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THE CONTACT SYSTEM ON THE (m, ℓ)-JET SPACES

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Abstract. This paper is a continuation of [8], where we give a construction of the canonical Pfaff system $\Omega(M_m^\ell)$ on the space of (m, ℓ)-velocities of a smooth manifold M. Here we show that the characteristic system of $\Omega(M_m^\ell)$ agrees with the Lie algebra of $\mathrm{Aut}(\mathbb{R}_m^\ell)$, the structure group of the principal fibre bundle $\check{M}_m^\ell \longrightarrow \mathcal{J}_m^\ell(M)$, hence it is projectable to an irreducible contact system on the space of (m, ℓ)-jets ($= \ell$-th order contact elements of dimension m) of M. Furthermore, we translate to the language of Weil bundles the structure form of jet fibre bundles defined by Goldschmidt and Sternberg in [2].

1. The characteristic system of $\Omega(M_m^\ell)$

It is well known that $\mathrm{Aut}(\mathbb{R}_m^\ell)$ is a Lie group whose Lie algebra is isomorphic to $\mathrm{Der}(\mathbb{R}_m^\ell, \mathbb{R}_m^\ell)$ (see [4, 5]); we are going to prove this result in a form which we will need later.

The elements of $\mathrm{Aut}(\mathbb{R}_m^\ell)$ are, in particular, linear automorphisms of $\mathbb{R}_m^\ell$; therefore if $\bar{\xi}$ is the infinitesimal generator of a 1-parameter subgroup $\{\tau_t\}$ of $\mathrm{Aut}(\mathbb{R}_m^\ell)$, we can associate to it the linear map ξ from $\mathbb{R}_m^\ell$ into itself which applies each vector $P \in \mathbb{R}_m^\ell$ into the element

$$\xi P = -\lim_{t \to 0} \frac{\tau_t P - P}{t} = -\bar{\xi}_P I \,, \tag{1.1}$$

where $I \colon \mathbb{R}_m^\ell \longrightarrow \mathbb{R}_m^\ell$ is the identity, which we understand as a vector valued function.

The mapping which assigns to each $\bar{\xi}$ the linear map ξ defined by (1.1) is an injective homomorphism of Lie algebras between the Lie algebra of $\mathrm{Aut}(\mathbb{R}_m^\ell)$ and the set of linear endomorphisms of $\mathbb{R}_m^\ell$, endowed with a Lie algebra structure by the commutator. Since $\{\tau_t\}$ is a group of automorphisms of $\mathbb{R}_m^\ell$ as an $\mathbb{R}$-algebra, and not only as a vector space, ξ is a derivation, as one can check easily, hence equation (1.1) establishes an injective mapping from the Lie algebra of $\mathrm{Aut}(\mathbb{R}_m^\ell)$ into $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}_m^\ell, \mathbb{R}_m^\ell)$; but the dimensions of $\mathrm{Aut}(\mathbb{R}_m^\ell)$ and $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}_m^\ell, \mathbb{R}_m^\ell)$ agree, and therefore the map $\xi \longmapsto \bar{\xi}$ is an isomorphism.

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We can summarize the former discussion as follows:

Proposition 1.1. *There is a canonical isomorphism between the Lie algebra of* $\mathrm{Aut}(\mathbb{R}^\ell_m)$ *and* $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^\ell_m)$*; the image of a tangent vector field* $\bar{\xi}$ *on* $\mathbb{R}^\ell_m$*, infinitesimal generator of a* 1*-parameter subgroup of automorphisms of* $\mathbb{R}^\ell_m$*, is the* $\mathbb{R}$*-derivation* ξ *from* $\mathbb{R}^\ell_m$ *into itself defined by (1.1).*

The group $\mathrm{Aut}(\mathbb{R}^\ell_m)$ acts on M^ℓ_m by composition; let $\{\tau_t\}$ be a 1-parameter subgroup of $\mathrm{Aut}(\mathbb{R}^\ell_m)$, $\{\tau'_t\}$ the 1-parameter group of automorphisms of M^ℓ_m attached to it and ξ' the infinitesimal generator of $\{\tau'_t\}$. For each $p^\ell_m \in M^\ell_m$ and each $f \in C^\infty(M)$ we have:

$$\xi'_{p^\ell_m} f = \lim_{t\to 0} \frac{f(\tau'_t p^\ell_m) - f(p^\ell_m)}{t} = \lim_{t\to 0} \frac{\tau_t(f(p^\ell_m)) - f(p^\ell_m)}{t} = -\xi(f(p^\ell_m))\,. \tag{1.2}$$

In particular, if p^ℓ_m is proper (regular), ξ' vanishes at p^ℓ_m only when $\xi = 0$, hence the Lie algebra of tangent vector fields in M^ℓ_m associated to the action of $\mathrm{Aut}(\mathbb{R}^\ell_m)$ is isomorphic to $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^\ell_m)$.

Theorem 1.2. *In the open subset* $\check{M}^\ell_m$ *of regular points of* M^ℓ_m *the characteristic system of the Pfaff system* $\Omega(M^\ell_m)$ *is the module of tangent vector fields generated by the Lie algebra of the group* $\mathrm{Aut}(\mathbb{R}^\ell_m)$ *acting in* M^ℓ_m*. Therefore, in* $\check{M}^\ell_m$ *this characteristic system is regular with rank* $m\binom{m+\ell}{m} - m$*.*

Proof. First we will show that each vector field ξ' of the Lie algebra of $\mathrm{Aut}(\mathbb{R}^\ell_m)$ acting in M^ℓ_m belongs to the characteristic system of $\Omega(M^\ell_m)$. It suffices to prove that ξ' annihilates $\Omega(M^\ell_m)$ and that this Pfaff system is invariant under the action of $\mathrm{Aut}(\mathbb{R}^\ell_m)$.

