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PROLONGATION OF SECOND ORDER CONNECTIONS TO VERTICAL WEIL BUNDLES

ANTONELLA CABRAS AND IVAN KOLÁŘ

ABSTRACT. We study systematically the prolongation of second order connections in the sense of C. Ehresmann from a fibered manifold into its vertical bundle determined by a Weil algebra A. In certain situations we deduce new properties of the prolongation of first order connections. Our original tool is a general concept of a B-field for another Weil algebra B and of its A-prolongation.

The simpliest example of a Weil functor is the jet functor T^r_k of k-dimensional velocities of order r. Recently it has been clarified that the theory of Weil bundles represents a unified technique for studying many geometric problems related with product preserving functors, see [11] for a survey. Our starting point is a paper by W. M. Mikulski and the second author, who studied the prolongation of a first order connection on an arbitrary fibered manifold $Y \to M$ with respect to a vertical functor on the category $\mathcal{FM}$ of all fibered manifolds and their morphisms, [12]. In the case of the vertical Weil functor V^A they obtained an interesting naturality result: the operator transforming a first order connection Γ on Y into its vertical prolongation $\mathcal{V}^A\Gamma$ on V^AY is the only natural one.

On the other hand our research is based on the results from [3], where we developed systematically the theory of second order connections in the sense of C. Ehresmann on an arbitrary fibered manifold. So the main subject of the present paper are the geometric properties of prolongation of the second order connections to the vertical Weil bundles. However, at several places we deduce further geometric properties of prolongation of the first order connections. Since our problems are of general character, our procedures are always coordinate free. Nevertheless, occasionally we outline the coordinate expressions as well. At these places we use

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heavily the fact that the A-prolongation of a real valued function is a function with values in the Weil algebra A.

In Section 1 we introduce an original tool, the concept of B-field for another Weil algebra B and of its A-prolongation. Sections 2 and 3 are devoted to the vertical A-prolongation of a first order connection Γ. In particular, we deduce an explicit formula for the curvature of $\mathcal{V}^A\Gamma$ in terms of the curvature of Γ and an analogous characterization of the lifting map of $\mathcal{V}^A\Gamma$. The first result on the prolongation of second order connections is Proposition 4, which reads that the prolongation of the product of the first order connections is the product of their prolongations. In Section 5 we introduce the lifting map of a second order connection Δ. It lifts the elements of the second iterated tangent bundle TTM of the base M to TTY. The sections $M \to TTM$ are $\mathbb{D}\otimes\mathbb{D}$-fields on M, where $\mathbb{D}$ is the algebra of dual numbers. Using the ideas of Section 1, we clarify that the lifting of $\mathbb{D}\otimes\mathbb{D}$-fields with respect to Δ has similar properties to lifting of vector fields with respect to a first order connection. This enables us to prove, in Corollary 3, that the canonical decomposition of a second order connection is preserved under the vertical A-prolongation. The last two sections are devoted to the basic properties of the first order absolute differentiation with respect to the prolongated connection and of the second order one.

All manifolds and maps are assumed to be infinitely differentiable and all manifolds are paracompact. Unless otherwise specified, we use the terminology and notation from [11].

1. Prolongation of Weil fields

Let A be a Weil algebra and T^A be the corresponding Weil functor, [11], [18]. For a manifold M, we consider each element of T^AM in the form of an A-jet j^Ag, $g:\mathbb{R}^k\to M$, where k is the width of A, [11]. For a smooth map $f:M\to N$, we define $T^Af:T^AM\to T^AN$ by

$$T^Af(j^Ag)=j^A(f\circ g)\,. \tag{1}$$

Let B be another Weil algebra. Consider $X\in T^B(T^AM)$, $X=j^B\varphi$, where $\varphi:\mathbb{R}^l\to T^AM$, $l=$ the width of B. Every $\varphi(t)\in T^AM$, $t\in\mathbb{R}^l$, is of the form $j^A\psi(\tau,t)$, $\tau\in\mathbb{R}^k$, where ψ is a map $\mathbb{R}^k\times\mathbb{R}^l\to M$. Hence $X=j^B(j^A\psi(\tau,t))$ and we can define the exchange diffeomorphism $i_M^{B,A}:T^B(T^AM)\to T^A(T^BM)$ by

$$i_M^{B,A}(X)=j^A(j^B\psi(\tau,t))\,. \tag{2}$$

Write $\pi_{A,M}:T^AM\to M$ for the bundle projection. Consider the bundle projection $\pi_{B,T^AM}:T^BT^AM\to T^AM$ and the induced map $T^B\pi_{A,M}:T^BT^AM\to T^BM$. One verifies easily that $i_M^{B,A}$ exchanges the related projections, i.e.

$$T^A\pi_{B,M}\circ i_M^{B,A}=\pi_{B,T^AM}\,,\quad \pi_{A,T^BM}\circ i_M^{B,A}=T^B\pi_{A,M}\,. \tag{3}$$

Definition 1. A section $\xi : M \to T^B M$ will be called a Weil field of type B or a B-field on M. The B-field $\mathcal{T}^A\xi = i_M^{B,A} \circ T^A\xi : T^A M \to T^B(T^A M)$ will be called the A-prolongation of ξ.

If B is the algebra $\mathbb{D}$ of dual numbers, then $T^{\mathbb{D}} = T$ is the tangent functor, ξ is a vector field and $\mathcal{T}^A\xi$ is the flow prolongation of ξ, [11]. We remark that one has defined the flow prolongation of a vector field with respect to an arbitrary natural bundle, while in the case of a B-field we have to consider Weil bundles only.

Consider a fibered manifold $p : Y \to M$. Its vertical Weil bundle $V^A Y \to M$ is defined by

$$V^A Y = \bigcup_{x \in M} T^A(Y_x)\,.$$

Let $W \to N$ be another fibered manifold and $f : Y \to W$ be an $\mathcal{FM}$-morphism with the base map $\underline{f} : M \to N$. Then $V^A f : V^A Y \to V^A W$ is defined fiberwise.

