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ON CONVERGENCE OF QUADRATURE-DIFFERENCES METHOD FOR LINEAR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS ON THE INTERVAL

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Abstract. Here we propose and justify the quadrature-differences method for the full linear singular integro-differential equations with Cauchy kernel on the interval $(-1,1)$. We consider equations of zero, positive and negative indices. It is shown, that the method converges to exact solution and the error estimate depends on the sharpness of derivative approximation and the smoothness of the coefficients and the right-hand side of the equation.

Introduction

In the papers [4] - [7] the quadrature-differences methods for the various classes of the periodic singular integro-differential equations with Hilbert kernels were justified. The convergence of the methods was proved and the errors estimates were obtained. Here we propose and justify the same method for the full linear singular integro-differential equations with Cauchy kernel on the interval $(-1,1)$.

It is known (see e.g. [3], [8], [15]), that the theories of the singular integral equations in periodic (with Hilbert kernel) and non-periodic (with Cauchy kernel) cases differ a lot due to the discontinuity of the contour in the latter case. Therefore the calculation schemes and the justifications of the method in this cases have essential distinctions. Thus, if for the equations with Hilbert kernels the same uniform grid is used both for the approximation of the derivatives and integrals and as collocation nodes, then for the equations with Cauchy kernel we have to use two different grids - the roots of the special polynomials. For the first class of the equations the problem is stated in Hölder space and therefore the usual technique of the compact approximation [18] for the justification is used and the rate of convergence grows up with the growing of the smoothness of the coefficients and the right-hand side of the equation infinitely. For the second class of the equations the derivative of the desired function has in general integrable singularities at the end points of the contour, therefore the problem is stated in the spaces of weighted

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quadratically integrable functions, so called "second kind" [18] of the theory of the approximation methods is used and the rate of convergence is restricted by the order of smoothness of the desired function, coefficients and the right-hand side of the equation.

In the paper equations of zero, positive and negative indices are considered, the covergence of the method is proved and the rate of convergence is obtained.

1. Formulation of the problem

Consider linear singular integro-differential equation of the form

$$\sum_{\nu=0}^{1}(a_\nu(t)x^{(\nu)}(t)+b_\nu(t)(Sx^{(\nu)})(t)+(Th_\nu x^{(\nu)})(t))=f(t)\,,\quad -1<t<1\,, \tag{1}$$

with initial condition

$$x(\xi_0)=0,\qquad -1\le\xi_0\le 1, \tag{2}$$

where $x(t)$ is desired unknown and $a_\nu(t),b_\nu(t),h_\nu(t,\tau),\ \nu=0,1,\ f(t)$ are given continuous functions of their arguments $t,\tau\in[-1,1]$, $b_1(t)$ is a polynomial of some order $n_0\ge 0$, singular integrals

$$(Sx^{(\nu)})(t)=\frac{1}{\pi}\int_{-1}^{1}\frac{x^{(\nu)}(\tau)d\tau}{\tau-t}\,,\qquad \nu=0,1\,,$$

are to be interpreted as the Cauchy-Lebesgues principal value and

$$(Th_\nu x^{(\nu)})(t)=\frac{1}{\pi}\int_{-1}^{1}h_\nu(t,\tau)x^{(\nu)}(\tau)d\tau\,,\quad \nu=0,1\,,$$

are regular integrals.

First we consider in details zero index equation ($\kappa=0$) and then point out the changes in the calculation scheme and justification for the cases of positive ($\kappa>0$) and negative ($\kappa<0$) indices.

2. Calculation scheme

Let's define, following Muskhelishvili [15], the index and the canonical function of the equation (1). To do this denote by $\theta(t)=\pi^{-1}\arg(a_1(t)+ib_1(t))$, $t\in[-1,1]$, some continuous and one-valued branch of the multi-valued function $\pi^{-1}\arg(a_1(t)+ib_1(t))$. Then the canonical function of the equation (1) will be

$$Z(t)=(1-t)^{\gamma_1}(1+t)^{\gamma_2}\exp(-\int_{-1}^{1}\frac{\theta(\tau)d\tau}{\tau-t})\,,\quad t\in(-1,1)\,,$$

where $\gamma_1=\lambda_1-\theta(1)$, $\gamma_2=\lambda_2+\theta(-1)$, and λ_1,λ_2 are the integers subordinate to the conditions $\gamma_1,\gamma_2\in(-1,1)$. The integer $\kappa=-(\lambda_1+\lambda_2)$ is called the index of the equation (1) and the numbers γ_1 and γ_2 determine the class of possible solutions of the problem (1), (2) (see [15], [8]).

Now we will define two weight-functions

$$\rho(t)=Z(t)(a_1^2(t)+b_1^2(t))^{-1/2}$$

and

$$\bar{\rho}(t) = Z^{-1}(t)(a_1^2(t) + b_1^2(t))^{-1/2}, \quad (a_1^2(t) + b_1^2(t))^{1/2} > 0\,,$$

and two sequences of polynomials $\{\phi_n(t)\}_{n=0}^{\infty}$ and $\{\psi_n(t)\}_{n=0}^{\infty}$ with the following properties:

$$\tag{3} \begin{aligned} &\int_{-1}^{1} \rho(\tau)\phi_k(\tau)\phi_l(\tau)d\tau = \sigma_k\delta_{k,l}\,,\ k \geq l\,, \\ &\int_{-1}^{1} \bar{\rho}(\tau)\psi_k(\tau)\psi_l(\tau)d\tau = \zeta_k\delta_{k,l}\,,\ k \geq l\,, \end{aligned}$$

$$\tag{4} a_1(t)\rho(t)\phi_{n+1}(t) + b_1(t)(S\rho\phi_{n+1})(t) = (-1)^{\kappa}\frac{\sigma_{n+1}\beta_{n+1-\kappa}}{\zeta_{n+1-\kappa}\alpha_{n+1}}\psi_{n+1-\kappa}(t)\,,$$

where $\alpha_{n+1} > 0$ and $\beta_{n+1-\kappa} > 0$ are the senior coefficients of the polynomials $\phi_{n+1}(t)$ and $\psi_{n+1-\kappa}(t)$ correspondently, and $\delta_{k,l}$ is Kronecker symbol. The existence of the polynomials satisfying (3), due to the positiveness and integrability of the weight-functions $\rho(t)$ and $\bar{\rho}(t)$, was shown in [17]. Moreover, it was shown there that each of the polynomials $\{\phi_n(t)\}_{n=0}^{\infty}$, $\{\psi_n(t)\}_{n=0}^{\infty}$ has just n real simple roots on the interval $(-1,1)$. Identity (4), which plays the crucial role in the following account, was obtained by Elliott [3].