Let $p^\ell_m \in \check{M}^\ell_m$ and let ξ' belong to the Lie algebra generated in M^ℓ_m by the action of $\mathrm{Aut}(\mathbb{R}^\ell_m)$; then from equation (1.2) follows that $\xi'_{p^\ell_m} f = -\xi(f(p^\ell_m))$ for each $f \in C^\infty(M)$, where $\xi \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^\ell_m)$, hence $\xi'_{p^{\ell-1}_m} f = \overline{\xi}_{(p^\ell_m)} f$, where $\overline{\xi} \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^{\ell-1}_m)$ is the composition of $-\xi$ with the canonical projection $\mathbb{R}^\ell_m \longrightarrow \mathbb{R}^{\ell-1}_m$. By Corollary 4.3 of [8], ξ' annihilates $\Omega(M^\ell_m)$.

Next we show the invariance of $\Omega(M^\ell_m)$ under $\mathrm{Aut}(\mathbb{R}^\ell_m)$.

Let $\sigma \in \mathrm{Aut}(\mathbb{R}^\ell_m)$; if $f \in C^\infty(M)$ and $p^\ell_m \in M^\ell_m$, then for each m-index α we have $\sigma^*(f_\alpha) = (\sigma \circ f)_\alpha$, where in the right side $\sigma \circ f$ is considered as a mapping from M^ℓ_m into $\mathbb{R}^\ell_m$. On the other hand, since σ is an $\mathbb{R}$-linear endomorphism of $\mathbb{R}^\ell_m$, the real components of $\sigma \circ f$ are a linear span, with real coefficients, of the real components of f. From this fact follows that, if $\bar{D}_{p^\ell_m} \in T_{p^\ell_m} M^\ell_m$ is the tangent vector attached to the derivation $D_{p^\ell_m} \in \mathcal{T}_{p^\ell_m} M^\ell_m$ by the canonical isomorphism between these two spaces, then

$$\sigma_* \bar{D}_{p^\ell_m} = \overline{\sigma \circ D_{p^\ell_m}}\,,$$

that is to say, $\sigma_* D_{p^\ell_m} = \sigma \circ D_{p^\ell_m}$ when σ_* is considered as a morphism from $\mathcal{T}_{p^\ell_m} M^\ell_m$ into $\mathcal{T}_{\sigma(p^\ell_m)} M^\ell_m$.

Let ω be an $(m+1)$–form on M; in the notations of [8] we have:

$$\begin{aligned}\langle(\sigma^*\widehat{\omega})_{p^\ell_m}, D_{p^\ell_m}\rangle &= \langle\widehat{\omega}_{\sigma(p^\ell_m)}, \sigma_*\left(D_{p^\ell_m}\right)\rangle \\ &= \omega_{\bar{\sigma}\left(p^{\ell-1}_m\right)}\left(\xi_{1(\sigma(p^\ell_m))}, \ldots, \xi_{m(\sigma(p^\ell_m))}, \bar{\sigma}\circ D_{p^{\ell-1}_m}\right),\end{aligned}$$

where $\bar{\sigma}\colon \mathbb{R}^{\ell-1}_m \longrightarrow \mathbb{R}^{\ell-1}_m$ is the canonical factorization of σ, operating on $M^{\ell-1}_m$.

For each $\xi \in \mathrm{Der}_{\mathbb{R}}\left(\mathbb{R}^\ell_m, \mathbb{R}^{\ell-1}_m\right)$, $\overline{\xi} = \bar{\sigma}^{-1}\circ\xi\circ\sigma$ is another derivation from $\mathbb{R}^\ell_m$ into $\mathbb{R}^{\ell-1}_m$ and furthermore $\xi_{(\sigma(p^\ell_m))} = \bar{\sigma}\circ\overline{\xi}_{(p^\ell_m)}$, hence we have:

$$\begin{aligned}\langle(\sigma^*\widehat{\omega})_{p^\ell_m}, D_{p^\ell_m}\rangle &= \omega_{\bar{\sigma}\left(p^{\ell-1}_m\right)}\left(\bar{\sigma}\circ\overline{\xi}_{1(p^\ell_m)}, \ldots, \bar{\sigma}\circ\overline{\xi}_{m(p^\ell_m)}, \bar{\sigma}\circ D_{p^{\ell-1}_m}\right) \\ &= \bar{\sigma}\left(\omega_{p^{\ell-1}_m}\left(\overline{\xi}_{1(p^\ell_m)}, \ldots, \overline{\xi}_{m(p^\ell_m)}, D_{p^{\ell-1}_m}\right)\right).\end{aligned}$$

But, if $\{\xi_1,\ldots,\xi_m\}$ is a basis of the $\mathbb{R}^{\ell-1}_m$-module $\mathrm{Der}_{\mathbb{R}}\left(\mathbb{R}^\ell_m, \mathbb{R}^{\ell-1}_m\right)$, $\{\overline{\xi}_1,\ldots,\overline{\xi}_m\}$ is another basis, hence $\sigma^*\left(\widehat{\omega}\right) = \bar{\sigma}\circ(u\widehat{\omega})$, where u is an invertible element of $\mathbb{R}^{\ell-1}_m$; then the real components of $\sigma^*(\widehat{\omega})$ are linear spans, with real coefficients, of those of $\widehat{\omega}$, and hence they belong to $\Omega\left(M^\ell_m\right)$.

