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SOME EQUALITIES FOR GENERALIZED INVERSES OF MATRIX SUMS AND BLOCK CIRCULANT MATRICES

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ABSTRACT. Let $A_1, A_2, \cdots, A_n$ be complex matrices of the same size. We show in this note that the Moore-Penrose inverse, the Drazin inverse and the weighted Moore-Penrose inverse of the sum $\sum_{t=1}^{n} A_t$ can all be determined by the block circulant matrix generated by $A_1, A_2, \cdots, A_n$. In addition, some equalities are also presented for the Moore-Penrose inverse and the Drazin inverse of a quaternionic matrix.

Let C be a circulant matrix over the complex number field $\mathbb{C}$ with the form

$$C = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}. \tag{1}$$

Then it is well known (see, e.g., [1] and [3]) that C satisfies the following similarity factorization equality

$$U^*CU = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{2}$$

where U is a fixed unitary matrix with the form

$$U = (u_{pq})_{n\times n}, \qquad u_{pq} = \frac{1}{\sqrt{n}}\omega^{(p-1)(q-1)}, \qquad \omega \text{ is the } n\text{th root of unity}, \tag{3}$$

and

$$\lambda_t = a_0 + a_1\omega^{(t-1)} + a_2(\omega^{(t-1)})^2 + \cdots + a_{n-1}(\omega^{(t-1)})^{n-1}, \quad t = 1, \cdots, n. \tag{4}$$

In particular,

$$\lambda_1 = a_0 + a_1 + \cdots + a_{n-1}. \tag{5}$$

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Observe that U in Eq.(3) has no relation with $a_0, \ldots, a_{n-1}$ in Eq.(1). Thus Eq.(2) can directly be extended to block circulant matrix as follows.

Lemma 1. *Let*

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix} \tag{6}$$

be a block circulant matrix over the complex number field $\mathbb{C}$, *where* $A_t \in \mathbb{C}^{r\times s}$, $t = 1, \cdots, n$. *Then* A *satisfies the following factorization equality*

$$U_r^* A U_s = \mathrm{diag}(J_1, J_2, \cdots, J_n)\,, \tag{7}$$

where U_r *and* U_s *are two fixed block unitary matrices*

$$U_r = (u_{pq} I_r)_{n\times n}\,, \qquad U_s = (u_{pq} I_s)_{n\times n}\,, \tag{8}$$

u_{pq} *is as in Eq.*(3), *meanwhile*

$$J_t = A_1 + A_2\omega^{(t-1)} + A_3(\omega^{(t-1)})^2 + \cdots + A_n(\omega^{(t-1)})^{n-1}\,, \qquad t = 1, \cdots, n\,. \tag{9}$$

Especially, the block entries in the first block rows and first block columns of U_r *and* U_s *are all identity matrices, and* J_1 *is*

$$J_1 = A_1 + A_2 + \cdots + A_n\,. \tag{10}$$

Observe that J_1 in Eq.(7) is the sum of $A_1, A_2, \cdots, A_n$. Thus Eq.(7) implies that the sum $\sum_{t=1}^n A_t$ is closely linked to its corresponding block circulant matrix through a unitary factorization equality. Recall a fundamental fact in the theory of generalized inverses of matrices (see, e.g., [2]) that

$$(PAQ)^\dagger = Q^* A^\dagger P^*, \qquad \text{if } P \text{ and } Q \text{ are unitary.} \tag{11}$$

Then from Eq.(7), we can directly find the following result.