Consider the subbundles $V^B(T^A Y \to T^A M) \subset T^B(T^A M)$ and $T^A(V^B Y) \subset T^A(T^B Y)$.

Lemma 1. *$i_Y^{B,A}$ maps $V^B(T^A Y \to T^A M)$ into $T^A(V^B Y)$.*

Proof. By locality, it suffices to consider a product bundle $Y = M \times N$. We have

$$T^A Y = T^A M \times T^A N\,,\ V^B(T^A Y \to T^A M) = T^A M \times T^B T^A N\,,$$
$$V^B Y = M \times T^B N\,,\ T^A(V^B Y) = T^A M \times T^A T^B N\,.$$

In this situation, $i_Y^{B,A}$ is reduced to the exchange diffeomorphism $i_N^{B,A} : T^B T^A N \to T^A T^B N$. □

The restricted and corestricted map will be denoted by

$$i_{V,Y}^{B,A} : V^B(T^A Y \to T^A M) \to T^A(V^B Y)\,.$$

If we consider $T^A(V^B Y)$ as a fibered manifold over $T^A M$, then $i_{V,Y}^{B,A}$ is an $\mathcal{FM}$-morphism over $\mathrm{id}_{T^A M}$. If we further restrict $i_{V,Y}^{B,A}$ to $V^B(V^A M \to M)$, then its values lie in $V^A(V^B Y \to M)$. The restricted and corestricted map

$$i_{Y,V}^{B,A} : V^B V^A Y \to V^A V^B Y$$

represents the exchange diffeomorphism applied fiberwise.

A B-field $\xi : Y \to T^B Y$ is called projectable, if there is a B-field $\underline{\xi} : M \to T^B M$ satisfying $T^B p \circ \xi = \underline{\xi} \circ p$. In this case,

$$\mathcal{V}^A\xi := i_{V,Y}^{A,B} \circ V^A\xi : V^A Y \to T^B(V^A Y)$$

is a projectable B-field on $V^A Y$ over $\underline{\xi}$, which will be called the vertical A-prolongation of ξ. If $B = \mathbb{D}$, then $V^{\mathbb{D}} = V$ is the vertical tangent functor and

ξ is a projectable vector field on Y. One verifies easily that $\mathcal{V}^A\xi$ coincides with the flow prolongation of ξ with respect to the functor V^A.

Consider another fibered manifold $q : Z \to M$ and an $\mathcal{FM}$-morphism $f : Y \times_M Z \to W$ with the base map $\underline{f} : M \to N$. Then we define the vertical A-prolongation with respect to the first factor

$$V_1^A f : V^A Y \times_M Z \to V^A W$$

fiberwise. In other words, if we denote by $f_z : Y_x \to W_{\underline{f}(x)}$, $z \in Z$, $q(z) = x$, the map $y \mapsto f(y, z)$, then

$$(V_1^A f)_z = T^A(f_z) : (V^A Y)_x \to (V^A W)_{\underline{f}(x)}\,.$$

In particular, if we have a section $s : M \to Z$ and write

$$f(s) : Y \to W\,,\ \ f(s)(y) = f(y, s(py))\,,$$

then $(V_1^A f)(s) = V^A(f(s)) : V^A Y \to V^A W$.

Consider the special case $W = V^B Y$ and f with the property $\pi_{B,Y}(f(y, z)) = y$. Then we introduce

$$\mathcal{V}_1^A f := i_{Y,V}^{A,B} \circ V_1^A f : V^A Y \times_M Z \to V^B V^A Y\,.$$

By construction, we have $(\mathcal{V}_1^A f)(s) = \mathcal{V}^A(f(s))$ for every section $s : M \to Z$.

2. On the prolongation of first order connections

According to [12], the construction of the vertical A-prolongation of a first order connection on Y can be based on the canonical exchange isomorphism

$$i_Y^{1,A} : V^A(J^1 Y \to M) \to J^1(V^A Y \to M)\,,$$

[10], see also [8] for the special case $A = \mathbb{D}$. Every $X \in V^A(J^1 Y \to M)$ over $x \in M$ is of the form $X = j^A\varphi(\tau)$, $\varphi : \mathbb{R}^k \to J_x^1 Y$, $\varphi(\tau) = j_x^1\psi(\tau, u)$, $u \in M$. Then we define

$$i_Y^{1,A}(X) = j_x^1 j^A \psi(\tau, u)\,. \tag{4}$$

Lemma 2. *If $f : Y \to Z$ is a base-preserving morphism, then the following diagram commutes*

$$\begin{tikzcd} V^A J^1 Y \arrow[r, "V^A J^1 f"] \arrow[d, "i_Y^{1,A}"'] & V^A J^1 Z \arrow[d, "i_Z^{1,A}"] \\ J^1 V^A Y \arrow[r, "J^1 V^A f"] & J^1 V^A Z \end{tikzcd} \tag{5}$$

Proof. We have $(V^A J^1 f)(j^A j^1_x \psi(\tau,u)) = j^A(J^1 f(j^1_x \psi(\tau,u)) = j^A(j^1_x f(\psi(\tau,u)))$. On the other hand, $(J^1 V^A f)(j^1_x j^A \psi(\tau,u)) = j^1_x j^A f(\psi(\tau,u))$. □

Every connection $\Gamma : Y \to J^1 Y$ induces a connection $\mathcal{V}^A \Gamma$ on $V^A Y \to M$ by

$$\mathcal{V}^A \Gamma = i^{1,A}_Y \circ V^A \Gamma : V^A Y \to J^1 V^A Y\,. \tag{6}$$

It will be useful to have a kind of coordinate expression of $\mathcal{V}^A \Gamma$. Locally Y is of the form $\mathbb{R}^{m,n} = \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ with the coordinates x^i on $\mathbb{R}^m$ and y^p on $\mathbb{R}^n$. Then y^p_i are the induced coordinates on $J^1\mathbb{R}^{m,n}$ and the equations of Γ are

$$y^p_i = F^p_i(x,y)\,. \tag{7}$$

Since $T^A\mathbb{R} = A$, we have $V^A\mathbb{R}^{m,n} = \mathbb{R}^m \times A^n$. So every element of $V^A\mathbb{R}^{m,n}$ is of the form $(x^1,\dots,x^m,\eta^1,\dots,\eta^n)$, $x^i \in \mathbb{R}$, $\eta^p \in A$. Hence η^p represents a kind of vector valued coordinate function on $V^A\mathbb{R}^{m,n}$. The induced vector valued coordinates on $J^1 V^A \mathbb{R}^{m,n}$ are η^p_i.

