak{p}$-second submodule of M.*

Proof. "If" By 1.10(b).

"Only if" Assume that N is a minimal $\mathfrak{p}$-secondary submodule of M. If $r \in \mathrm{W}(N)$ then $rN \neq N$. Since rN is a quotient of N, hence a $\mathfrak{p}$-secondary submodule of M, and N is a minimal $\mathfrak{p}$-secondary submodule of M, we have that $rN = 0$ and hence $r \in \mathrm{Ann}(N)$. Thus N is a (minimal) $\mathfrak{p}$-second submodule of M. □

2. Prime and second modules

Definition 2.1. Let M be an R-module. Then

(a) We say M is a *prime module* if the zero submodule of M is a prime submodule of M, see also [8].
(b) We say M is a *second module* if M is a second submodule of itself.

Note that N is a prime submodule of M if and only if M/N is a prime module. In addition, the ring R is a prime R-module if and only if R is an integral domain. Also R is a second R-module if and only if R is a field.

Proposition 2.2. *Let $\mathfrak{p} \in \operatorname{Spec} R$. Then the following hold:*

(a) *The sum of $\mathfrak{p}$-second modules is a $\mathfrak{p}$-second module.*
(b) *Every product of $\mathfrak{p}$-second module is a $\mathfrak{p}$-second module.*
(c) *Every non-zero quotient of a $\mathfrak{p}$-second module is likewise $\mathfrak{p}$-second.*

Proof. "(a)" Let $M_1, M_2, \ldots, M_n$ be $\mathfrak{p}$-second modules. Then for any $1 \leq i \leq n$ we have $\mathrm{Ann}(M_i) = \mathfrak{p}$ and M_i is a divisible $R/\mathfrak{p}$-module, by (1.3). Therefore $\mathrm{Ann}(\sum M_i) = \mathfrak{p}$ and $\sum M_i$ is a divisible $R/\mathfrak{p}$-module. Thus $\sum M_i$ is a $\mathfrak{p}$-second module, by (1.3).

The proof of parts (b) and (c) are similar. □

Theorem 2.3. *Let M be a prime module. Then the following are equivalent:*

(i) *M is a second module,*
(ii) *M is an injective $R/Ann(M)$-module.*

Proof. Since M is a prime module we have that $\mathfrak{p} = \mathrm{Ann}(M) \in \mathrm{Spec}(R)$ and M is a torsion-free $R/\mathfrak{p}$-module, by [1; Theorem 1].

"(i)$\Longrightarrow$(ii)" Since $\mathfrak{p} = \mathrm{Ann}(M)$ and M is a second module we have that M is a divisible $R/\mathfrak{p}$-module, by (1.3). Thus M is a vector space over the field of fractions of $R/\mathfrak{p}$, and hence M is an injective $R/\mathfrak{p}$-module.

"(ii)$\Longrightarrow$(i)" Since M is an injective $R/\mathfrak{p}$-module we have that M is a divisible $R/\mathfrak{p}$-module, by [7; 3.23]. Now the assertion follows from (1.3). □

Theorem 2.4. *Let M be a second module. Then the following are equivalent:*

(i) *M is a prime,*
(ii) *M is a flat $R/Ann(M)$-module.*

Proof. Since M is a second module we have that $\mathfrak{p} = \mathrm{Ann}(M) \in \mathrm{Spec}(R)$ and M is a divisible $R/\mathfrak{p}$-module, by [1; Theorem 1].

"(i)$\Longrightarrow$(ii)" Since $\mathfrak{p} = \mathrm{Ann}(M)$ and M is a prime module we have that M is a torsion-free $R/\mathfrak{p}$-module, by [1; Theorem 1]. Thus M is a vector space over the field of fractions of $R/\mathfrak{p}$, and hence M is a flat $R/\mathfrak{p}$-module.

"(ii)$\Longrightarrow$(i)" Since M is a flat $R/\mathfrak{p}$-module we have that M is a torsion-free $R/\mathfrak{p}$-module, by [7; 4.33]. Now the assertion follows from [1; Theorem 1]. □

In [9], Vasconcelos has shown an interesting application of the Nakayama Lemma. If M is a finitely generated R-module and $\varphi : M \longrightarrow M$ is a surjective R-homomorphism then φ is injective, cf. [5; 2.4]. We use this result in the next theorem.

Theorem 2.5. *Let M be a finitely generated R-module. If M is a second module then M is a prime module.*

Proof. Immediate from [5; 2.4].

The next result is a dual of (2.5) in a certain sense.

Theorem 2.6. *Let M be an Artinian R-module. If M is a prime module then M is a second module.*

Proof. Immediate from the Fitting's Lemma.

Corollary 2.7. *Let M be a finitely generated and Artinian module. Then M is a prime module if and only if M is a second module.*

3. Functorial Results

In [10] we have some functorial results for the category of modules such that the zero submodule has a primary decomposition (or being zero) and the category of modules that has secondary representation (or being zero). In this chapter we bring the similar results for the class $\mathcal{P}$ of all prime R-modules (or being zero), and the class $\mathcal{S}$ of all second R-module (or being zero).

Theorem 3.1. *Let* T *be a linear functor over the category of R-modules. Then the following hold*:

(a) *If* T *is a left exact and covariant and if $M \in \mathcal{P}$ then* $\mathrm{T}(M) \in \mathcal{P}$. *In particular, if F is a flat R-module and $M \in \mathcal{P}$ then $M \otimes F \in \mathcal{P}$, and if $M \in \mathcal{P}$ then* $\mathrm{Hom}(N, M) \in \mathcal{P}$ *for any R-module N.*

(b) *If* T *is right exact and covariant and if $M \in \mathcal{P}$ then* $\mathrm{T}(M)\mathcal{S}$. *In particular, if E is an injective R-module and $M \in \mathcal{P}$ then* $\mathrm{Hom}(M, E) \in \mathcal{S}$.

(c) *If* T *is right exact and covariant and if $M \in \mathcal{S}$ then* $\mathrm{T}(M) \in \mathcal{S}$. *In particular, if $M \in \mathcal{S}$ then $M \otimes N \in \mathcal{S}$ for any R-module N. In addition, if P is a projective R-module and $M \in \mathcal{S}$ then* $\mathrm{Hom}(P, M) \in \mathcal{S}$.

(d) *If* T *is left exact and contravariant and if $M \in \mathcal{S}$ then* $\mathrm{T}(M) \in \mathcal{P}$. *In particular, if $M \in \mathcal{S}$ then* $\mathrm{Hom}(M, N) \in \mathcal{P}$ *for any N.*

Proof. "(a)" Since for any $a \in R$ the homothety $M \xrightarrow{a\cdot} M$ is either injective or zero, we have that $\mathrm{T}(M) \xrightarrow{a\cdot} \mathrm{T}(M)$ is either injective or zero. Thus $\mathrm{T}(M)$ is either prime or zero.

"(b)" Since for any $a \in R$ the homothety $M \xrightarrow{a\cdot} M$ is either injective or zero, we have that $\mathrm{T}(M) \xrightarrow{a\cdot} \mathrm{T}(M)$ is either surjective or zero. Thus $\mathrm{T}(M)$ is either second or zero.

The proof of (c) is similar to the proof (a), and the proof of (d) is similar to the proof (b). □

Acknowledgments. This work was down within the framework of the associateship Scheme of the International Centre for Theoretical Physics, Trieste. The author would like to thank Chin-Pi Lu, University of Colorado, for his useful comments. The author would also like to thank the referee for his/her suggestions.

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