Let

$$\tag{5} \{\tau_k \mid \phi_{n+1}(\tau_k) = 0\,,\ k = 0, 1, \ldots, n\}\,,$$

$$\tag{6} \{t_j \mid \psi_{n+1-\kappa}(t_j) = 0\,,\ j = 0, 1, \ldots, n-\kappa\}\,,$$

be the grids on $[-1, 1]$. By

$$\tag{7} \{\tau_k \mid k = -1, 0, \ldots, n\}\,.$$

we denote the grid (5) with the node $\tau_{-1} = -1$ added.

We'll seek an approximate solution of the equation (1) as a vector

$$\tag{8} \mathbf{x}_n = (x_{-1}, \ldots, x_n)$$

of values of unknown function in the nodes of the grid (7). Derivatives and values of the unknown function in the nodes of the grids (5), (6) and for the initial condition in the point ξ_0 we'll approximate by any numerical formulas

$$\begin{aligned} x'(\tau_k) &\sim [D_n^{(1)}\mathbf{x}_n]_{\tau_k}\,, && k = 0, 1, \ldots, n\,, \\ x(t_j) &\sim [D_n^{(0)}\mathbf{x}_n]_{t_j}\,, && j = 0, 1, \ldots, n-\kappa\,, \\ x(\xi_0) &\sim [D_n^{(0)}\mathbf{x}_n]_{\xi_0}\,. \end{aligned}$$

which use only the nodes (7) and the components of the vector (8).

Singular integral $(Sx)(t)$ is to be approximated by the quadrature. To do this we'll integrate polynomial

$$(Q_{n-\kappa}D_n^{(0)}\mathbf{x}_n)(\tau) = \sum_{j=0}^{n-\kappa}[D_n^{(0)}\mathbf{x}_n]_{t_j}l_j(\tau)\,,$$

$$l_j(\tau)=\frac{\psi_{n+1-\kappa}(\tau)}{(\tau-t_j)\psi'_{n+1-\kappa}(t_j)}\,,\quad j=0,1,\dots,n-\kappa\,,$$

$$(SQ_{n-\kappa}D_n^{(0)}\mathbf{x}_n)(t)=\sum_{j=0}^{n-\kappa}[D_n^{(0)}\mathbf{x}_n]_{t_j}(Sl_j)(t)\,,\tag{9}$$

$$(Sl_j)(t)=\frac{(S\psi_{n+1-\kappa})(t)-(S\psi_{n+1-\kappa})(t_j)}{(t-t_j)\psi'_{n+1-\kappa}(t_j)}\,,j=0,1,\dots,n-\kappa\,.$$

To approximate regular integral $(Th_0x)(t)$ we'll integrate polynomial

$$(Q_{n-\kappa}h_0D_n^{(0)}\mathbf{x}_n)(t,\tau)=\sum_{j=0}^{n-\kappa}[D_n^{(0)}\mathbf{x}_n]_{t_j}h_0(t,t_j)l_j(\tau)\,,$$

$$(TQ_{n-\kappa}h_0D_n^{(0)}\mathbf{x}_n)(t)=\sum_{j=0}^{n-\kappa}[D_n^{(0)}\mathbf{x}_n]_{t_j}h_0(t,t_j)Tl_j\,,\tag{10}$$

$$Tl_j=(S\psi_{n+1-\kappa})(t_j)/\psi'_{n+1-\kappa}(t_j)\,,\quad j=0,1,\dots,n-\kappa\,.$$

Coefficients of the quadrature formulas (9), (10) depend on the integrals $(S\psi_{n+1-\kappa})(t)$, which, according to the relations [1]

$$(S1)(t)=\frac{1}{\pi}\ln\left|\frac{1-t}{1+t}\right|\,,$$

$$(S\tau^k)(t)=\frac{t^k}{\pi}\ln\left|\frac{1-t}{1+t}\right|+\frac{2}{\pi}\sum_{j=0}^{\left[\frac{k-1}{2}\right]}\frac{t^{k-(2j+1)}}{2j+1}\,,\quad k=1,2,\dots,$$

could be calculated explicitly for all fixed n.

To approximate the dominant part of the equation (1)

$$(Ux')(t)=a_1(t)x'(t)+b_1(t)(Sx')(t)$$

we'll applicate the operator U to the polynomial

$$(P_n\rho^{-1}D_n^{(1)}\mathbf{x}_n)(\tau)=\sum_{k=0}^{n}\rho^{-1}(\tau_k)[D_n^{(1)}\mathbf{x}_n]_{\tau_k}\bar{l}_k(\tau)\,,$$

$$\bar{l}_k(\tau)=\frac{\phi_{n+1}(\tau)}{(\tau-\tau_k)\phi'_{n+1}(\tau_k)}\,,\quad k=0,1,\dots,n\,,$$

multiplied to the weight-function $\rho(\tau)$,

$$(U\rho P_n\rho^{-1}D_n^{(1)}\mathbf{x}_n)(t)=\sum_{k=0}^{n}\rho^{-1}(\tau_k)[D_n^{(1)}\mathbf{x}_n]_{\tau_k}(U\rho\bar{l}_k)(t)\,,\tag{11}$$

where, using (4),

$$(U\rho\bar{l}_k)(t)=(-1)^\kappa\frac{\sigma_{n+1}\beta_{n+1-\kappa}(\psi_{n+1-\kappa}(t)-\psi_{n+1-\kappa}(\tau_k))}{\zeta_{n+1-\kappa}\alpha_{n+1}(t-\tau_k)\phi'_{n+1}(\tau_k)}\,,\tag{12}$$

[1] $[\frac{k-1}{2}]$ denotes the largest integer not exceeding $\frac{k-1}{2}$.