In general, consider a function $f : Y \to \mathbb{R}$. If we interpret $\mathbb{R}$ as a fibered manifold $\mathbb{R} \to pt$, where pt is a point, then f is an $\mathcal{FM}$-morphism. So we have an induced A-valued function $V^A f : V^A Y \to A$. Since the addition or multiplication in A is defined to be the prolongation of the addition or multiplication of reals, we have, for another function $g : Y \to \mathbb{R}$,

$$V^A(f+g) = V^A f + V^A g\,, \tag{8}$$

$$V^A(fg) = (V^A f)(V^A g) \tag{9}$$

with addition or multiplication in A on the right hand side, respectively. Furthermore, if we consider a function $\varphi : M \to \mathbb{R}$, then we find similarly

$$V^A(\varphi f) = \varphi V^A f\,. \tag{10}$$

In this notation, (7) implies the following coordinate form of $\mathcal{V}^A \Gamma$

$$\eta^p_i = V^A(F^p_i)\,. \tag{11}$$

For every vector field X on M, we denote by $\Gamma X : Y \to TY$ its Γ-lift. The curvature of Γ can be interpreted as a map

$$C_\Gamma : Y \times_M (TM \times_M TM) \to VY\,. \tag{12}$$

For every two vector fields X_1, X_2 on M, we have

$$C_\Gamma(X_1,X_2) = [\Gamma X_1, \Gamma X_2] - \Gamma([X_1,X_2])\,. \tag{13}$$

If we apply $\mathcal{V}^A_1$ to (12), we obtain

$$\mathcal{V}^A_1 C_\Gamma : V^A Y \times_M (TM \times_M TM) \to VV^A Y\,.$$

Proposition 1. *We have* $C_{\mathcal{V}^A\Gamma} = \mathcal{V}^A C_\Gamma$.

Proof. Applying V^A to (13), we obtain

$$(V_1^A C_\Gamma)(X_1, X_2) = V^A([\Gamma X_1, \Gamma X_2]) - V^A\Gamma([X_1, X_2])\,.$$

Using $i^{A,\mathbb{D}}_{V,Y}$, we find

$$(\mathcal{V}_1^A C_\Gamma)(X_1, X_2) = \mathcal{V}^A([\Gamma X_1, \Gamma X_2]) - (\mathcal{V}^A\Gamma)([X_1, X_2])\,.$$

But $\mathcal{V}^A$ is the flow prolongation of vector fields that preserves the bracket, [11]. Hence $\mathcal{V}^A([\Gamma X_1, \Gamma X_2]) = [(\mathcal{V}^A\Gamma)X_1, (\mathcal{V}^A\Gamma)X_2]$. This completes the proof. □

If Γ is given by (7), then the coordinate form of C_Γ is

$$\left(\frac{\partial F_i^p}{\partial x^j} + \frac{\partial F_i^p}{\partial y^q} F_j^q\right) dx^i \wedge dx^j\,. \tag{14}$$

Using Proposition 1, we deduce the following coordinate expression of $C_{\mathcal{V}^A\Gamma}$

$$\left[V^A\left(\frac{\partial F_i^p}{\partial x^j}\right) + V^A\left(\frac{\partial F_i^p}{\partial y^q}\right) V^A(F_j^q)\right] dx^i \wedge dx^j\,, \tag{15}$$

where the product of the second and third terms is in A.

Since $\beta : J^1Y \to Y$ is an affine bundle with associated vector bundle $VY \otimes T^*M$, the difference of two connections Γ, $\overline{\Gamma} : Y \to J^1Y$ is a map $\delta(\Gamma, \overline{\Gamma}) : Y \to VY \otimes T^*M$ that is called the deviation of Γ and $\overline{\Gamma}$. If we write $\delta(\Gamma, \overline{\Gamma}) : Y \times_M TM \to VY$, we can construct

$$\mathcal{V}_1^A \delta(\Gamma, \overline{\Gamma}) : V^AY \times_M TM \to VV^AY\,.$$

On the other hand, we have the deviation $\delta(\mathcal{V}^A\Gamma, \mathcal{V}^A\overline{\Gamma}) : V^AY \times_M TM \to VV^AY$.

Proposition 2. *It holds* $\delta(\mathcal{V}^A\Gamma, \mathcal{V}^A\overline{\Gamma}) = \mathcal{V}_1^A\delta(\Gamma, \overline{\Gamma})$.

Proof. For every vector field X on M, we have

$$\delta(\Gamma, \overline{\Gamma})(X) = \Gamma X - \overline{\Gamma} X\,.$$

Then we proceed in the same way as in the proof of Proposition 1. □

3. The lifting map in the first order

We are going to describe the prolongation of the lifting map of a first order connection on Y in a way we shall need in the second order. We start with the general situation. The lifting of tangent vectors from the base to the horizontal subspaces of Y can be interpreted as an $\mathcal{FM}$-morphism $\Lambda_Y : J^1Y \times_M TM \to TY$ over id_{TM}, provided we consider $J^1Y \times_M TM$ as a fibered manifold over TM. If we take $j^1_x\sigma(u) \in J^1Y$ and $\frac{\partial}{\partial t}\big|_0\gamma(t) \in T_xM$, then

$$\Lambda_Y(j^1_x\sigma(u), \frac{\partial}{\partial t}\Big|_0\gamma(t)) = \frac{\partial}{\partial t}\Big|_0\sigma(\gamma(t))\,.$$

On V^AY, we have $\Lambda_{V^AY} : J^1V^AY \times_M TM \to TV^AY$.