The former discussion shows that the Lie algebra generated in M^ℓ_m by $\mathrm{Aut}\left(\mathbb{R}^\ell_m\right)$ is contained in the characteristic system of $\Omega\left(M^\ell_m\right)$. Let us show finally that the $C^\infty\left(\check{M}^\ell_m\right)$-module of vector fields on $\check{M}^\ell_m$ generated by this Lie algebra is the full characteristic system of $\Omega(M^\ell_m)$. According to a classical theorem of Elie Cartan, given a manifold Z solution of a Pfaff system Ω and a vector field belonging to the characteristic system of Ω which is not tangent to Z at a point P, we can find a solution manifold of Ω containing a neighbourhood of P in Z and whose dimension is equal to $\dim Z + 1$. In particular, each tangent vector at P which is the value at P of a vector field belonging to the characteristic system of Ω must be tangent to every locally maximal solution of Ω containing P. If we apply this result to our case and take into account the assertion of Theorem 4.5 of [8], it is sufficient to show that for each $p^\ell_m \in \check{M}^\ell_m$ there are m-dimensional submanifolds $W_1,\ldots,W_k$ of M whose manifolds of (m,ℓ)-velocities ${W_i}^\ell_m$ $(1\le i\le k)$ contain p^ℓ_m and such that $\cap_{i=1}^k T_{p^\ell_m}{W_i}^\ell_m$ is equal to the value at p^ℓ_m of the Lie algebra of $\mathrm{Aut}(\mathbb{R}^\ell_m)$ acting in M^ℓ_m.

Let us take local coordinates $y_1,\ldots,y_n \in C^\infty(M)$ in a neighbourhood U of $p = p^0_m$ such that

$$\begin{aligned} y_i(p^\ell_m) &= x_i \qquad (i=1,\ldots,m)\\ y_{m+j}(p^\ell_m) &= 0 \qquad (j=1,\ldots,n-m)\end{aligned}$$

Consider the m-dimensional manifolds $W_0,\ldots,W_m$, contained in U, defined by the equations

$$\begin{aligned} &W_0 : \{y_{m+1}=0,\ldots,y_n=0\}\\ &W_i : \{y_{m+1}=y_i^{\ell+1}, y_{m+2}=0,\ldots,y_n=0\} \qquad (i=1,\ldots,m)\end{aligned}$$

From Proposition 3.4 of [8] follows that $\cap_{i=0}^{m} \mathcal{T}_{p^\ell_m} W^\ell_{im}$ is the set of derivations

$$\eta_1(x)\left(\frac{\partial}{\partial y_1}\right)_{p^\ell_m} + \cdots + \eta_m(x)\left(\frac{\partial}{\partial y_m}\right)_{p^\ell_m}, \quad \text{where} \quad \eta_1, \ldots, \eta_m \in \mathfrak{m}(\mathbb{R}^\ell_m),$$

that, as one can deduce easily from proposition 1.1, agrees with the value at p^ℓ_m of the Lie algebra of $\operatorname{Aut}(\mathbb{R}^\ell_m)$ acting in M^ℓ_m. □

2. The contact system on $\mathcal{J}^\ell_m(M)$

The canonical projection $\check{M}^\ell_m \longrightarrow \mathcal{J}^\ell_m(M)$ allows to consider the exterior differential forms in $\mathcal{J}^\ell_m(M)$ as forms in $\check{M}^\ell_m$; we will use this fact in the sequel.

Definition 2.1. We will call *contact system* in $\mathcal{J}^\ell_m(M)$, and denote by $\Omega(\mathcal{J}^\ell_m(M))$, the intersection of the contact system $\Omega(\check{M}^\ell_m)$ with $\mathcal{E}^1(\mathcal{J}^\ell_m(M)) =$ module of smooth 1-forms on $\mathcal{J}^\ell_m(M)$.

Theorem 2.2. *The contact system $\Omega(\mathcal{J}^\ell_m(M))$ is regular with rank $(n-m)\binom{\ell+m-1}{m}$. When considered as a subset of $\mathcal{E}^1(\check{M}^\ell_m)$, it spans the contact system $\Omega(\check{M}^\ell_m)$. Furthermore, $\Omega(\mathcal{J}^\ell_m(M))$ is irreducible.*

Proof. The second assertion is a consequence of the first one and Proposition 4.2 of [8]. Then, Theorem 1.2 says that the characteristic system of $\Omega(\check{M}^\ell_m)$ is vertical for the projection $\check{M}^\ell_m \longrightarrow \mathcal{J}^\ell_m(M)$, and therefore $\Omega(\mathcal{J}^\ell_m(M))$ is irreducible.

It remains to compute the rank of $\Omega(\mathcal{J}^\ell_m(M))$; we will do it in each open subset from a covering of $\mathcal{J}^\ell_m(M)$. Using the notations from [7], let us consider an open subset U of M with coordinates $y_1, \ldots, y_n$ and the open subset $\underline{U}^\ell_m$ of U^ℓ_m of regular points with respect to $\mathbb{R}[y_1, \ldots, y_m]$; let us denote its image in $\mathcal{J}^\ell_m(M)$ by $\underline{\mathcal{J}}^\ell_m(U)$, endowed with the local coordinates $\{y_{i0}, Y_{m+j,\beta}\}$. Let $\mathcal{Y}^\ell_m$ be the image of the section $\eta\colon \underline{\mathcal{J}}^\ell_m(U) \longrightarrow \underline{U}^\ell_m$ which associates to $\mathfrak{p}^\ell_m$ the point p^ℓ_m defined by the equations

$$y_i(p^\ell_m) = y_i(p) + x_i \qquad (1 \le i \le m)$$

$$y_{m+j}(p^\ell_m) = \sum_{|\alpha| \le \ell} \frac{1}{\alpha!} Y_{m+j,\alpha}(\mathfrak{p}^\ell_m) x^\alpha \qquad (1 \le j \le n-m)$$

$\mathcal{Y}^\ell_m$ is a closed submanifold of $\underline{U}^\ell_m$, and $\eta\colon \underline{\mathcal{J}}^\ell_m(U) \longrightarrow \mathcal{Y}^\ell_m$ is a diffeomorphism which defines a local trivialization over $\underline{\mathcal{J}}^\ell_m(U)$ of the principal fibre bundle $\check{M}^\ell_m \longrightarrow \mathcal{J}^\ell_m(M)$. Since $\operatorname{Aut}(\mathbb{R}^\ell_m)$ is at the same time the structure group of this bundle and the group whose Lie algebra generates the characteristic system of $\Omega(\check{M}^\ell_m)$, from the classical theory of Elie Cartan about the reduction of a Pfaff system to the ring of first integrals of its characteristic system follows that $\eta^*: \mathcal{E}^1(\mathcal{Y}^\ell_m) \longrightarrow \mathcal{E}^1(\underline{\mathcal{J}}^\ell_m(U)$ applies the specialization of $\Omega(\check{M}^\ell_m)$ to $\mathcal{Y}^\ell_m$ into $\Omega(\mathcal{J}^\ell_m(M)) = \mathcal{E}^1(\mathcal{J}^\ell_m(M)) \cap \Omega(\check{M}^\ell_m)$. Thus, our problem is reduced to compute the rank of the specialization $\Omega(\mathcal{Y}^\ell_m)$ of $\Omega(\check{M}^\ell_m)$ to $\mathcal{Y}^\ell_m$.