for $k = 0, 1, \dots, n$. To approximate the regular integral $(Th_1 x')(t)$ we'll integrate the polynomial

$$(P_n \rho^{-1} h_1 D_n^{(1)} \mathbf{x}_n)(t, \tau) = \sum_{k=0}^{n} \rho^{-1}(\tau_k)[D_n^{(1)} \mathbf{x}_n]_{\tau_k} h_1(t, \tau_k) \bar{l}_k(\tau)\,,$$

also multiplied to weight-function $\rho(\tau)$,

$$(T\rho P_n \rho^{-1} h_1 D_n^{(1)} \mathbf{x}_n)(t) = \sum_{k=0}^{n} \rho^{-1}(\tau_k)[D_n^{(1)} \mathbf{x}_n]_{\tau_k} h_1(t, \tau_k) T\rho \bar{l}_k\,, \tag{13}$$

where $T\rho\bar{l}_k$, $k = 0, 1, \dots, n$, are coefficients of the Gauss type quadrature formula and for τ_k, which are not the roots of the polynomial $b_1(t)$, the following relationship is valid:

$$T\rho\bar{l}_k = (-1)^{\kappa} \frac{\sigma_{n+1}\beta_{n+1-\kappa}\psi_{n+1-\kappa}(\tau_k)}{\zeta_{n+1-\kappa}\alpha_{n+1} b_1(\tau_k)\phi'_{n+1}(\tau_k)}\,.$$

Substituting the numerical derivative formulas, the values of the quadratures (9)-(11),(13) and the right-hand side in the nodes of the grid (6) in the equation (1) and the numerical formula for the point ξ_0 in the initial condition (2), we'll obtain the system of linear algebraic equations

$$\begin{aligned}
&\sum_{k=0}^{n} \rho^{-1}(\tau_k)[D^{(1)} \mathbf{x}_n]_{\tau_k} (U\rho\bar{l}_k)(t_m) + a_0(t_m)[D_n^{(0)} \mathbf{x}_n]_{t_m} \\
&\quad + b_0(t_m) \sum_{j=0}^{n-\kappa} [D_n^{(0)} \mathbf{x}_n]_{t_m} (Sl_j)(t_m) + \sum_{j=0}^{n-\kappa} [D_n^{(0)} \mathbf{x}_n]_{t_m} h_0(t_m, t_j) Tl_j \\
&\quad + \sum_{k=0}^{n} \rho^{-1}(\tau_k)[D_n^{(1)} \mathbf{x}_n]_{\tau_k} h_1(t_m, \tau_k) T\rho\bar{l}_k = f(t_m)\,,
\end{aligned} \tag{14}$$

for $m = 0, 1, \dots, n - \kappa$,

$$[D_n^{(0)} \mathbf{x}_n]_{\xi_0} = 0\,, \tag{15}$$

of the quadrature-differences method.

3. Justification

Let's denote by $W^1_{2,\rho}$ ($W^0_{2,\rho} = L_{2,\rho}$) the set of absolutely continuous on $[-1, 1]$ functions, which have quadratically integrable with the weight-function $\rho(\tau)$ derivatives and let's define the following couples of spaces X, X_n; Y, Y_n; $Z, Z_{n-\kappa}$:

$$X = \{x \in W^1_{2,\rho^{-1}} \mid x(\xi_0) = 0\}\,,\ Y = L_{2,\rho^{-1}}\,,\ Z = L_{2,\bar{\rho}}\,,$$

with the norms

$$\|x\|_X = \left(\int_{-1}^{1} \rho^{-1}(\tau) |x'(\tau)|^2 \, d\tau \right)^{1/2}, \quad x \in X\,,$$

$$\|y\|_Y = \left(\int_{-1}^{1} \rho^{-1}(\tau)|y(\tau)|^2 \, d\tau \right)^{1/2}, \quad y \in Y ,$$

$$\|z\|_Z = \left(\int_{-1}^{1} \bar{\rho}(\tau)|z(\tau)|^2 \, d\tau \right)^{1/2}, \quad z \in Z ;$$

$X_n = \{\mathbf{x}_n\}$ - the set of $n+2$-components vectors of the form (8) satisfying to the condition

$$[D_n^{(0)}\mathbf{x}_n]_{\xi_0} = 0 ,$$

$Y_n = \{\mathbf{y}_n\}$ - the set of $n+1$-components vectors, $Z_{n-\kappa} = \{\mathbf{z}_{n-\kappa}\}$ - the set of $n+1-\kappa$-components vectors with the norms

$$\|\mathbf{x}_n\|_{X_n} = \left(\int_{-1}^{1} \rho(\tau)|(P_n \rho^{-1} \bar{D}_n^{(1)} \mathbf{x}_n)(\tau)|^2 \, d\tau \right)^{1/2}, \quad \mathbf{x}_n \in X_n ,$$

$$\|\mathbf{y}_n\|_{Y_n} = \left(\int_{-1}^{1} \rho(\tau)|(P_n \rho^{-1} \mathbf{y}_n)(\tau)|^2 \, d\tau \right)^{1/2}, \quad \mathbf{y}_n \in Y_n ,$$

$$\|\mathbf{z}_{n-\kappa}\|_{Z_{n-\kappa}} = \left(\int_{-1}^{1} \bar{\rho}(\tau)|(Q_{n-\kappa} \mathbf{z}_{n-\kappa})(\tau)|^2 \, d\tau \right)^{1/2}, \quad \mathbf{z}_{n-\kappa} \in Z_{n-\kappa} ,$$

where operator $\bar{D}_n^{(1)} : X_n \to Y_n$ is defined by the formula

$$[\bar{D}_n^{(1)}\mathbf{x}_n]_{\tau_k} = \frac{x_k - x_{k-1}}{\tau_k - \tau_{k-1}}, \quad k = 0, 1, \ldots, n .$$

Here and further the product of vector and function is equal to vector, components of which are the products of vector components and function values in the same nodes and

$$(P_n \mathbf{y}_n)(\tau) = \sum_{k=0}^{n} [\mathbf{y}_n]_{\tau_k} \bar{l}_k(\tau) , \; \bar{l}_k(\tau) = \frac{\phi_{n+1}(\tau)}{(\tau - \tau_k)\phi'_{n+1}(\tau_k)}, \quad k = 0, 1, \ldots, n ,$$

$$(Q_{n-\kappa} \mathbf{z}_{n-\kappa})(t) = \sum_{j=0}^{n-\kappa} [\mathbf{z}_{n-\kappa}]_{t_j} l_j(t) , \; l_j(t) = \frac{\psi_{n+1-\kappa}(t)}{(t - t_j)\psi'_{n+1-\kappa}(t_j)} , \; j = 0, 1, \ldots, n-\kappa ,$$

are Lagrange interpolative operators. We'll need also the following operators

$$p_n^1 : \; X \to X_n , \qquad p_n^1 x = (x(\tau_{-1}), x(\tau_0), \ldots, x(\tau_n)) ,$$

$$p_n : \; Y \to Y_n , \qquad p_n y = (y(\tau_0), y(\tau_1), \ldots, y(\tau_n)) ,$$

$$q_{n-\kappa} : \; Z \to Z_{n-\kappa} , \quad q_{n-\kappa} z = (z(t_0), z(t_1), \ldots, z(t_{n-\kappa})) .$$