Proposition 3. *The following diagram commutes*

$$\begin{tikzcd} V^A J^1 Y \times_M TM \arrow[r, "V^A\Lambda_Y"] \arrow[d, "i_Y^{1,A}"'] & V^A(TY \to TM) \arrow[d, "i_{V,Y}^{A,\mathbb{D}}"] \\ J^1 V^A Y \times_M TM \arrow[r, "\Lambda_{V^A Y}"] & T(V^A Y) \end{tikzcd}$$

Proof. Take $(j^A j^1_x \psi(\tau, u), \frac{\partial}{\partial t}\big|_0 \gamma(t)) \in V^A J^1 Y \times_M TM$. Clockwise, we first obtain $j^A \frac{\partial}{\partial t}\big|_0 \psi(\tau, \gamma(t))$ and then $\frac{\partial}{\partial t}\big|_0 j^A \psi(\tau, \gamma(t))$. Counterclockwise, we first construct $j^1_x j^A \psi(\tau, u)$ and then $\frac{\partial}{\partial t}\big|_0 j^A \psi(\tau, \gamma(t))$. $\square$

The lifting map $\Lambda\Gamma$ of a first order connection $\Gamma\colon Y \to J^1 Y$ can be defined as

$$\Lambda\Gamma = \Lambda_Y \circ (\Gamma \times_M \mathrm{id}_{TM}) : Y \times_M TM \to TY\,.$$

If we add $\mathcal{V}^A\Gamma = i_Y^{1,A} \circ V^A\Gamma$ to the left of (16), we obtain

Corollary 1. *We have*

$$\Lambda(\mathcal{V}^A\Gamma) = i_{V,Y}^{A,\mathbb{D}} \circ V^A(\Lambda\Gamma)\,. \tag{17}$$

4. Prolongation of second order connections

Consider the second non-holonomic prolongation $\widetilde{J}^2 Y = J^1(J^1 Y \to M)$ of Y, [6], [13]. Beside the target jet projection $\beta_1 : \widetilde{J}^2 Y \to J^1 Y$ we have the first jet prolongation $J^1\beta : \widetilde{J}^2 Y \to J^1 Y$ of the $\mathcal{FM}$-morphism $\beta : J^1 Y \to Y$. The local coordinates x^i, y^p on Y and the corresponding coordinates y^p_i on $J^1 Y$ induce the additional coordinates y^p_{0i}, y^p_{ij} on $\widetilde{J}^2 Y$.

A second order non-holonomic connection on Y in the sense of C. Ehresmann is a section $\Delta : Y \to \widetilde{J}^2 Y$, [7], [9]. So the coordinate expression of Δ is

$$y^p_i = F^p_i(x, y)\,,\ y^p_{0i} = G^p_i(x, y)\,,\ y^p_{ij} = H^p_{ij}(x, y)\,. \tag{18}$$

The second order exchange isomorphism of Y is defined by

$$i_Y^{2,A} = J^1 i_Y^{1,A} \circ i_{J^1 Y}^{1,A} : V^A \widetilde{J}^2 Y \to \widetilde{J}^2 V^A Y\,, \tag{19}$$

where $J^1 i_Y^{1,A} : J^1(V^A J^1 Y) \to \widetilde{J}^2 V^A Y$ is the first jet prolongation of $i_Y^{1,A}$ and $i_{J^1 Y}^{1,A} : V^A \widetilde{J}^2 Y \to J^1 V^A(J^1 Y)$ is the first order exchange isomorphism of $J^1 Y$.

Definition 2. The vertical A-prolongation of $\Delta : Y \to \widetilde{J}^2 Y$ is

$$\mathcal{V}^A\Delta := i_Y^{2,A} \circ V^A Y : V^A Y \to \widetilde{J}^2 V^A Y\,.$$

If η^p, η^p_i, η^p_{0i}, η^p_{ij} are the induced A-valued coordinates on $\widetilde{J}^2 V^A Y$, then the coordinate form of $\mathcal{V}^A\Delta$ is

$$\eta^p_i = V^A(F^p_i)\,,\ \eta^p_{0i} = V^A(G^p_i)\,,\ \eta^p_{ij} = V^A(H^p_{ij})\,. \tag{20}$$

The product $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^2 Y$ of two first order connections Γ, $\overline{\Gamma}$ on Y is defined by $\Gamma * \overline{\Gamma} := J^1\Gamma \circ \overline{\Gamma}$, where $J^1\Gamma : J^1 Y \to \widetilde{J}^2 Y$ is the first jet prolongation of the $\mathcal{FM}$-morphism $\Gamma : Y \to J^1 Y$.

Proposition 4. *We have* $\mathcal{V}^A(\Gamma * \overline{\Gamma}) = (\mathcal{V}^A\Gamma) * (\mathcal{V}^A\overline{\Gamma})$.