Lemma 2. *Let* A *be given in Eq.*(6), U_r *and* U_s *be given in Eq.*(8). *Then*

(a) *The Moore-Penrose inverse of* A *satisfies*

$$U_s^* A^\dagger U_r = \mathrm{diag}(J_1^\dagger, J_2^\dagger, \cdots, J_n^\dagger)\,. \tag{12}$$

(b) *If* $r = s$, *then the Drazin inverse of* A *satisfies*

$$U_r^* A^D U_r = \mathrm{diag}(J_1^D, J_2^D, \cdots, J_n^D)\,. \tag{13}$$

(c) *Suppose that* $M \in \mathbb{C}^{r\times r}$, *and* $N \in \mathbb{C}^{s\times s}$ *are two positive definite Hermitian matrices. Then the weighted Moore-Penrose inverse of* A *satisfies*

$$U_s^* A^\dagger_{\widehat{M},\widehat{N}} U_r = \mathrm{diag}((J_1)^\dagger_{M,N}, (J_2)^\dagger_{M,N}, \cdots, (J_n)^\dagger_{M,N})\,, \tag{14}$$

where $\widehat{M} = \mathrm{diag}(M, M, \cdots, M)$ *and* $\widehat{N} = \mathrm{diag}(N, N, \cdots, N)$.

Proof. Since U_r and U_s in Eq.(7) are unitary, we have

$$(U_r^* A U_s)^\dagger = U_s^* A^\dagger U_r \tag{15}$$

by Eq.(11). On the other hand, it is easily seen that

$$[\mathrm{diag}(J_1, J_2, \cdots, J_n)]^\dagger = \mathrm{diag}(J_1^\dagger, J_2^\dagger, \cdots, J_n^\dagger)\,.$$

Thus Eq.(12) follows. Secondly, noting

$$(U_r^* A U_r)^D = (U_r^{-1} A U_r)^D = U_r^{-1} A^D U_r = U_r^* A^D U_r\,,$$

and

$$[\mathrm{diag}(J_1, J_2, \cdots, J_n)]^D = \mathrm{diag}(J_1^D, J_2^D, \cdots, J_n^D)\,,$$

we have Eq.(13). To prove Eq.(14), we apply the following well-known identity (see [2])

$$A_{M,N}^\dagger = N^{-\frac{1}{2}}(M^{\frac{1}{2}} A N^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}\,,$$

and Eq.(11) to the left-hand side of Eq.(7),

$$\begin{aligned}
(U_r^* A U_s)_{\widehat{M},\widehat{N}}^\dagger &= \widehat{N}^{-\frac{1}{2}}(\widehat{M}^{\frac{1}{2}} U_r^* A U_s \widehat{N}^{-\frac{1}{2}})^\dagger \widehat{M}^{\frac{1}{2}} \\
&= \widehat{N}^{-\frac{1}{2}}(U_r^* \widehat{M}^{\frac{1}{2}} A \widehat{N}^{-\frac{1}{2}} U_s)^\dagger \widehat{M}^{\frac{1}{2}} \\
&= \widehat{N}^{-\frac{1}{2}} U_s^* (\widehat{M}^{\frac{1}{2}} A \widehat{N}^{-\frac{1}{2}})^\dagger U_r \widehat{M}^{\frac{1}{2}} \\
&= U_s^* \widehat{N}^{-\frac{1}{2}} (\widehat{M}^{\frac{1}{2}} A \widehat{N}^{-\frac{1}{2}})^\dagger \widehat{M}^{\frac{1}{2}} U_r \\
&= U_s^* A_{\widehat{M},\widehat{N}}^\dagger U_r\,,
\end{aligned}$$

where two simple facts

$$U_r^* \widehat{M}^{\frac{1}{2}} = U_r^* \widehat{M}^{\frac{1}{2}}\,, \qquad U_s \widehat{N}^{-\frac{1}{2}} = \widehat{N}^{-\frac{1}{2}} U_s$$

are used in the above deduction. On the other hand,

$$\begin{aligned}
&[\mathrm{diag}(J_1, \cdots, J_n)]_{\widehat{M},\widehat{N}}^\dagger \\
&= \widehat{N}^{-\frac{1}{2}}[\widehat{M}^{\frac{1}{2}} \mathrm{diag}(J_1, \cdots, J_n) \widehat{N}^{-\frac{1}{2}}]^\dagger \widehat{M}^{\frac{1}{2}} \\
&= \widehat{N}^{-\frac{1}{2}}[\mathrm{diag}((M^{\frac{1}{2}} J_1 N^{-\frac{1}{2}})^\dagger, \cdots, (M^{\frac{1}{2}} J_n N^{-\frac{1}{2}})^\dagger)] \widehat{M}^{\frac{1}{2}} \\
&= \mathrm{diag}(N^{-\frac{1}{2}}(M^{\frac{1}{2}} J_1 N^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}, \cdots, N^{-\frac{1}{2}}(M^{\frac{1}{2}} J_n N^{-\frac{1}{2}})^\dagger M^{\frac{1}{2}}) \\
&= \mathrm{diag}((J_1)_{M,N}^\dagger, \cdots, (J_n)_{M,N}^\dagger)\,.
\end{aligned}$$