By definition of η, the points of $\mathcal{Y}^\ell_m$ are determined by the equations $y_i(p^\ell_m) = y_{i0}(p^\ell_m) + x_i$ $(i = 1, \ldots, m)$. If we take as a basis of the $\mathbb{R}^{\ell-1}_m$-module $\operatorname{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^{\ell-1}_m)$

the derivations $\xi_k = \frac{\partial}{\partial x_k}$ $(k = 1, \ldots, m)$ it follows that $\xi_{k\,(p^\ell_m)} y_i = \delta_{ki}$. Then, for each tangent vector $\bar{D}_{p^\ell_m} \in T_{p^\ell_m} \mathcal{Y}^\ell_m$ and each $f \in C^\infty(M)$ we have:

$$(2.1) \quad (m+1)!\,(dy_1 \wedge \cdots \wedge dy_m \wedge df)_{p^{\ell-1}_m} \left(\xi_{1\,(p^\ell_m)}, \ldots, \xi_{m\,(p^\ell_m)}, D_{p^{\ell-1}_m}\right) = D_{p^{\ell-1}_m} f - \xi_{1\,(p^\ell_m)} f \cdot D_{p^{\ell-1}_m} y_1 - \cdots - \xi_{m\,(p^\ell_m)} f \cdot D_{p^{\ell-1}_m} y_m \,,$$

where $D_{p^\ell_m} \in \mathcal{T}_{p^\ell_m} M^\ell_m$ is the derivation corresponding to the tangent vector $\bar{D}_{p^\ell_m}$. Since $\bar{D}_{p^\ell_m}$ is tangent to $\mathcal{Y}^\ell_m$, then $\bar{D}_{p^\ell_m} y_{i\alpha} = 0$ $(1 \le i \le m; 1 \le |\alpha| \le \ell)$, hence $D_{p^{\ell-1}_m} y_i = \bar{D}_{p^{\ell-1}_m} y_{i0} \in \mathbb{R}$. On the other hand we have

$$\xi_{i\,(p^\ell_m)} f = \sum_{|\beta| \le \ell-1} \frac{1}{\beta!} f_{\beta+\epsilon_i}\left(p^\ell_m\right) x^\beta \qquad (1 \le i \le m)$$

$$D_{p^{\ell-1}_m} f = \sum_{|\beta| \le \ell-1} \frac{1}{\beta!} \bar{D}_{p^{\ell-1}_m} f_\beta x^\beta$$

and replacing in (2.1) we get:

$$(m+1)!\,(dy_1 \wedge \cdots \wedge dy_m \wedge df)_{p^{\ell-1}_m} \left(\xi_{1\,(p^\ell_m)}, \ldots, \xi_{m\,(p^\ell_m)}, D_{p^{\ell-1}_m}\right) = \sum_{|\beta| \le \ell-1} \frac{1}{\beta!} \left[d_{p^{\ell-1}_m} f_\beta - \sum_{i=1}^m f_{\beta+\epsilon_i}\left(p^\ell_m\right) d_{p^{\ell-1}_m} y_{i0} \right] \left(\bar{D}_{p^{\ell-1}_m}\right) x^\beta \,.$$

From the former calculus follows that, up to some factors, the real components of the specialization to $\mathcal{Y}^\ell_m$ of the 1–form $\widehat{\omega}$ in $\check{M}^\ell_m$ with values in $\mathbb{R}^{\ell-1}_m$ associated to the $(m+1)$–form $\omega = dy_1 \wedge \cdots \wedge dy_m \wedge df$ are the 1-forms

$$\omega_\beta = df_\beta - \sum_{i=1}^m f_{\beta+\epsilon_i} dy_{i0} \qquad (|\beta| \le \ell - 1)$$

Replacing f by each one of the coordinates $y_{m+1}, \ldots, y_n$ we obtain $(n-m)\binom{m+\ell-1}{m}$ 1–forms on $\mathcal{Y}^\ell_m$ whose values at each point are linearly independent, hence the rank of $\Omega\left(\mathcal{Y}^\ell_m\right)$ is $\ge (n-m)\binom{m+\ell-1}{m}$ and, since it must be less than or equal to this number (which is the rank of $\Omega\left(\check{M}^\ell_m\right)$), we finish the proof. □

Remark. If we use η^* to pass the 1–forms

$$(2.2) \qquad \omega_{m+j,\beta} = dy_{m+j,\beta} - \sum_{i=1}^m y_{m+j,\beta+\epsilon_i} dy_{i0} \quad (i \le j \le n-m; |\beta| \le \ell - 1)$$

from $\mathcal{Y}^\ell_m$ to $\underline{\mathcal{J}}^\ell_m(U)$, we obtain in this open subset the following basis of the contact system:

$$(2.3) \qquad \theta_{m+j,\beta} = dY_{m+j,\beta} - \sum_{i=1}^m Y_{m+j,\beta+\epsilon_i} dy_{i0} \quad (i \le j \le n-m; |\beta| \le \ell - 1)$$

Theorem 2.3. *For each $r \ge 1$, the specialization (by means of the Taylor immersion) to $\mathcal{J}^{\ell+r}_m(M)$ of the contact system in $\mathcal{J}^r_m\left(\mathcal{J}^\ell_m(M)\right)$, considered as a jet space $\mathcal{J}^r_m$ of the manifold $\mathcal{J}^\ell_m(M)$, is the contact system in $\mathcal{J}^{\ell+r}_m(M)$.*

