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LINEAR VOLTERRA-STIELTJES INTEGRAL EQUATIONS IN THE SENSE OF THE KURZWEIL-HENSTOCK INTEGRAL

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Abstract. In 1990, Hönig proved that the linear Volterra integral equation

$$x(t) - (K)\int_{[a,t]} \alpha(t,s)\,x(s)\,ds = f(t)\,, \qquad t \in [a,b]\,,$$

where the functions are Banach space-valued and f is a Kurzweil integrable function defined on a compact interval $[a,b]$ of the real line $\mathbb{R}$, admits one and only one solution in the space of the Kurzweil integrable functions with resolvent given by the Neumann series. In the present paper, we extend Hönig's result to the linear Volterra-Stieltjes integral equation

$$x(t) - (K)\int_{[a,t]} \alpha(t,s)\,x(s)\,dg(s) = f(t)\,, \qquad t \in [a,b]\,,$$

in a real-valued context.

1. Introduction

J. Kurzweil (in 1957) and, independently, R. Henstock (in 1961) gave a Riemannian definition of the equivalent integrals of Denjoy and Perron defined in the beginning of the twentieth century. The Kurzweil-Henstock integral (also called the *generalized Riemann integral* or the *Riemann complete integral*) encompasses the integrals of Riemann and Lebesgue as well as its improper integrals (see [7], [14], [15], [18], [19] or [21]). It also has good convergence properties ([15]). Though non-complete, the space of all equivalence classes of Kurzweil-Henstock integrable functions, endowed with the Alexiewicz norm, has good functional analytic properties ([5]).

The Kurzweil-Henstock integral has been shown to be useful in the study of Integral Equations (see for instance [4], [11] and [24]). In [11], Theorem 3.5, Hönig proved that the abstract linear Volterra-Kurzweil integral equation

$$x(t) - (K)\int_{[a,t]} \alpha(t,s)\,x(s)\,ds = f(t)\,, \qquad t \in [a,b]\,,$$

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admits a unique solution in the space of the Kurzweil integrable functions with resolvent given by the Neumann series. In the following pages, we improve Hönig's result by considering a more general equation

$$x(t) - (K)\int_{[a,t]} \alpha(t,s)\, x(s)\, dg(s) = f(t)\ , \qquad t \in [a,b]\ .$$

However the functions involved are real-valued, since we apply some results of the Kurzweil-Henstock integration theory that hold in the real-valued case but *fail* in a general Banach space-valued context. Such results are:

1. the spaces of Henstock and of Kurzweil integrals coincide;
2. a Riemann integrable function is also Henstock integrable;
3. the Fundamental Theorem of Calculus holds for the Kurzweil integral.

For examples in which the above results are not valid, see [2] and [4].

2. Basic definitions and properties

Let E and F be normed spaces and $L(E,F)$ be the space of linear continuous functions from E to F. We write $L(E) = L(E,E)$ and $E' = L(E,\mathbb{R})$, where $\mathbb{R}$ denotes the real line. Let X and Y be Banach spaces and $[a,b]$ be a compact interval of $\mathbb{R}$. We denote by $C([a,b],X)$ the space of all continuous functions from $[a,b]$ to X endowed with the usual supremum norm $\|\cdot\|_\infty$. We define

$$C_a([a,b],X) = \{f \in C([a,b],X)\,;\, f(a) = 0\}\ .$$

When $X = \mathbb{R}$, we replace $C([a,b],X)$ and $C_a([a,b],X)$ by $C([a,b])$ and $C_a([a,b])$ respectively.

2.1. **Variation and semi-variation of a function.** Any finite set of closed non-overlapping subintervals $[t_{i-1},t_i]$ of $[a,b]$ such that the union of all intervals $[t_{i-1},t_i]$ equals $[a,b]$ is called a *division* of $[a,b]$. In this case, we write $d = (t_i) \in D_{[a,b]}$, where $D_{[a,b]}$ denotes the set of all divisions of $[a,b]$. By $|d|$ we mean the number of subintervals in which $[a,b]$ is divided through a given $d \in D_{[a,b]}$.

Definition 1. Given a function $f : [a,b] \to E$, E a normed space, and $d = (t_i) \in D_{[a,b]}$, we define

$$V_d(f) = V_{d,[a,b]}(f) = \sum_{i=1}^{|d|} \|f(t_i) - f(t_{i-1})\|$$

and the *variation* of f is given by

$$V(f) = V_{[a,b]}(f) = \sup\left\{V_d(f)\,;\, d \in D_{[a,b]}\right\}\ .$$

If $V(f) < \infty$, then f is a function of *bounded variation* and we write $f \in BV([a,b],E)$. When $E = \mathbb{R}$, we replace $BV([a,b],E)$ by $BV([a,b])$.

In Definition 1 we have $\|f(t)\| \le \|f(a)\| + V_{[a,t]}(f)$, for every $t \in [a,b]$, where $V_{[a,t]}(f)$ denotes the variation of f on the interval $[a,t]$. Hence $\|f\|_\infty \le \|f(a)\| + V(f)$. Analogously $\|f\|_\infty \le \|f(b)\| + V(f)$.

Definition 2. Let E and F be normed spaces. Given a function $\alpha : [a,b] \to L(E,F)$ and $d = (t_i) \in D_{[a,b]}$ we define

$$SV_d(\alpha) = SV_{d,[a,b]}(\alpha) = \sup\left\{\left\|\sum_{i=1}^{|d|}[\alpha(t_i) - \alpha(t_{i-1})]\,x_i\right\|; x_i \in F, \|x_i\| \le 1\right\}.$$

Then the *semi-variation* of α is defined as

$$SV(\alpha) = SV_{[a,b]}(\alpha) = \sup\left\{SV_d(\alpha); d \in D_{[a,b]}\right\}.$$

If $SV(f) < \infty$, then α is a function of *bounded semi-variation* and we write $\alpha \in SV([a,b], L(E,F))$.