Theorem 1. *Let for $\kappa = 0$ the problem* (1), (2) *and the calculation scheme* (5) - (15) *of the method satisfy the following conditions:*

A.1) *functions $a_\nu(t)$, $b_\nu(t)$, $h_\nu(t, \tau)$, $\nu = 0, 1$, and $f(t)$ by $t, \tau \in [-1, 1]$ belong to Hölder space H_μ, $0 < \mu \leq 1$;*

A.2) *$a_1^2(t) + b_1^2(t) \neq 0$ on $[-1, 1]$;*

A.3) *$b_1(t)$ is a polynomial of some order $n_0 \geq 0$;*

A.4) *$b_0(\pm) = 0$;*

A.5) *the problem* (1), (2) *has a unique solution* $x^*(t)$ *for any right-hand side* $f(t) \in Z$*;*

B.1) *all formulas of numerical differentiation used in calculation scheme* (5)-(15) *converge to the exact values of the approximated functions and their derivatives in the nodes of corresponding grids;*

B.2) *discrete Cauchy problem*

$$[D_n^{(1)}\mathbf{x}_n]_{\tau_k} = [\mathbf{y}_n]_{\tau_k}\,,\ k = 0,1,\ldots,n\,,\ [\mathbf{x}_n]_{\tau_{-1}} = 0\,,$$

has a unique solution $\mathbf{x}_n$ *for any* $\mathbf{y}_n \in Y_n$ *and* [2]

$$\|\mathbf{x}_n\|_{X_n} \le C\|\mathbf{y}_n\|_{Y_n}\,.$$

Then for n large enough the system of equations (14), (15) *is uniquely solvable and approximate solutions* $\mathbf{x}_n^*$ *converge to the exact solution* $x^*(\tau) \in X$ *of the problem* (1), (2) *with the error estimate*

$$\|\mathbf{x}_n^* - p_n^1 x^*\|_{X_n} \le C(n^{-\gamma} + \varepsilon_n(x^*))\,,$$

where

$$\gamma = \min\{\mu, 1+\gamma_1, 1+\gamma_2\}\,,$$

$$\varepsilon_n(x^*) = \max\Big\{ \max_{\sigma\in\{t_j\}_{j=0}^{n-\kappa}\cup\{\xi_0\}} |[D_n^{(0)}p_n^1 x^*]_\sigma - x^*(\sigma)|,$$

$$\max_{\sigma\in\{\tau_k\}_{k=0}^{n}} |\rho^{-1}(\sigma)[D_n^{(1)}p_n^1 x^*]_\sigma - \rho^{-1}(\sigma)x^{*'}(\sigma)|\Big\}\,.$$

Proof. It's known (see e.g. [3], [8], [15]) that if the right-hand side of the equation (1) belongs to H_μ or $L_{2,\bar\rho}$, the derivative of the solution of the problem (1), (2) has the form $x^{*'}(t) = \rho(t)\omega(t)$, where $\omega(t) \in H_\mu$ or $\omega(t) \in L_{2,\rho}$ correspondently, i.e. $x^*(t) \in W^1_{2,\rho^{-1}}$. Thus we'll consider the problem (1), (2) as operator equation

$$Kx \equiv UD^{(1)}x + Vx = f\,,\ K : X \to Z\,, \tag{16}$$

where

$$Uy = a_1 y + b_1 Sy\,,\ U : Y \to Z\,,\ Vx = \sum_{\nu=0}^{1} A_\nu D^{(\nu)}x\,,\ V : X \to Z\,,$$

$$A_0 D^{(0)}x = a_0 D^{(0)}x + b_0 S D^{(0)}x + Th_0 D^{(0)}x\,,$$

$$A_1 D^{(1)}x = Th_1 D^{(1)}x\,,\ D^{(1)}x = x'\,,\ D^{(0)}x = x\,.$$

Here, as it was shown in [3], [8], [15] and [9], $K : X \to Z$ is a linear bounded operator, $V : X \to Z$ is a compact operator and $U : Y \to Z$ is continuously invertible.

Let's consider η an arbitrary constant, which is not an eigenvalue of the problem

$$D^{(1)}x + \eta\rho x = 0\,,\ x(\xi_0) = 0\,,$$

and make a substitution

$$z = U(D^{(1)}x + \eta\rho x) \tag{17}$$

[2]Here and further C denotes generic constants, independent from n.

in the equation (16). Due to invertibility of the operator $U : Y \to Z$

$$x = GU^{-1}z\ ,\ D^{(1)}x = U^{-1}z - \eta\rho GU^{-1}z\ , \tag{18}$$

where $G : Y \to X$ is the inverse to

$$Fx = D^{(1)}x + \eta\rho x\ ,\ F : X \to Y\ ,$$

the equation (16) will take the form

$$Bz \equiv z + VGU^{-1}z - \eta U\rho\, GU^{-1}z = f\ ,\ B : Z \to Z\ , \tag{19}$$

being still equivalent to the original one. The equivalence here means, that solvability of one of them yields solvability of another and their solutions are joined by the relationships (17), (18).