Proof. In the notations of [7], the local equations of the Taylor immersion $\varphi\colon \mathcal{J}_m^{\ell+r}(M) \longrightarrow \mathcal{J}_m^r\left(\mathcal{J}_m^\ell(M)\right)$ are:

$$\begin{aligned} y_{i00} &= y_{i0} && (1 \le i \le m) \\ \mathbf{Y}_{m+j,\alpha,\beta} &= Y_{m+j,\alpha+\beta} && (1 \le j \le n-m; |\alpha| \le \ell, |\beta| \le r) \end{aligned}$$

According to the former remark, the contact system in $\mathcal{J}_m^r\left(\mathcal{J}_m^\ell(M)\right)$ has the local basis

$$\Theta_{m+j,\alpha,\beta} = d\mathbf{Y}_{m+j,\alpha,\beta} - \sum_{i=1}^{m} \mathbf{Y}_{m+j,\alpha,\beta+\epsilon_i}\, dy_{i00} \qquad (|\alpha| \le \ell, |\beta| \le r-1)$$

and if we specialize these one-forms to $\mathcal{J}_m^{\ell+r}(M)$ we obtain:

$$\theta_{m+j,\alpha+\beta} = \varphi^*\left(\Theta_{m+j,\alpha,\beta}\right) = dY_{m+j,\alpha+\beta} - \sum_{i=1}^{m} Y_{m+j,\alpha+\beta+\epsilon_i}\, dy_{i0}\,,$$

which span the contact system in the corresponding open subset of $\mathcal{J}_m^{\ell+r}(M)$. $\square$

Let us denote by $\Omega\left(\mathcal{J}_m^\ell(M)\right)^\perp$ the distribution of tangent vector fields on $\mathcal{J}_m^\ell(M)$ which annihilate the contact system. The vector fields

$$\begin{aligned} \partial_i^{(\ell)} &= \frac{\partial}{\partial y_{i0}} + \sum_{j=1}^{n-m} \sum_{|\beta| \le \ell-1} Y_{m+j,\beta+\epsilon_i} \frac{\partial}{\partial Y_{m+j,\beta}} && (1 \le i \le m) \\ &\frac{\partial}{\partial Y_{m+j,\alpha}} && (1 \le j \le n-m; |\alpha| = \ell) \end{aligned}$$

form a basis of this distribution in the open subset $\underline{\mathcal{J}}_m^\ell(U)$ and for each point $\mathfrak{p}_m^\ell \in \underline{\mathcal{J}}_m^\ell(U)$ the derivations $\left(\frac{\partial}{\partial Y_{m+j,\alpha}}\right)_{\mathfrak{p}_m^\ell}$ $(1 \le j \le n-m; |\alpha| = \ell)$ are a basis of the vector space $Q_{\mathfrak{p}_m^\ell}\underline{\mathcal{J}}_m^\ell(U)$ (notations of [7]).

From the calculus in local coordinates for the prolongation of an ideal made in [7] follows that the prolongation of an ideal I from $C^\infty\left(\mathcal{J}_m^{\ell-1}(M)\right)$ to $C^\infty\left(\mathcal{J}_m^\ell(M)\right)$ is locally generated by I_0 and $\partial_i^{(\ell)} I_0$, $i = 1, \ldots, m$, with the notations used there. Taking in account that the vector fields $\frac{\partial}{\partial Y_{m+j,\alpha}}$, $(|\alpha| = \ell)$ annihilate I_0, we obtain the following

Theorem 2.4. *The prolongation of an ideal I from $C^\infty\left(\mathcal{J}_m^{\ell-1}(M)\right)$ to $C^\infty\left(\mathcal{J}_m^\ell(M)\right)$ is the ideal locally generated by I_0 and the sets $D(I_0)$, where D runs through the module of tangent vector fields which annihilate the contact system $\Omega\left(\mathcal{J}_m^\ell(M)\right)$.*

Remark. Let $\pi\colon M \longrightarrow X$ be a fibre bundle, $m = \dim X$, and denote by $\mathcal{J}^\ell(X,M)$ the fibre bundle of jets of local cross-sections of π. If s is a local cross-section of π defined in a neighbourhood of $x \in X$ and $\mathfrak{p}^\ell = j_x^\ell s$, then the image of the tangent linear map $\left(j^\ell s\right)_*\colon T_x X \longrightarrow T_{\mathfrak{p}^\ell}\mathcal{J}^\ell(X,M)$ annihilates $\Omega\left(\mathcal{J}^\ell(X,M)\right)_{\mathfrak{p}^\ell}$ and, when s varies without changing $j_x^\ell s$, the image of $\left(j^\ell s\right)_*$ runs through the full space $\Omega\left(\mathcal{J}^\ell(X,M)\right)_{\mathfrak{p}^\ell}^\perp$. Therefore we can describe the contact system in the following way: its value at each point $\mathfrak{p}^\ell \in \mathcal{J}^\ell(X,M)$ is the set of 1-forms at $\mathfrak{p}^\ell$ which annihilate all the spaces $\left(j^\ell s\right)_*(T_xX)$, when s runs through the family of local sections of π defined in a neighbourhood of $x=\pi^\ell\left(\mathfrak{p}^\ell\right)$ such that $j^\ell_x s=\mathfrak{p}^\ell$.

3. The contact 1–form on $\mathcal{J}^\ell(X,M)$

In this section we want to translate to our language the 1–form on $\mathcal{J}^\ell(X,M)$ and valued in the vertical tangent bundle $V\mathcal{J}^{\ell-1}(X,M)$ defined in [2, p. 206] by Goldschmidt and Sternberg.