Proof. If we apply Lemma 2 to the morphism Γ, we obtain $(J^1V^A\Gamma) \circ i_Y^{1,A} = i_{J^1Y}^{1,A} \circ V^AJ^1\Gamma$. Hence $\mathcal{V}^A(\Gamma * \overline{\Gamma}) = i_Y^{2,A} \circ V^A(\Gamma * \overline{\Gamma}) = J^1 i_Y^{1,A} \circ i_{J^1Y}^{1,A} \circ V^AJ^1\Gamma \circ V^A\overline{\Gamma} = J^1 i_Y^{1,A} \circ J^1V^A\Gamma \circ i_Y^{1,A} \circ V^A\overline{\Gamma} = J^1\mathcal{V}^A\Gamma \circ \mathcal{V}^A\overline{\Gamma} = \mathcal{V}^A\Gamma * \mathcal{V}^A\overline{\Gamma}$. □

Remark 1. In general, an r-th order non-holonomic connection on Y is a section $\Gamma : Y \to \widetilde{J}^rY$, where the r-th order non-holonomic prolongation of Y is defined by the iteration $\widetilde{J}^rY = J^1(\widetilde{J}^{r-1}Y \to M)$. We outline how the previous results can be extended to such connections. First of all, we introduce $i_Y^{r,A} : V^A\widetilde{J}^rY \to \widetilde{J}^rV^AY$ by the induction

$$i_Y^{r,A} = J^1 i_Y^{r-1,A} \circ i_{\widetilde{J}^{r-1}Y}^{1,A} ,$$

where $i_Y^{r-1,A} : V^A\widetilde{J}^{r-1}Y \to \widetilde{J}^{r-1}V^AY$. Since $i_Y^{r,A}$ is a composition of natural transformations, it is a natural transformation too. By induction, one deduces

$$i_Y^{r,A} = \widetilde{J}^l i_Y^{k,A} \circ i_{\widetilde{J}^kY}^{l,Y} \tag{21}$$

for all k and l satisfying $k + l = r$. Using $i_Y^{r,A}$, we introduce

$$\mathcal{V}^A\Gamma = i_Y^{r,A} \circ V^A\Gamma : V^AY \to \widetilde{J}^rV^AY .$$

If $\overline{\Gamma} : Y \to \widetilde{J}^sY$ is another s-th order non-holonomic connection on Y, then the product $\Gamma * \overline{\Gamma}$ is defined by

$$\Gamma * \overline{\Gamma} = \widetilde{J}^s\Gamma \circ \overline{\Gamma} : Y \to \widetilde{J}^{r+s}Y ,$$

[17], [9]. Even in this case, we have

$$\mathcal{V}^A(\Gamma * \overline{\Gamma}) = (\mathcal{V}^A\Gamma) * (\mathcal{V}^A\overline{\Gamma}) .$$

Indeed, applying naturality $V^A\widetilde{J}^s \to \widetilde{J}^sV^A$ to Γ, we obtain $i_{\widetilde{J}^rY}^{s,A} \circ V^A\widetilde{J}^s\Gamma = \widetilde{J}^sV^A\Gamma \circ i_Y^{s,A}$. Hence $i_Y^{r+s,A} \circ V^A(\widetilde{J}^s\Gamma \circ \overline{\Gamma}) = \widetilde{J}^s i_Y^{r,A} \circ i_{\widetilde{J}^rY}^{s,A} \circ V^A\widetilde{J}^s\Gamma \circ V^A\overline{\Gamma} = \widetilde{J}^s i_Y^{r,A} \circ \widetilde{J}^sV^A\Gamma \circ i_Y^{s,A} \circ V^A\overline{\Gamma} = (\widetilde{J}^s\mathcal{V}^A\Gamma) \circ \mathcal{V}^A\overline{\Gamma}$.

5. The lifting map in the second order

The space of all non-holonomic 2-jets from a manifold M into another manifold N is defined as the second non-holonomic prolongation of the product fibered manifold $M \times N \to M$, [6]. It is well known that every $X \in \widetilde{J}_x^2(M,N)_y$ determines a map $\lambda X : TT_xM \to TT_yN$, where TT is the second iterated tangent functor, [15], [2]. We recall a direct geometric construction of λX.

Let $X = j_x^1\sigma(u)$, $\sigma : M \to J^1(M,N)$. We have $\sigma(u) = j_u^1\varrho(u,v)$, where ϱ is a local map $M \times N \to N$, so that

$$X = j_x^1 j_u^1 \varrho(u,v) . \tag{22}$$

Consider $D \in TT_xM$, $D = \frac{\partial}{\partial t}\big|_0 \gamma(t)$, $\gamma(t) = \frac{\partial}{\partial s}\big|_0 \delta(t,s)$, where $\gamma(t)$ is a tangent vector to M at $\delta(t,0)$. Then one defines

$$\lambda X(D) = \frac{\partial}{\partial t}\Big|_0 \frac{\partial}{\partial s}\Big|_0 \sigma(\delta(t,0), \delta(t,s)) \in TT_yN\,. \tag{23}$$

We shall write $\lambda(X,D)$ instead of $\lambda X(D)$ in the sequel.

If we have some local coordinates x^i and y^p on M and N, then y^p_i, y^p_{0i}, y^p_{ij} are the additional coordinates on $\widetilde{J}^2(M,N)$. Further, write $x^i_1 = dx^i$ in the first step and $x^i_2 = dx^i$, $x^i_3 = dx^i_1$ in the second step for the induced coordinates on TTM. Let y^p_1, y^p_2, y^p_3 be the coordinates induced on TTN in the same way.. Then the coordinate form of $\lambda(X,D)$ is, see [15], [2],

$$y^p_1 = y^p_i x^i_1\,, \;\; y^p_2 = y^p_{0i} x^i_2\,, \;\; y^p_3 = y^p_{ij} x^i_1 x^j_2 + y^p_i x^i_3\,. \tag{24}$$

In the case of $\widetilde{J}^2Y$, (23) induces a lifting map

$$\lambda_Y : \widetilde{J}^2Y \times_M TTM \to TTY\,, \tag{25}$$

which is an $\mathcal{FM}$-morphism over id_{TTM}, provided we consider $\widetilde{J}^2Y \times_M TTM$ as a fibered manifold over TTM. If we take V^AY instead of Y, we have

$$\lambda_{V^AY} : \widetilde{J}^2V^AY \times_M TTM \to TTV^AY\,.$$

We recall that the Weil algebra of TT is $\mathbb{D} \otimes \mathbb{D}$.