So we have Eq.(14). □

The main results of this note are presented below.

Theorem 3. *Let $A_1, A_2, \cdots, A_n \in \mathbb{C}^{r\times s}$ be given. Then the Moore-Penrose inverse of their sum satisfies the identity*

$$(16)\quad (A_1 + A_2 + \cdots + A_n)^\dagger = \frac{1}{n}[I_s, I_s, \cdots, I_s] \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^\dagger \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix}.$$

Proof. Pre-multiplying $[I_s, 0, \cdots, 0]$ and post-multiplying $[I_r, 0, \cdots, 0]^T$ on the both sides of Eq.(12) immediately yield Eq.(16). □

Similarly we can establish the following two theorems.

Theorem 4. *Let $A_1, A_2, \cdots, A_n \in \mathbb{C}^{r\times r}$ be given. Then the Drazin inverse of their sum satisfies the equality*

(17)
$$(A_1 + A_2 + \cdots + A_n)^D = \frac{1}{n}[I_r, I_r, \cdots, I_r]\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^D \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix}.$$

In particular, if the block circulant matrix in it is nonsingular, then

(18)
$$(A_1 + A_2 + \cdots + A_n)^{-1} = \frac{1}{n}[I_r, I_r, \cdots, I_r]\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^{-1} \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix}.$$

Theorem 5. *Let $A_1, A_2, \cdots, A_n \in \mathbb{C}^{r\times s}$ be given, $M \in \mathbb{C}^{r\times r}$ and $N \in \mathbb{C}^{s\times s}$ be two positive definite Hermitian matrices. Then the weighted Moore-Penrose inverse of their sum satisfies*

(19)
$$(A_1 + A_2 + \cdots + A_n)^{\dagger}_{M,N} = \frac{1}{n}[I_s, I_s, \cdots, I_s]\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ A_n & A_1 & \cdots & A_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{bmatrix}^{\dagger}_{\widehat{M},\widehat{N}} \begin{bmatrix} I_r \\ I_r \\ \vdots \\ I_r \end{bmatrix},$$

where $\widehat{M} = \mathrm{diag}(M, M, \cdots, M)$ and $\widehat{N} = \mathrm{diag}(N, N, \cdots, N)$.

Eqs.(16)–(18) show that the expressions of the Moore-Penrose inverse, the Drazin inverse, and the weighted Moore-Penrose inverse of the sum $\sum_{t=1}^n A_t$ can all be determined through the block circulant matrix A generated by $A_1, A_2, \cdots, A_n$. Using them one can establish various valuable expressions for generalized inverses of matrices. Some related work was presented in the author's [6].

Note that any complex matrix can be written as $A + iB$. Some interesting equalities can also be derived from Eqs.(16)–(18) for generalized inverses of a complex matrix $A + iB$.

Corollary 6. *Let $A + iB \in \mathbb{C}^{r\times s}$ with $A,\ B \in \mathbb{R}^{r\times s}$. Then the Moore-Penrose inverse of $A + iB$ satisfies the equality*

(20)
$$(A + iB)^{\dagger} = \frac{1}{2}[I_s, iI_s]\begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{\dagger}\begin{bmatrix} I_r \\ -iI_r \end{bmatrix}.$$

Proof. According to Eq.(16), we first see that

(21)
$$(A + iB)^{\dagger} = \frac{1}{2}[I_s, I_s]\begin{bmatrix} A & iB \\ iB & A \end{bmatrix}^{\dagger}\begin{bmatrix} I_r \\ I_r \end{bmatrix}.$$