In [7] we show that the tangent space to $\mathcal{J}^\ell_m(M)$ at a point $\mathfrak{p}^\ell_m$ is isomorphic to

$$\mathrm{Der}_{\mathbb{R}}\left(C^\infty(M), C^\infty(M)/\mathfrak{p}^\ell_m\right)/\mathrm{Der}_{\mathbb{R}}\left(C^\infty(M)/\mathfrak{p}^\ell_m, C^\infty(M)/\mathfrak{p}^\ell_m\right)$$

Let $\mathfrak{p}^\ell\in\mathcal{J}^\ell(X,M)$ and $x=\pi^\ell(\mathfrak{p}^\ell)$; let $D_{\mathfrak{p}^\ell}$ be a tangent vector to $\mathcal{J}^\ell(X,M)$ at $\mathfrak{p}^\ell$ and $D\in\mathrm{Der}_{\mathbb{R}}\left(C^\infty(M), C^\infty(M)/\mathfrak{p}^\ell\right)$ be a derivation whose class modulo $\mathrm{Der}_{\mathbb{R}}\left(C^\infty(M)/\mathfrak{p}^\ell, C^\infty(M)/\mathfrak{p}^\ell\right)$ is attached to $D_{\mathfrak{p}^\ell}$ by the above isomorphism; let us denote by $\bar D$ the projection of the derivation D to $\mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M)/\mathfrak{p}^{\ell-1})$.

The restriction of $\bar D$ to $C^\infty(X)$ sends $\mathfrak{m}_x^{\ell+1}$ to zero, hence $\bar D$ gives rise to a derivation $\widetilde D$ from $C^\infty(X)/\mathfrak{m}_x^{\ell+1}$ into $C^\infty(X)/\mathfrak{m}_x^{\ell}$. If we identify $C^\infty(M)/\mathfrak{p}^\ell$ with $C^\infty(X)/\mathfrak{m}_x^{\ell+1}$, the specialization to $C^\infty(X)$ of the homomorphism $\mathfrak{p}^\ell: C^\infty(M)\longrightarrow C^\infty(M)/\mathfrak{p}^\ell$ is the canonical factor map, hence the restriction of $\bar D$ to $C^\infty(X)$ factors as $\widetilde D\circ\mathfrak{p}^\ell$, and $\bar D-\widetilde D\circ\mathfrak{p}^\ell$ is a $C^\infty(X)$-derivation, that is to say an element of $V_{\mathfrak{p}^{\ell-1}}\mathcal{J}^{\ell-1}(X,V)$ which depends only on $D_{\mathfrak{p}^\ell}$; indeed, if D vanishes at $\mathfrak{p}^\ell$ then $\bar D$ factorizes, via the quotient map $\mathfrak{p}^\ell: C^\infty(M)\longrightarrow C^\infty(M)/\mathfrak{p}^\ell$, as $\widetilde D\circ\mathfrak{p}^\ell$, and $\bar D-\widetilde D\circ\mathfrak{p}^\ell=0$.

Thus we have defined a mapping

$$\begin{aligned}\omega^{(\ell)}_{\mathfrak{p}^\ell}: T_{\mathfrak{p}^\ell}\mathcal{J}^\ell(X,M)&\longrightarrow V_{\mathfrak{p}^{\ell-1}}\mathcal{J}^{\ell-1}(X,M)\\ D_{\mathfrak{p}^\ell}&\longmapsto \bar D-\widetilde D\circ\mathfrak{p}^\ell\end{aligned}$$

which we call the *contact 1–form* on $\mathcal{J}^\ell(X,M)$ and agrees with the one defined by Goldschmidt and Sternberg in [2]. Note that $\omega^{(\ell)}$ depends on the fibration $\pi: M\longrightarrow X$ whereas the contact system on $\mathcal{J}^\ell_m(V)$ does not. We are going to study the relationship between these two constructions.

The following result shows that the kernel of $\omega^{(\ell)}$ does not depend on the fibration $\pi: M\longrightarrow X$.

Proposition 3.1. *The value of $\omega^{(\ell)}$ at each point $\mathfrak{p}^\ell\in\mathcal{J}^\ell(X,V)$ is the epimorphism $\omega^{(\ell)}_{\mathfrak{p}^\ell}: T_{\mathfrak{p}^\ell}\mathcal{J}^\ell(X,M)\longrightarrow V_{\mathfrak{p}^{\ell-1}}\mathcal{J}^{\ell-1}(X,V)$ which for the vertical vectors is the natural projection and whose kernel is the set of classes of derivations from $C^\infty(M)$ into $C^\infty(M)/\mathfrak{p}^\ell$ which send $\mathfrak{p}^\ell$ to $\mathfrak{p}^{\ell-1}/\mathfrak{p}^\ell$.*

Proof. The first statement is immediate. On the other hand, $\ker\omega^{(\ell)}_{\mathfrak{p}^\ell}$ is the set of classes of derivations D from $C^\infty(M)$ into $C^\infty(M)/\mathfrak{p}^\ell$ such that $\bar D=\widetilde D\circ\mathfrak{p}^\ell$, that is to say, $\bar D$ sends $\mathfrak{p}^\ell$ to zero or, what is the same, D applies $\mathfrak{p}^\ell$ into $\mathfrak{p}^{\ell-1}/\mathfrak{p}^\ell$. Conversely, if D sends $\mathfrak{p}^\ell$ to $\mathfrak{p}^{\ell-1}/\mathfrak{p}^\ell$, then $\bar D-\widetilde D\circ\mathfrak{p}^\ell$ annihilates $\mathfrak{p}^\ell$; since it annihilates $C^\infty(X)$ too, it must vanish, because $C^\infty(X)+\mathfrak{p}^\ell=C^\infty(M)$. □

Let $p_m^\ell \in \check{M}_m^\ell$; from Corollary 4.3 of [8] follows that $D_{p_m^\ell} \in \mathcal{T}_{p_m^\ell}\check{M}_m^\ell$ annihilates the contact system $\Omega(\check{M}_m^\ell)$ if and only if its projection to $\mathcal{T}_{p_m^{\ell-1}}\check{M}_m^{\ell-1}$ has the form $D_{p_m^{\ell-1}} = \xi \circ p_m^\ell$, for some $\xi \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}_m^\ell, \mathbb{R}_m^{\ell-1})$.