Proposition 5. *The following diagram commutes*

$$\begin{array}{ccc} V^A\widetilde{J}^2Y \times_M TTM & \xrightarrow{V^A\lambda_Y} & V^A(TTY \to TTM) \\ \Big\downarrow i^{2,A}_Y \quad \Big\| & & \Big\downarrow i^{A,\mathbb{D}\otimes\mathbb{D}}_{V,Y} \\ \widetilde{J}^2V^AY \times_M TTM & \xrightarrow{\lambda_{V^AY}} & TT(V^AY) \end{array} \tag{26}$$

Proof. If we consider $X \in V^A\widetilde{J}^2Y$ in the form $X = j^A j^1_x j^1_u \psi(\tau,u,v)$, then

$$i^{2,A}_Y(X) = J^1 i^{1,A}_Y(i^{1,A}_{J^1Y}(X)) = j^1_x j^1_u j^A \psi(\tau,u,v)\,. \tag{27}$$

Indeed, $i^{1,A}_{J^1Y}$ exchanges j^A and j^1_x and $J^1 i^{1,A}_Y$ exchanges j^A and j^1_u. Consider further $D = \frac{\partial}{\partial t}\big|_0 \frac{\partial}{\partial s}\big|_0 \delta(t,s) \in TT_xM$. Clockwise, we first obtain $j^A \frac{\partial}{\partial t}\big|_0 \frac{\partial}{\partial s}\big|_0 \psi(\tau, \delta(t,0), \delta(t,s))$ and then

$$\frac{\partial}{\partial t}\Big|_0 \frac{\partial}{\partial s}\Big|_0 j^A \psi(\tau, \delta(t,0), \delta(t,s))\,. \tag{28}$$

Counterclockwise, we first find $j^1_x j^1_u j^A \psi(\tau,u,v)$ and then (28). $\square$

The lifting map of $\Delta : Y \to \widetilde{J}^2Y$ is defined by

$$\lambda\Delta = \lambda_Y \circ (\Delta \times_M \mathrm{id}_{TTM}) : Y \times_M TTM \to TTY\,.$$

If we add $\mathcal{V}^A\Delta = i^{2,A}_Y \circ V^A\Delta$ to the left of (26), we obtain the following characterization of the lifting map of $\mathcal{V}^A\Delta$.

Corollary 2. *We have* $\lambda(\mathcal{V}^A\Delta) = i_{V,Y}^{A,\mathbb{D}\otimes\mathbb{D}} \circ V^A(\lambda\Delta)$. □

Consider a section $\xi : M \to TTM$, i.e. a $\mathbb{D}\otimes\mathbb{D}$-field on M. Its Δ-lift can be introduced by

$$\Delta(\xi) = \lambda_Y \circ (\Delta \times_M \xi)\,.$$

This is a projectable $\mathbb{D}\otimes\mathbb{D}$-field on Y, so that we have defined its vertical A-prolongation

$$\mathcal{V}^A(\Delta\xi) : V^AY \to TTV^AY\,.$$

On the other hand, we can construct the $\mathcal{V}^A\Delta$-lift $(\mathcal{V}^A\Delta)(\xi) : V^AY \to TTV^AY$. The following result demonstrates that the lifting of $\mathbb{D}\otimes\mathbb{D}$-fields in the theory of second order connections plays an analogous role to the lifting of vector fields in the theory of first order connections.

Proposition 6. *We have* $(\mathcal{V}^A\Delta)(\xi) = \mathcal{V}^A(\Delta\xi)$.

Proof. Add the map $V^A\Delta \times_M \xi$ to the left of the top row of (26). Clockwise we obtain $\mathcal{V}^A(\Delta\xi)$ and counterclockwise we find $(\mathcal{V}^A\Delta)(\xi)$. □

Consider another connection $\overline{\Delta} : Y \to \widetilde{J}^2Y$ satisfying

$$\beta_1\Delta = \beta_1\overline{\Delta}\,,\ J^1\beta(\Delta) = J^1\beta(\overline{\Delta})\,. \tag{29}$$

In [3] we deduced that there is a section $\Sigma : Y \to VY\otimes\overset{2}{\otimes}T^*M$ such that $\overline{\Delta} = \Delta+\Sigma$. Let

$$y_i^p = F_i^p(x,y)\,,\ y_{0i}^p = G_i^p(x,y)\,,\ y_{ij}^p = \overline{H}_{ij}^p(x,y) \tag{30}$$

be the coordinate expression of $\overline{\Delta}$. Then $\Sigma = (\overline{H}_{ij}^p - H_{ij}^p)$.

A $\mathbb{D}\otimes\mathbb{D}$-field $\xi : M \to TTM$ induces two vector fields $\xi_1 = \pi_{\mathbb{D},TM}\circ\xi$ and $\xi_2 = T\pi_{\mathbb{D},M}\circ\xi$ on M. If $\xi = (\xi_1^i, \xi_2^i, \xi_3^i)$ is the coordinate expression of ξ, then $\xi_1 = (\xi_1^i)$ and $\xi_2 = (\xi_2^i)$. By (24) we find the coordinate forms of the corresponding lifts

$$\Delta(\xi) = (F_i^p\xi_1^i, G_i^p\xi_2^i.H_{ij}^p\xi_1^i\xi_2^j + F_i^p\xi_3^i)\,,$$
$$\overline{\Delta}(\xi) = (F_i^p\xi_1^i, G_i^p\xi_2^i, \overline{H}_{ij}^p\xi_1^i\xi_2^j + F_i^p\xi_3^i)\,.$$

This implies

$$\overline{\Delta}(\xi) - \Delta(\xi) = \Sigma(\xi_1,\xi_2)\,. \tag{31}$$

Now we apply V^A. If we take into account the difference in notation on both sides, we obtain

$$V^A(\overline{\Delta}(\xi)) - V^A(\Delta(\xi)) = V_1^A\Sigma(\xi_1,\xi_2)\,. \tag{32}$$