Now let's rewrite the system of equations (14), (15) as an operator equation

$$K_{n-\kappa}\mathbf{x}_n \equiv U_{n-\kappa}D_n^{(1)}\mathbf{x}_n + V_{n-\kappa}\mathbf{x}_n = \mathbf{f}_{n-\kappa}\ ,\ K_{n-\kappa} : X_n \to Z_{n-\kappa}\ , \tag{20}$$

where

$$\begin{aligned}
U_{n-\kappa}\mathbf{y}_n &= q_{n-\kappa}U\rho P_n\rho^{-1}\mathbf{y}_n\ ,\ U_{n-\kappa} : Y_n \to Z_{n-\kappa}\ ,\\
V_{n-\kappa}\mathbf{x}_n &= q_{n-\kappa}\sum_{\nu=0}^{1} A_{\nu n}D_n^{(\nu)}\mathbf{x}_n\ ,\ V_{n-\kappa} : X_n \to Z_{n-\kappa}\ ,\\
A_{0n}D_n^{(0)}\mathbf{x}_n &= a_0Q_{n-\kappa}D_n^{(0)}\mathbf{x}_n + b_0SQ_{n-\kappa}D_n^{(0)}\mathbf{x}_n + TQ_{n-\kappa}h_0D_n^{(0)}\mathbf{x}_n\ ,\\
A_{1n}D_n^{(1)}\mathbf{x}_n &= T\rho P_n\rho^{-1}h_1D_n^{(1)}\mathbf{x}_n\ , \qquad \mathbf{f}_{n-\kappa} = q_{n-\kappa}f\ ,
\end{aligned}$$

and make a substitution

$$\mathbf{z}_{n-\kappa} = U_{n-\kappa}F_n\mathbf{x}_n\ , \tag{21}$$

where

$$[F_n\mathbf{x}_n]_{\tau_k} = [D_n^{(1)}\mathbf{x}_n]_{\tau_k} + \eta\rho(\tau_k)[\mathbf{x}_n]_{\tau_k}\ ,\ k = 0, 1, \ldots, n,\ F_n : X_n \to Y_n\ .$$

The operator $U_{n-\kappa} : Y_n \to Z_{n-\kappa}$ is invertible explicitly for all n, beginning from some $n_1,\ n_1 \geq \max\{n_0, \kappa\}$ (see [3]) and

$$\mathbf{x}_n = G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa},\ D_n^{(1)}\mathbf{x}_n = U_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} - \eta\rho\, G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa}\ ,$$

where $G_n :\ Y_n \to X_n$ is the inverse to F_n. The invertibility of F_n for all n beginning from some $n_2,\ n_2 \geq n_1$ follows from the conditions B.1), B.2) of the Theorem 1 and the choice of η (see [18]). Moreover, for any $y(t) = \rho(t)\omega(t),\ \omega(t) \in H_\mu$

$$\|p_n^1Gy - G_np_ny\|_{X_n} \leq C\ \varepsilon_n(Gy)\ . \tag{22}$$

So by the substitution (21) we'll get an equation

$$\begin{aligned}
B_{n-\kappa}\mathbf{z}_{n-\kappa} &\equiv \mathbf{z}_{n-\kappa} + V_{n-\kappa}G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} - \eta U_{n-\kappa}\rho\, G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa}\\
&= \mathbf{f}_{n-\kappa}\ , \quad B_{n-\kappa} : Z_{n-\kappa} \to Z_{n-\kappa}\ .
\end{aligned} \tag{23}$$

which is equivalent to (20).

Now to prove the unique solvability of the equation (23), we have to establish, according to the Theorem 6.1 [18], the following:

a) $\|Q_{n-\kappa}\mathbf{f}_{n-\kappa} - f\|_Z \to 0$ for $n \to \infty$;
b) the sequence of operators $(B_{n-\kappa})$ approximates operator B compactly;
c) $B : Z \to Z$ is invertible.

The validity of a) follows immediately from the estimations [16]

$$\|Q_{n-\kappa}\mathbf{f}_{n-\kappa} - f\|_Z \le CE_{n-\kappa}(f), \tag{24}$$

$$E_{n-\kappa}(f) \le C(n-\kappa)^{-\mu}\,,\ f(t) \in H_\mu\,,\ n > \kappa\,, \tag{25}$$

where $E_{n-\kappa}(f)$ is the best uniform approximation of the function $f(t)$ by the polynomials of order not higher than $n-\kappa$ on $[-1,1]$.

To check b) we'll show first that the sequence $(B_{n-\kappa})$ approximates the operator B with respect to $Q_{n-\kappa}$. For arbitrary $\mathbf{z}_{n-\kappa} \in Z_{n-\kappa}$, we'll write

$$\begin{aligned} &\|Q_{n-\kappa}B_{n-\kappa}\mathbf{z}_{n-\kappa} - BQ_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \\ &\quad\le \|Q_{n-\kappa}V_{n-\kappa}G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} - VGU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \\ &\qquad + |\eta|\,\|Q_{n-\kappa}U_{n-\kappa}\rho\, G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} - U\rho\, GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \end{aligned} \tag{26}$$

and estimate each summand of the right-hand side independently.

To estimate the first summand we'll use the partial uniform best approximation $E^\tau_n(h)$ $(E^t_n(h))$ of the function $h(t,\tau)$ by the variable τ (t)

$$E^\tau_n(h) = \|E_n(h)\|_Z\,,\ (E^t_n(h) = \|E_n(h)\|_Z)\,.$$

Here, inside the norm symbol, we take first the best approximation by the variable τ (t) and then take norm by the other variable. Using boundness of the operator $S : Z \to Z$ [9], the equivalence

$$U^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} = p_nU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa} \tag{27}$$

and the estimations (24), (22) we'll obtain

$$\begin{aligned} &\|Q_{n-\kappa}V_{n-\kappa}G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} - VGU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \\ &\quad\le C\left(\varepsilon_n(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E^\tau_n(\rho^{-1}h_1D^{(1)}GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E^t_{n-\kappa}(h_1)\right. \\ &\qquad + E_{n-\kappa}(a_0GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E_{n-\kappa}(b_0SQ_{n-\kappa}D^{(0)}_nG_np_nU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) \\ &\qquad \left. + E_{n-\kappa}(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E^\tau_{n-\kappa}(h_0GU^{-1}\,Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E^t_{n-\kappa}(h_0)\right)\,. \end{aligned}$$

For the second summand, using once more (27), (22), (24) and the boundness of the operator $U : Y \to Z$, we'll have

$$\begin{aligned} &|\eta|\ \|Q_{n-\kappa}U_{n-\kappa}\rho\, G_nU^{-1}_{n-\kappa}\mathbf{z}_{n-\kappa} - U\rho\, GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z \\ &\quad\le C(\varepsilon_n(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + E_n(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}))\,. \end{aligned}$$

Finally, using the conditions A.1), A.4) of the Theorem 1 and the estimation (25) we'll obtain

$$\begin{aligned} \|Q_{n-\kappa}B_{n-\kappa}\mathbf{z}_{n-\kappa} - BQ_{n-\kappa}\mathbf{z}_{n-\kappa}\|_Z &\le C(\varepsilon_n(GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}) + (n-\kappa)^{-\gamma})\,, \\ \gamma &= \min\{\mu, 1+\gamma_1, 1+\gamma_2\}\,, \end{aligned}$$

which means the approximation of the operator B by the sequence of the operators $(B_{n-\kappa})$ with respect to $Q_{n-\kappa}$.