If $\ker p_m^\ell = \mathfrak{p}^\ell$, through the isomorphisms

$$C^\infty(M)/\mathfrak{p}^\ell \approx \mathbb{R}_m^\ell \quad \text{and} \quad C^\infty(M)/\mathfrak{p}^{\ell-1} \approx \mathbb{R}_m^{\ell-1}$$

defined by p_m^ℓ and $p_m^{\ell-1}$, respectively, $D_{p_m^\ell}$ corresponds to a derivation $D\colon C^\infty(M) \to C^\infty(M)/\mathfrak{p}^\ell$ whose projection $\bar{D}$ annihilates $\mathfrak{p}^\ell$. Conversely, if a derivation D from $C^\infty(M)$ into $C^\infty(M)/\mathfrak{p}^\ell$ applies $\mathfrak{p}^\ell$ into $\mathfrak{p}^{\ell-1}/\mathfrak{p}^\ell$, $\bar{D}$ factorizes as a derivation $\widetilde{D} : C^\infty(M)/\mathfrak{p}^\ell \longrightarrow C^\infty(M)/\mathfrak{p}^{\ell-1}$. If we take $p_m^\ell \in \check{M}_m^\ell$ such that $\ker p_m^\ell = \mathfrak{p}^\ell$, we have that D is identified with a derivation $D_{p_m^\ell} \in \mathcal{T}_{p_m^\ell}\check{M}_m^\ell$ and its projection, $\bar{D}$, with $D_{p_m^{\ell-1}} = \xi \circ p_m^\ell$, where $\xi \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}_m^\ell, \mathbb{R}_m^{\ell-1})$ is the derivation induced by $\widetilde{D}$. Summarizing:

Corollary 3.2. *In the open subset $\mathcal{J}^\ell(X,M)$ of $\mathcal{J}_m^\ell(M)$ the annihilator subspace of the contact system $\Omega(\mathcal{J}_m^\ell(V))$ agrees with the kernel of $\omega^{(\ell)}$.*

Remark. Let U be an open subset of M coordinated by functions $y_1, \dots, y_n$ which identify it with an open subset $U' \times U''$; in the notations of [7], in $\mathcal{J}^\ell(U',U)$ we have local coordinates $y_i, Y_{m+j,\alpha}$ $(1 \le i \le m,\ 1 \le j \le n-m,\ |\alpha| \le \ell)$. The expression of $\omega^{(\ell)}$ in these coordinates is as follows:

$$\omega^{(\ell)} = \sum_{j=1}^{n-m} \sum_{|\beta| \le \ell-1} \theta_{m+j,\beta} \otimes \frac{\partial}{\partial Y_{m+j,\beta}},$$

where the $\theta_{m+j,\beta}$ are the 1-forms given by (2.3), because the coordinate forms of $\omega^{(\ell)}$ have the same annihilator subspace than the contact system and both sides of the above equality agree when they are applied to vertical vectors.

Since $\omega_{\mathfrak{p}^\ell}^{(\ell)}$ is the natural projection for vertical vectors, $Q_{\mathfrak{p}^\ell}\mathcal{J}^\ell(X,V)$ is contained in $\ker \omega_{\mathfrak{p}^\ell}^{(\ell)}$.

If the projection onto $T_x(X)$ of $D_{\mathfrak{p}^\ell} \in \ker \omega_{\mathfrak{p}^\ell}^{(\ell)}$ is a vector $D_x \neq 0$, the class of $\bar{D} = \widetilde{D} \circ \mathfrak{p}^\ell$ depends only on D_x and $\mathfrak{p}^\ell$.

In fact, let Y, Y' be derivations from $C^\infty(X)/\mathfrak{m}_x^{\ell+1}$ into $C^\infty(X)/\mathfrak{m}_x^\ell$ such that $Y_x = Y'_x$; then the image of $Y - Y'$ is contained in $\mathfrak{m}_x/\mathfrak{m}_x^\ell$, like $(Y - Y') \circ \mathfrak{p}^\ell$, hence $(Y - Y') \circ \mathfrak{p}^\ell$ determines a derivation from $C^\infty(X)/\mathfrak{m}_x^\ell$ into $C^\infty(X)/\mathfrak{m}_x^\ell$, and consequently the derivations $Y \circ \mathfrak{p}^\ell$, $Y' \circ \mathfrak{p}^\ell$ from $C^\infty(M)$ into $C^\infty(X)/\mathfrak{m}_x^\ell$ have the same class modulo $\mathrm{Der}_{\mathbb{R}}\left(C^\infty(X)/\mathfrak{m}_x^\ell, C^\infty(X)/\mathfrak{m}_x^\ell\right)$.

Let us denote by $Y_x \circ \mathfrak{p}^\ell \in T_{\mathfrak{p}^{\ell-1}}\mathcal{J}^{\ell-1}(X,V)$ the vector representing the derivation $Y \circ \mathfrak{p}^\ell$; it is easy to show that $Y_x \circ \mathfrak{p}^\ell$ is the composition of $\mathfrak{p}^\ell$, understood as an epimorphism from $C^\infty(\mathcal{J}^{\ell-1}(X,M))$ into $C^\infty(X)/\mathfrak{m}_x^2$, and Y_x thought as a derivation from $C^\infty(X)/\mathfrak{m}_x^2$ into $\mathbb{R}$.