Next we apply $i_{V,Y}^{A,\mathbb{D}}$ to both sides. Since Σ is vertical valued, $i_{V,Y}^{A,\mathbb{D}}$ is reduced to $i_{Y,V}^{A,\mathbb{D}}$ on the right hand side. Thus,

$$\mathcal{V}^A(\overline{\Delta}(\xi)) - \mathcal{V}^A(\Delta(\xi)) = (\mathcal{V}_1^A\Sigma)(\xi_1,\xi_2)\,.$$

By Proposition 6, we have $\mathcal{V}^A(\Delta\xi) = (\mathcal{V}^A\Delta)(\xi)$ and the same for $\overline{\Delta}$. Thus, we have proved

Proposition 7. *If (29) holds and* $\overline{\Delta} = \Delta + \Sigma$, *then*

$$\mathcal{V}^A \overline{\Delta} = \mathcal{V}^A \Delta + \mathcal{V}_1^A \Sigma .$$

In [3] we deduced that every $\Delta : Y \to \widetilde{J}^2 Y$ can be uniquely written in the form

$$\Delta = \Gamma * \overline{\Gamma} + \Sigma \tag{33}$$

with $\Gamma, \overline{\Gamma} : Y \to J^1 Y$ and $\Sigma : Y \to VY \otimes \overset{2}{\otimes} T^* M$. By Propositions 4 and 7, we obtain

Corollary 3. *If* (33) *holds, then*

$$\mathcal{V}^A \Delta = (\mathcal{V}^A \Gamma) * (\mathcal{V}^A \overline{\Gamma}) + \mathcal{V}_1^A \Sigma .$$

Remark 2. An r-th order non-holonomic connection $\Gamma : Y \to \widetilde{J}^r Y$ determines a lifting map in a similar way. Every non-holonomic r-jet $X \in \widetilde{J}^r_x(M, N)_y$ induces a map

$$\lambda X : T \dots T_x M \to T \dots T_y N ,$$

where $T \dots T$ is the r-times iterated tangent functor, [16]. This defines the lifting map of Γ

$$\lambda \Gamma : Y \times_M T \dots TM \to T \dots TY .$$

The Weil algebra of $T \dots T$ is $\mathbb{D} \otimes \dots \otimes \mathbb{D} = \overset{r}{\otimes} \mathbb{D}$. Hence every $\overset{r}{\otimes}\mathbb{D}$-field ξ on M is lifted into a $\overset{r}{\otimes}\mathbb{D}$-field $\Gamma\xi$ on Y. Several properties of this operation are analogous to the second order case, but we shall not go into details here.

6. The first order absolute differentiation

The absolute differentiation with respect to a first order connection $\Gamma : Y \to J^1 Y$ can be viewed as a map

$$\nabla_\Gamma : J^1 Y \to VY \otimes T^* M .$$

The simpliest way of defining it is we use the fact that $\beta : J^1 Y \to Y$ is an affine bundle with associated vector bundle $VY \otimes T^* M$ and we set

$$\nabla_\Gamma (X) = X - \Gamma(\beta X) , \qquad X \in J^1 Y , \tag{34}$$

[14]. However, this approach is somewhat formal and cannot be extended to higher orders.

The definition of higher order absolute differentiation by C. Ehresmann is of jet character, [7], [9]. That is why we find it useful to discuss also the first order case from such a point of view. We shall use the standard identification $T(Y_x)\otimes T^*_x M = J^1_x(M, Y_x)$. We introduce

$$\mathcal{J}^1 Y = \bigcup_{x\in M} J^1_x(M, Y_x)\,.$$

Hence (34) is an $\mathcal{FM}$-morphism $J^1Y \to \mathcal{J}^1 Y$. From the jet viewpoint it is more convenient to construct the inverse map $\nabla_\Gamma^{-1} : \mathcal{J}^1 Y \to J^1 Y$. We have

$$\Gamma(y) = j^1_x \psi(u, y)\,,\ \psi(x, y) = y\,, \qquad u \in M, y \in Y_x\,.$$

If (7) are the equations of Γ, then

$$\frac{\partial \psi^p(x, y)}{\partial x^i} = F^p_i(x, y)\,.$$

For $Z = j^1_x \varphi(u) \in \mathcal{J}^1_x Y$ we define

$$\nabla_\Gamma^{-1}(Z) = j^1_x \psi(u, \varphi(u)) \in J^1_x Y\,. \tag{35}$$

Write z^p_i for the induced coordinates in $\mathcal{J}^1 Y$. Evaluating (35), we obtain

$$y^p_i = F^p_i(x, y) + z^p_i\,.$$

This is the coordinate form of (34).

Consider $W = j^A j^1_x \varphi(\tau, u) \in V^A \mathcal{J}^1 Y$, $\varphi : \mathbb{R}^k \times M \to Y_x$. We define an exchange isomorphism $I^{1,A}_Y : V^A \mathcal{J}^1 Y \to \mathcal{J}^1 V^A Y$ by

$$I^{1,A}_Y(W) = j^1_x j^A \varphi(\tau, u)\,. \tag{36}$$

In the case of $\mathcal{V}^A\Gamma$, we have $\nabla^{-1}_{\mathcal{V}^A\Gamma} : \mathcal{J}^1 V^A Y \to J^1 V^A Y$.