Let's assume now, that the sequence $(\mathbf{z}_{n-\kappa})$, $\mathbf{z}_{n-\kappa} \in Z_{n-\kappa}$ is bounded $\|\mathbf{z}_{n-\kappa}\|_{Z_{n-\kappa}} \le 1$. As the functions

$$Q_{n-\kappa}V_{n-\kappa}G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} \quad \text{and} \quad \eta Q_{n-\kappa}U_{n-\kappa}\rho\, G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa}$$

are polynomials and the derivatives of the functions

$$VGU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa} \quad \text{and} \quad \eta U\rho\, GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa}$$

are bounded in Z, then, according to Riesz theorem [11], the functions

$$\begin{aligned} Q_{n-\kappa}B_{n-\kappa}\mathbf{z}_{n-\kappa} - BQ_{n-\kappa}\mathbf{z}_{n-\kappa} &= Q_{n-\kappa}V_{n-\kappa}G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} \\ - \eta Q_{n-\kappa}U_{n-\kappa}\rho\, G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa} &- VGU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa} + \eta U\rho\, GU^{-1}Q_{n-\kappa}\mathbf{z}_{n-\kappa} \end{aligned}$$

form compact sequence in Z and thus the condition b) is valid. The validity of the condition c) follows from the condition A.5) of the Theorem 1 and the equivalence of the equations (16) and (19).

Therefore, according to the Theorem 6.1 [18], for all n, beginning from some n_3, $n_3 \ge n_2$, operators $B_{n-\kappa} : Z_{n-\kappa} \to Z_{n-\kappa}$ and thus operators $K_{n-\kappa} : X_n \to Z_{n-\kappa}$ are invertible and their inverses are bounded collectively and the approximate solutions $\mathbf{x}_n^* = G_nU_{n-\kappa}^{-1}\mathbf{z}_{n-\kappa}^*$ of the system of equations (14), (15) converge to the exact solution $x^* = GU^{-1}z^*$ of the problem (1), (2) with a rate

$$\begin{aligned} \|\mathbf{x}_n^* - p_n^1x^*\|_{X_n} &\le C\|q_{n-\kappa}Kx^* - K_{n-\kappa}p_n^1x^*\|_{Z_{n-\kappa}} \\ &\le C\big(\|q_{n-\kappa}Ux^{*'} - U_{n-\kappa}D_n^{(1)}p_n^1x^*\|_{Z_{n-\kappa}} \\ &\quad + \|q_{n-\kappa}(a_0x^* - a_0Q_{n-\kappa}D_n^{(0)}p_n^1x^*)\|_{Z_{n-\kappa}} \\ &\quad + \|q_{n-\kappa}(b_0Sx^* - b_0SQ_{n-\kappa}D_n^{(0)}p_n^1x^*)\|_{Z_{n-\kappa}} \\ &\quad + \|q_{n-\kappa}(Th_0x^* - TQ_{n-\kappa}h_0D_n^{(0)}p_n^1x^*)\|_{Z_{n-\kappa}} \\ &\quad + \|q_{n-\kappa}(Th_1x^{*'} - T\rho P_n\rho^{-1}h_1D_n^{(1)}p_n^1x^*)\|_{Z_{n-\kappa}}\big)\,. \end{aligned} \tag{28}$$

Using once more the boundness of the operators $U : Y \to Z$, $S : Z \to Z$, estimation (24), Hölder inequality and the error estimate of the Gauss type quadrature formula we'll find

$$\|q_{n-\kappa}Ux^{*'} - U_{n-\kappa}D_n^{(1)}p_n^1x^*\|_{Z_{n-\kappa}} \le C(E_{n-\kappa}(Ux^{*'}) + E_n(\rho^{-1}x^{*'}) + \varepsilon_n(x^*))\,,$$

$$\|q_{n-\kappa}(a_0x^* - a_0Q_{n-\kappa}D_n^{(0)}p_n^1x^*)\|_{Z_{n-\kappa}} \le C\varepsilon_n(x^*)\,,$$

$$\begin{aligned} \|q_{n-\kappa}(b_0Sx^* - b_0SQ_{n-\kappa}D_n^{(0)}p_n^1x^*)\|_{Z_{n-\kappa}} \le (&E_{n-\kappa}(b_0Sx^*) + E_{n-\kappa}(x^*) \\ &+ \varepsilon(x^*) + E_{n-\kappa}(b_0SQ_{n-\kappa}q_{n-\kappa}x^*))\,, \end{aligned}$$

$$\|q_{n-\kappa}(Th_0x^* - TQ_{n-\kappa}h_0D_n^{(0)}p_n^1x^*)\|_{Z_{n-\kappa}} \le C(E_{n-\kappa}^t(h_0) + E_{n-\kappa}^\tau(h_0) + \varepsilon(x^*))\,,$$

$$\begin{aligned} &\|q_{n-\kappa}(Th_1x^{*'} - T\rho P_n\rho^{-1}h_1D_n^{(1)}p_n^1x^*)\|_{Z_{n-\kappa}} \\ &\qquad \le C(E_{n-\kappa}^t(h_1) + E_n^\tau(\rho^{-1}h_1D^{(1)}x^*) + \varepsilon_n(x^*))\,. \end{aligned}$$

Thus, taking into account the smoothness of the functions in the right-hand side, the estimation (25) and the obvious inequality [3] $(n-\kappa)^{-\gamma} \le Cn^{-\gamma}$, we'll obtain the requested estimation

$$\|\mathbf{x}_n^* - p_n^1 x^*\|_{X_n} \le C(n^{-\gamma} + \varepsilon_n(x^*))\,, \quad \gamma = \min\{\mu, 1+\gamma_1, 1+\gamma_2\}\,.$$

Theorem 1 is proved. □

4. Equations of non-zero indices

It is known [15], that for the unique solvability of the problem (1), (2), in case, when $\kappa > 0$, the equations

$$\int_{-1}^{1} \tau^j x'(\tau)\, d\tau = 0, \quad j = 0, 1, \ldots, \kappa - 1\,, \tag{29}$$

should be added. So the equations

$$\sum_{k=0}^{n} \rho^{-1}(\tau_k)[D_n^{(1)}\mathbf{x}_n]_{\tau_k} \int_{-1}^{1} \tau^j \rho(\tau) \bar{l}_k(\tau)\, d\tau = 0\,, \quad j = 0, 1, \ldots, \kappa - 1\,. \tag{30}$$

should be added to the system of equations (14), (15). The justification of the method in this case is similar to the justification in $\kappa = 0$ case, except the definitions of the spaces X and X_n, where the conditions (29) and (30) should be added.