Moreover observe that

$$\begin{bmatrix} A & iB \\ iB & A \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & iI_r \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} I_s & 0 \\ 0 & -iI_s \end{bmatrix}.$$

We then get

$$\begin{bmatrix} A & iB \\ iB & A \end{bmatrix}^\dagger = \begin{bmatrix} I_s & 0 \\ 0 & iI_s \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^\dagger \begin{bmatrix} I_r & 0 \\ 0 & -iI_r \end{bmatrix}.$$

Putting it in Eq.(21) yields Eq.(20). □

Corollary 7. *Let $A + iB \in \mathbb{C}^{r\times r}$ with $A,\ B \in \mathbb{R}^{r\times r}$. Then the Drazin inverse of $A + iB$ satisfies the equality*

$$(A + iB)^D = \frac{1}{2}[I_r, iI_r] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^D \begin{bmatrix} I_r \\ -iI_r \end{bmatrix}. \tag{22}$$

In particular, if $A + iB$ is nonsingular, then

$$(A + iB)^{-1} = \frac{1}{2}[I_r, iI_r] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^{-1} \begin{bmatrix} I_r \\ -iI_r \end{bmatrix}. \tag{23}$$

Corollary 8. *Let $A + iB \in \mathbb{C}^{r\times s}$ with $A, B \in \mathbb{R}^{r\times s}$, $M \in \mathbb{C}^{r\times r}$ and $N \in \mathbb{C}^{s\times s}$ be two positive definite Hermitian matrices. Then the weighted Moore-Penrose inverse of $A + iB$ satisfies the equality*

$$(A + iB)^\dagger_{M,N} = \frac{1}{2}[I_s, iI_s] \begin{bmatrix} A & -B \\ B & A \end{bmatrix}^\dagger_{\widehat{M},\widehat{N}} \begin{bmatrix} I_r \\ -iI_r \end{bmatrix}, \tag{24}$$

where $\widehat{M} = \mathrm{diag}(M, M)$ and $\widehat{N} = \mathrm{diag}(N, N)$.

The results in the above three corollaries on complex matrices motivate us to find the following interesting results on generalized inverses of quaternionic matrices.

Theorem 9. *Let $A = A_0 + iA_1 + jA_2 + kA_3$ be a quaternionic matrix, where $A_0, \ldots, A_3 \in \mathbb{R}^{m\times n}$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. Then*

(a) *The Moore-Penrose inverse of A satisfies the equality*

$$A^\dagger = \frac{1}{4}[I_n, iI_n, jI_n, kI_n] \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}^\dagger \begin{bmatrix} I_m \\ -iI_m \\ -jI_m \\ -kI_m \end{bmatrix}. \tag{25}$$

(b) *If $m = n$, then the Drazin inverse of A satisfies the equality*

$$A^D = \frac{1}{4}[I_n, iI_n, jI_n, kI_n] \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}^D \begin{bmatrix} I_n \\ -iI_n \\ -jI_n \\ -kI_n \end{bmatrix}. \tag{26}$$

(c) *In particular, if A is nonsingular, then the inverse of A satisfies the equality*

$$A^{-1} = \frac{1}{4}[I_n, iI_n, jI_n, kI_n] \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}^{-1} \begin{bmatrix} I_n \\ -iI_n \\ -jI_n \\ -kI_n \end{bmatrix}. \tag{27}$$

The equalities (25)–(27) can be derived from the following universal factorization equality for a quaternionic matrix

$$V_m \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} V_n = \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{bmatrix}, \tag{28}$$

where

$$V_t = \frac{1}{2} \begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ -iI_t & I_t & kI_t & -jI_t \\ -jI_t & -kI_t & I_t & iI_t \\ -kI_t & jI_t & -iI_t & I_t \end{bmatrix}, \qquad t = m, n \tag{29}$$

is a unitary quaternionic matrix, that is, $V_tV_t^* = V_t^*V_t = I_t$. The equality was first established by the author in [7]. Based on it, one can easily extend various results in the real and complex matrix theory to the real quaternion algebra.

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