Let $p_m^\ell \in \check{M}_m^\ell$ such that $\ker p_m^\ell = \mathfrak{p}^\ell$; from Proposition 3.5 of [8] follows that $\mathrm{Der}(\mathbb{R}_m^\ell, \mathbb{R}_m^{\ell-1}) \approx T_{x_m^{\ell-1}}\check{X}_m^{\ell-1}$, where x_m^ℓ is the specialization of p_m^ℓ to $C^\infty(X)$, hence the projection of the morphism $p^{\ell}_{m*} : \mathrm{Der}(\mathbb{R}^{\ell}_{m}, \mathbb{R}^{\ell-1}_{m}) \longrightarrow T_{p^{\ell-1}_{m}} \check{M}^{\ell-1}_{m}$ to the jet space is the linear map

$$\begin{aligned} \mathfrak{p}^{\ell}_{*} : T_x X &\longrightarrow T_{\mathfrak{p}^{\ell-1}} \mathcal{J}^{\ell-1}(X,V) \\ Y_x &\longrightarrow \mathfrak{p}^{\ell}_{*}(Y_x) = Y_x \circ \mathfrak{p}^{\ell} \end{aligned}$$

If $\bar{\mathfrak{p}}^{\ell} \in \mathcal{J}^{\ell}(X,V)$ verifies that $\bar{\mathfrak{p}}^{\ell-1} = \mathfrak{p}^{\ell-1}$, then $\bar{\mathfrak{p}}^{\ell} - \mathfrak{p}^{\ell}$ is a derivation from $C^{\infty}(M)$ into $\mathfrak{m}^{\ell}_{x} / \mathfrak{m}^{\ell+1}_{x}$ which vanish if and only if $Y \circ (\bar{\mathfrak{p}}^{\ell} - \mathfrak{p}^{\ell}) = 0$ for each $Y \in \mathrm{Der}_{\mathbb{R}}(C^{\infty}(X)/\mathfrak{m}^{\ell+1}_{x}, C^{\infty}(X)/\mathfrak{m}^{\ell}_{x})$; that is to say, $\mathfrak{p}^{\ell}$ is completely determined by the couple $(\mathfrak{p}^{\ell-1}, \mathfrak{p}^{\ell}{}_{*})$.

The following result is a reformulation of Theorem 4.5 of [8].

Proposition 3.3. *Let F be a local cross-section of $\mathcal{J}^{\ell}(X,M) \longrightarrow X$ over an open subset $W \subset X$; if the contact form $\omega^{(\ell)}$ vanishes over $F(W)$, then there is a section $s : W \longrightarrow M$ of $\pi : V \longrightarrow X$ such that $j^{\ell}s = F$.*

Proof. We apply induction on ℓ; if $\ell = 1$ and $F\colon W \longrightarrow \mathcal{J}^{1}(X,V)$ is a local section, taking $s = \pi^{1} \circ F$ we obtain a local section of $\pi\colon M \longrightarrow X$. For each $x \in W$, $F(x)$ and $j^{1}_{x}s$ have the same projection $s(x) \in M$; but $j^{1}_{x}s$ is the composition of the map $s^{*} : C^{\infty}(M) \longrightarrow C^{\infty}(W)$ and the quotient modulo $\mathfrak{m}^{2}_{x}$, and for each $Y_x \in T_x(X)$ we have $s_{*}(Y_x) = Y_x \circ j^{1}_{x}s$, where in the right side Y_x is considered as a derivation from $C^{\infty}(X)/\mathfrak{m}^{2}_{x}$ into $\mathbb{R}$.

If we assume that the specialization of $\omega^{(1)}$ to $F(W)$ vanishes, then $\omega^{(1)}(F_{*}Y_x) = 0$ for each $Y_x \in T_x W$, hence

$$\pi^{1}_{*}(F_{*}Y_x) = s_{*}(Y_x) = Y_x \circ j^{1}_{x}s = Y_x \circ F(x)$$

and $j^{1}_{x}s = F(x)$.

Now suppose that the statement is proved up to $\ell - 1$; since $F(W)$ is a solution of $\omega^{(\ell)}$, $\pi^{\ell-1}_{\ell} \circ F(W)$ is a solution of $\omega^{(\ell-1)}$ and by the induction hypothesis there is a section s such that $j^{\ell-1}s = \pi^{\ell-1}_{\ell} \circ F$; then $j^{\ell-1}_{x}s = \pi^{\ell-1}_{\ell}(F(x))$ for each $x \in W$, and since $\omega^{(\ell)}(F_{*}Y_x) = 0$ by hypothesis for each $Y_x \in T_x W$, we have

$$(\pi^{\ell-1}_{\ell})_{*}(F_{*}(Y_x)) = (j^{\ell-1}s)_{*}(Y_x) = Y_x \circ j^{\ell}_{x}s = Y_x \circ F(x)\,,$$

and therefore $F(x) = j^{\ell}_{x}s$. □

References

[1] Ehresmann, C., *Introduction à la théorie des structures infinitesimales et des pseudo-groupes de Lie*, Colloque de Géometrie Différentielle, C.N.R.S. (1953), 97–110.

[2] Goldschmidt, H. and Sternberg, S., *The Hamilton–Cartan formalism in the calculus of variations*, Ann. Inst. Fourier (Grenoble) **23** (1973), 203–267.

[3] Grigore, D. R. and Krupka, D., *Invariants of velocities and higher order Grassmann bundles*, J. Geom. Phys. **24** (1998), 244–264.

[4] Jacobson, N., *Lie algebras*, John Wiley & Sons, Inc., New York, 1962.

[5] Kolář, I., Michor, P. W. and Slovák, J., *Natural operations in differential geometry*, Springer-Verlag, New York, 1993.

[6] Morimoto, A., *Prolongation of connections to bundles of infinitely near points*, J. Differential Geom. **11** (1976), 479–498.

[7] Muñoz, J., Muriel, F. J. and Rodríguez, J., *Weil bundles and jet spaces*, Czechoslovak Math. J. **50 (125)** (2000), 721–748.

[8] ———, *The contact system on the spaces of* (m,ℓ)*-velocities*, Proceedings of the 7th International Conference Differential Geometry and Applications (Brno, 1998) (1999), 263–272.

[9] Weil, A.,*Théorie des points proches sur les variétés différentiables*, Colloque de Géometrie Différentielle, C.N.R.S. (1953), 111–117.

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