Theorem 2. *Let for $\kappa > 0$ the problem* (1), (2), (29), *and the calculation scheme* (5) - (15), (30) *of the method satisfy to the conditions* A.1) - A.5), B.1), B.2) *of the Theorem 1. Then, for n large ehough, the system of equations* (14), (15), (30) *is uniquely solvable and approximate solutions $\mathbf{x}_n^*$ converge to the exact solution $x^*(\tau) \in X$ of the problem* (1), (2), (29) *with the error estimate*

$$\|\mathbf{x}_n^* - p_n^1 x^*\|_{X_n} \le C(n^{-\gamma} + \varepsilon_n(x^*))\,.$$

The case, when $\kappa < 0$ is more complicate, because the operator $U : Y \to Z$ and therefore the operator $K : X \to Z$ in this case are in general uninvertible and the condition $A.5$) of the Theorem 1 will not be satisfied. We may assume instead only the solvability of the concrete equation with the fixed coefficients and the right-hand side. Moreover, the system of equations (14), (15) in this case will contain $n+2$ unknown variables, but consist of $n+2-\kappa$ equations. So it will be overdetermined and thus, in general, unsolvable. It means, that the previously used proof can't be applied here. Nevertheless, we may reduce this case to the general one by a simple technique firstly used by V.V.Ivanov [10] and later by many authors (see e.g. [2], [12], [13]).

Instead of the equation (1) we'll consider equation

$$UD^{(1)}x + Vx + w = f\,, \tag{31}$$

[3] Here C depends on γ and n_3.

containing in the left-hand side polynomial

$$w(t) = \sum_{j=1}^{-\kappa} \chi_j t^{j-1}$$

with the coefficients χ_j, $j = 1, \dots, -\kappa$, ought to be determined. Equations (1) and (31) are closely connected. Indeed, if $x^*(\tau)$ is the solution of the problem (1), (2), then the couple (x^*, w), $w(t) \equiv 0$ will be the solution of the problem (31), (2). On the other hand, if the problem (1), (2) will be solvable for only one fixed right-hand side, then the corresponding problem (31), (2) will be solvable for any right-hand side $f(t) \in Z$, because to satisfy the conditions of solvability (see [15], [8]) one needs to find out only the coefficients χ_j, $j = 1, \dots, -\kappa$ of the polynomial $w(t)$ satisfying the equations

$$\int_{-1}^{1} \bar\rho(t) t^{j-1} (f(t) - (Vx)(t) - w(t))\, dt = 0\,, \quad j = 1, \dots, -\kappa\,,$$

and thus the proof of the Theorem 1 will be valid also for this case.

The system of the equations (14), (15) also should be slightly changed. We'll add the summands $w_n(t_j)$, $j = 0, 1, \dots, n-\kappa$ - the values of the approximating polynomial

$$w_n(t) = \sum_{j=1}^{-\kappa} \chi_{jn} t^{j-1}$$

in the nodes of the grid (6) to the left-hand sides of the equations of the system (14). In the operator form the system of the equations will take the following form

$$\begin{gathered} U_{n-\kappa} D_n^{(1)} \mathbf{x}_n + V_{n-\kappa} \mathbf{x}_n + \mathbf{w}_{n-\kappa} = \mathbf{f}_{n-\kappa}\,, \\ \mathbf{w}_{n-\kappa} = (w_n(t_0), \dots, w_n(t_{n-\kappa}))\,. \end{gathered} \tag{32}$$

Now the number of the unknown variables is $n+2-\kappa$, so it is equal to the number of equations. These changes now allow us to use the proof, like the one of the Theorem 1.

Theorem 3. *Let for* $\kappa < 0$ *the problem* (1), (2) *and the calculation scheme* (5) - (13), (32) *of the method satisfy to the conditions* A.1) - A.4), B.1), B.2) *of the Theorem 1. Let's assume also, that the problem* (1), (2) *has a unique solution* $x^*(t)$. *Then for* n, *large enough, the system of equations* (32) *is uniquely solvable and the approximate solutions* $\bar{\mathbf{x}}_n^* = (\mathbf{x}_n^*, \chi_{1n}^*, \dots, \chi_{-\kappa n}^*)$ *converge to the exact solution* $\bar x^* = (x^*, 0)$ *of the equation* (31) *with the error estimate*

$$\|\bar{\mathbf{x}}_n^* - \bar p_n^1 \bar x^*\|_{\bar X_n} = \|\mathbf{x}_n^* - p_n^1 x^*\|_{X_n} + \max_{1 \le j \le -\kappa} |\chi_{jn}| \le C(n^{-\gamma} + \varepsilon_n(x^*))\,,$$

$$\bar p_n^1 \bar x^* = (p_n^1 x^*, \chi_1^*, \dots, \chi_{-\kappa}^*)\,,\ \bar X_n = X_n \times R_{-\kappa}\,,\ \|\bar{\mathbf{x}}_n\|_{\bar X_n} = \|\mathbf{x}_n\|_{X_n} + \max_{1 \le j \le -\kappa} |\chi_{jn}|\,.$$

The proof of the Theorem 3 is, in general, similar to the proof of the Theorem 1, so we'll give it briefly, paying attention only to the major differences.