Proposition 8. *The following diagram commutes*

$$\begin{tikzcd}
V^A \mathcal{J}^1 Y \arrow[r, "V^A(\nabla_\Gamma^{-1})"] \arrow[d, "I^{1,A}_Y"'] & V^A J^1 Y \arrow[d, "i^{1,A}_Y"] \\
\mathcal{J}^1 V^A Y \arrow[r, "\nabla^{-1}_{\mathcal{V}^A\Gamma}"] & J^1 V^A Y
\end{tikzcd}$$

Proof. By the definition of $\mathcal{V}^A\Gamma$, we have

$$\mathcal{V}^A\Gamma(j^A \varrho(\tau)) = j^A_x j^A \psi(u, \varrho(\tau))\,, \qquad j^A \varrho(\tau) \in V^A_x Y\,.$$

Hence $\nabla^{-1}_{\mathcal{V}^A\Gamma}(I^{1,A}_Y(W)) = j^1_x j^A \psi(u, \varphi(\tau, u)) = i^{1,A}_Y(V^A(\nabla_\Gamma^{-1})(W))$. □

Remark 3. It is instructive to characterize the absolute differentiation with respect to Γ from the groupoidal point of view, [7], [9]. Assume Y is locally trivial and denote by GY the groupoid of all diffeomorphisms between the individual fibers of Y. This is a smooth space in the sense of A. Frölicher. If $f : Y_x \to Y_u$ is a diffeomorphism, then $T^A f : V_x^A Y \to V_u^A Y$ is also a diffeomorphism. In this sense the groupoid GY acts smoothly on $V^A Y$ as well. To define a first order element of connection C on Y at $x \in M$ in the sense of C. Ehresmann, [7], [9], we have to consider locally a smooth family of diffeomorphisms $\Theta(u) : Y_x \to Y_u$, $u \in M$, and define

$$C(y) = j_x^1[\Theta(u)(y)]\,, \qquad y \in Y_x\,.$$

The corresponding map $\nabla_C^{-1} : \mathcal{J}_x^1 Y \to J_x^1 Y$ is of the form

$$\nabla_C^{-1}(j_x^1 \varphi(u)) = j_x^1 \Theta(u)(\varphi(u))\,, \qquad \varphi : M \to Y_x\,.$$

If we replace $\Theta(u)$ by $T^A \Theta(u)$, we obtain another construction of the connection $\mathcal{V}^A \Gamma$ as well as another approach to the absolute differentiation on $V^A Y$ with respect to $\mathcal{V}^A \Gamma$.

7. The second order absolute differentiation

In the second order, we define

$$\widetilde{\mathcal{J}}^2 Y = \bigcup_{x \in M} \widetilde{J}_x^2(M, Y_x)\,.$$

The absolute differentiation with respect to $\Delta : Y \to \widetilde{J}^2 Y$ is a map

$$\nabla_\Delta : \widetilde{J}^2 Y \to \widetilde{\mathcal{J}}^2 Y\,,$$

whose inverse map can be introduced as follows. We have

$$\Delta(y) = j_x^1 j_u^1 \psi(u, v, y)\,, \qquad u, v \in M, y \in Y_x\,.$$

Consider $Z \in \widetilde{J}_x^2(M, Y_x)$ of the form

$$Z = j_x^1 j_u^1 \varphi(u, v)\,, \qquad \varphi : M \times M \to Y_x\,.$$

Then we define

$$\nabla_\Delta^{-1}(Z) = j_x^1 j_u^1 \psi(u, v, \varphi(u, v)) \in \widetilde{J}_x^2 Y\,. \tag{37}$$

If (18) is the coordinate expression of Δ and z_{0i}^p, z_{ij}^p are the additional coordinates on $\widetilde{\mathcal{J}}^2 Y$, then the evaluation of (37) yields

$$y_i^p = F_i^p + z_i^p\,,\ y_{0i}^p = G_i^p + z_{0i}^p\,,\ y_{ij}^p = H_{ij}^p + \frac{\partial F_i^p}{\partial y^q} z_{0j}^q + \frac{\partial G_j^p}{\partial y^q} z_i^q + z_{ij}^p\,. \tag{38}$$

These formulae coincide with (19) from [3]. This clarifies that (37) is equivalent to the approach from [3].

Consider $W = j^A j_x^1 j_u^1 \varphi(\tau, u, v) \in V^A \widetilde{J}^2 Y, \varphi : \mathbb{R}^k \times M \times M \to Y_x$. We define an exchange isomorphism $I_Y^{2,A} : V^A \widetilde{\mathcal{J}}^2 Y \to \widetilde{\mathcal{J}}^2 V^A Y$ by

$$I_Y^{2,A}(W) = j_x^1 j_u^1 j^A \varphi(\tau, u, v)\,. \tag{39}$$

The construction of $\nabla_{\mathcal{V}^A \Delta}$ from ∇_Δ is described in the following assertion.

Proposition 9. *The following diagram commutes*

$$\begin{tikzcd} V^A\widetilde{J}^2Y \arrow[r, "V^A(\nabla_\Delta^{-1})"] \arrow[d, "I_Y^{2,A}"'] & V^A\widetilde{J}^2Y \arrow[d, "i_Y^{2,A}"] \\ \widetilde{J}^2V^AY \arrow[r, "\nabla^{-1}_{\mathcal{V}^A\Delta}"] & \widetilde{J}^2V^AY \end{tikzcd}$$

Proof. By the definition of $\mathcal{V}^A\Delta$ and by (27), we have

$$\mathcal{V}^A\Delta(j^A\varrho(\tau)) = j^1_x j^1_u j^A\psi(u,v,\varrho(\tau)), \qquad j^A\varrho(\tau)\in V^A_xY\,.$$

Hence $\nabla^{-1}_{\mathcal{V}^A\Delta}(I_Y^{2,A}(W)) = j^1_x j^1_u j^A\psi(u,v,\varphi(\tau,u,v)) = i_Y^{2,A}(V^A(\nabla_\Delta^{-1})(W))$. □

Remark 4. The groupoidal approach from Remark 3 can be applied to the second order absolute differentiation as well.

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Antonella Cabras
Dipartimento di Matematica Applicata "G. Sansone"
Via S. Marta 3
50139 Firenze, ITALY
E-mail: cabras@dma.unifi.it

Ivan Kolář
Department of Algebra and Geometry
Faculty of Science, Masaryk University
Janáčkovo nám. 2a
662 95 Brno, CZECH REPUBLIC
E-mail: kolar@math.muni.cz