Let's rewrite the equation (31) in operator form

$$\bar K \bar x \equiv \bar U(D^{(1)} x, w) + Vx = f,\ \bar K : \bar X \to Z\,, \tag{33}$$

where

$$\bar{X} = \{\bar{x} \mid \bar{x} = (x, w), x \in X\},\ \|\bar{x}\|_{\bar{X}} = \|x\|_X + \max_{1 \le j \le -\kappa} |\chi_j|\,,$$

$$\bar{Y} = \{\bar{y} \mid \bar{y} = (y, w), y \in Y\}\,,\ \|\bar{y}\|_{\bar{Y}} = \|y\|_Y + \max_{1 \le j \le -\kappa} |\chi_j|\,,$$

$$\bar{U}(y, w) = Uy + w\,,\ \bar{U} : \bar{Y} \to Z\,.$$

The operator $\bar{U}$ is invertible and

$$\bar{U}^{-1} z = (U^{-1}(z - w), w)\,,$$

where $w(t)$ is a polynomial, which coefficients can be found from the equations

$$\int_{-1}^{1} \bar{\rho}(t) t^{j-1} (z(t) - w(t)) dt = 0\,, \qquad j = 1, \ldots, -\kappa\,.$$

Thus the substitution

$$z = \bar{U}(D^{(1)} x + \eta \rho x, w) \tag{34}$$

allows us to reduce the equation (33) to the equivalent equation

$$\bar{B} z \equiv z + V G \bar{U}^{-1} z - \eta U \rho\, G \bar{U}^{-1} z = f\,,\ \bar{B} : Z \to Z\,. \tag{35}$$

Let's rewrite the equation (32) in the same way

(36)

$$\bar{K}_{n-\kappa} \bar{\mathbf{x}}_n \equiv \bar{U}_{n-\kappa}(D_n^{(1)} \mathbf{x}_n, \mathbf{w}_{n-\kappa}) + V_{n-\kappa} \mathbf{x}_n = \mathbf{f}_{n-\kappa}\,,\ \bar{K}_{n-\kappa} : \bar{X}_n \to Z_{n-\kappa}\,,$$

where

$$\bar{X}_n = X_n \times R_{-\kappa}\,,\ \|\bar{\mathbf{x}}_n\|_{\bar{X}_n} = \|\mathbf{x}_n\|_{X_n} + \max_{1 \le j \le -\kappa} |\chi_{jn}|\,,$$

$$\bar{Y}_n = Y_n \times R_{-\kappa}\,,\ \|\bar{\mathbf{y}}_n\|_{\bar{Y}_n} = \|\mathbf{y}_n\|_{Y_n} + \max_{1 \le j \le -\kappa} |\chi_{jn}|\,,$$

$$\bar{U}_{n-\kappa}(\mathbf{y}_n, \mathbf{w}_{n-\kappa}) = U_{n-\kappa} \mathbf{y}_n + \mathbf{w}_{n-\kappa}\,,\ \bar{U}_{n-\kappa} : \bar{Y}_n \to Z_{n-\kappa}\,,$$

with $\mathbf{w}_{n-\kappa} = (w_n(t_0), \ldots, w_n(t_{n-\kappa}))$ - be a vector of the values of the polynomial $w_n(t)$, which coefficients can be found from the equations

$$\int_{-1}^{1} \bar{\rho}(t) t^{j-1} (Q_{n-\kappa} \mathbf{z}_{n-\kappa} - w_n(t)) dt = 0\,, \qquad j = 1, \ldots, -\kappa\,. \tag{37}$$

Now we'll use the substitution

$$\mathbf{z}_{n-\kappa} = \bar{U}_{n-\kappa}(D_n^{(1)} \mathbf{x}_n + \eta \rho \mathbf{x}_n, \mathbf{w}_{n-\kappa})$$

which allows us to reduce the equation (36) to the equivalent equation

$$\begin{aligned} \bar{B}_{n-\kappa} \mathbf{z}_{n-\kappa} &\equiv \mathbf{z}_{n-\kappa} + V_{n-\kappa} G_n \bar{U}_{n-\kappa}^{-1} \mathbf{z}_{n-\kappa} - \eta U_{n-\kappa} \rho\, G_n \bar{U}_{n-\kappa}^{-1} \mathbf{z}_{n-\kappa} \\ &= \mathbf{f}_{n-\kappa}\,,\ \bar{B}_{n-\kappa} : Z_{n-\kappa} \to Z_{n-\kappa}\,. \end{aligned} \tag{38}$$

Besides, according to the proof of the Theorem 1, we have to check that the conditions a) - c) are satisfied. The condition a) may be checked like the one in the proof of the Theorem 1. In order to check b), we have previously to calculate $\mathbf{w}_{n-\kappa}$ for the chosen $\mathbf{z}_{n-\kappa}$ according to the formula (37) and then to follow the proof of the Theorem 1 taking $\mathbf{z}_{n-\kappa} - \mathbf{w}_{n-\kappa}$ instead of $\mathbf{z}_{n-\kappa}$. The validity of the condition c) follows from the invertability of the operator $\bar{K}$. Indeed, for the given right-hand side of the equation (35) we'll obtain the right-hand side of the equation (33). Then, due to the invertability of $\bar{K}$, we'll find the couple (x, w) and via (34) will obtain z.

The error estimate

$$\|\bar{\mathbf{x}}_n^* - \bar{p}_n^1 \bar{x}^*\|_{\bar{X}_n} = \|\mathbf{x}_n^* - p_n^1 x^*\|_{X_n} + \max_{1 \le j \le -\kappa} |\chi_{jn}| \le C \|q_{n-\kappa} \bar{K} \bar{x}^* - \bar{K}_{n-\kappa} \bar{p}_n^1 \bar{x}^*\|_{Z_{n-\kappa}}$$

$$= C \|q_{n-\kappa} K x^* - K_{n-\kappa} p_n^1 x^*\|_{Z_{n-\kappa}} \le C(n^{-\gamma} + \varepsilon_n(x^*)),$$

obtained just as in the proof of the Theorem 1, finishes the proof in this case.

Remark 1. The Theorems 1 - 3 might be extended to the case of the mostly general boundary condition [18]

$$u(x) \equiv \int_{-1}^{1} x^{(\nu)}(\tau)\, d\zeta(\tau) = 0,$$

where $\zeta(\tau)$ is a given function of the bounded variation and integral is interpreted as Stieltjes one. This boundary condition might be approximated by any differences condition

$$u_n(\mathbf{x}_n) = 0,$$

satisfying $u_n(p_n^1 x) \to u(x)$ for $n \to \infty$ for any $x \in X$. The Theorems 1 - 3 will remain valid, but the value $|u_n(p_n^1 x^*)|$ should substitute $|[D_n^{(0)} p_n^1 x^*]_{\xi_0}|$ in the definition of $\varepsilon_n(x^*)$.

Remark 2. The condition $A.3)$ of the Theorem 1 is only sufficient and, as it was shown in [1], [13], [14], can be reduced.

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