It is clear that $BV([a,b], L(E,F)) \subset SV([a,b], L(E,F))$. Also $SV([a,b], L(E,\mathbb{R})) = BV([a,b], E')$ (see [9]). Let X and Y be Banach spaces and $c \in [a,b]$. We also consider the spaces

$$BV_c([a,b], X) = \{f \in BV([a,b], X); f(c) = 0\},$$

$$BV_c^+([a,b], X) = \{f \in BV_c([a,b], X); f \text{ is right continuous}\}$$

$$SV_c([a,b], L(X,Y)) = \{\alpha \in SV([a,b], L(X,Y)); \alpha(c) = 0\}$$

all of which are complete when endowed with the norm given by the variation (in the first two cases) and the semi-variation (in the last case).

For details and properties of all these spaces, see [9].

2.2. **The vector integrals of Kurzweil and of Henstock.**

2.2.1. *Definitions and terminology.* A pair $d = (\xi_i, t_i)$ is a *tagged division* of $[a,b]$, if $(t_i) \in D_{[a,b]}$ and $\xi_i \in [t_{i-1}, t_i]$, for every i. We denote by $TD_{[a,b]}$ the set of all tagged divisions of $[a,b]$. Any subset of a tagged division of $[a,b]$ is a *tagged partial division* of $[a,b]$ and we write $d \in TPD_{[a,b]}$ in this case. A *gauge* of a set $E \subset [a,b]$ is any function $\delta : E \to]0,\infty[$. Given a gauge δ of $[a,b]$, we say that $d = (\xi_i, t_i) \in TPD_{[a,b]}$ is δ*-fine*, if $[t_{i-1}, t_i] \subset \{t \in [a,b]; |t - \xi_i| < \delta(\xi_i)\}$, for every i.

Throughout this paper, X and Y always denote Banach spaces.

Definition 3. Consider functions $\alpha : [a,b] \to L(X,Y)$ and $f : [a,b] \to X$.
(i) We say that α is *Kurzweil f-integrable* if there exists $I \in Y$ (we write $I = (K)\int_{[a,b]} \alpha(t)\, df(t)$) such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\left\|\sum_{i=1}^{|d|} \alpha(\xi_i)[f(t_i) - f(t_{i-1})] - I\right\| < \varepsilon.$$

In this case, we write $\alpha \in K_f([a,b], L(X,Y))$.
(ii) We say that f is *Kurzweil α-integrable* if there exists $I \in Y$ (we write $I = (K)\int_{[a,b]} d\alpha(t)\, f(t)$) such that given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that

$$\left\|\sum_{i=1}^{|d|} [\alpha(t_i) - \alpha(t_{i-1})] f(\xi_i) - I\right\| < \varepsilon,$$

whenever $d = (\xi_i, t_i) \in TD_{[a,b]}$ is δ-fine. In this case, $f \in K^{\alpha}([a,b], X)$.

If δ is a constant function in the definition of $\alpha \in K_f([a,b], L(X,Y))$, then we obtain the Riemann-Stieltjes integral $\int_{[a,b]} \alpha(t)\, df(t)$ and we write $\alpha \in R_f([a,b], L(X,Y))$. Similarly, when we consider only constant gauges δ in the definition of $f \in K^{\alpha}([a,b], X)$, we obtain the Riemann-Stieltjes integral $\int_{[a,b]} d\alpha(t)\, f(t)$ and we write $f \in R^{\alpha}([a,b], X)$.

The vector integral of Henstock is more restrictive than that of Kurzweil.

Definition 4. Let $\alpha : [a,b] \to L(X,Y)$ and $f : [a,b] \to X$ be functions. We say that α is *Henstock f-integrable* (we write $\alpha \in H_f([a,b], L(X,Y))$) if there exists a function $A_f : [a,b] \to Y$ (called the *associated function* of α) such that for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_{i=1}^{|d|} \|\alpha(\xi_i)[f(t_i) - f(t_{i-1})] - [A_f(t_i) - A_f(t_{i-1})]\| < \varepsilon .$$

In an analogous way we define the Henstock α-integrability of $f : [a,b] \to X$ and we write $f \in H^{\alpha}([a,b], X)$ (see [2]).

Clearly $H_f([a,b], L(X,Y)) \subset K_f([a,b], L(X,Y))$ and $H^{\alpha}([a,b], X) \subset K^{\alpha}([a,b], X)$. If we identify the isomorphic spaces $L(\mathbb{R}, \mathbb{R})$ and $\mathbb{R}$ (see [17], p. 269-270), then all the spaces $K_f([a,b], L(\mathbb{R}))$, $K_f([a,b], \mathbb{R})$, $H_f([a,b], L(\mathbb{R}))$ and $H_f([a,b], \mathbb{R})$ can also be identified, since $K_f([a,b], \mathbb{R}) = H_f([a,b], \mathbb{R})$ (see for instance [19]). For simplicity of notation we replace $K_f([a,b], \mathbb{R})$ and $H_f([a,b], \mathbb{R})$ respectively by $K_f([a,b])$ and $H_f([a,b])$. And we write either $K_f([a,b])$ or $H_f([a,b])$ depending on which definition we want to emphasize.

Given $f : [a,b] \to X$ and $\alpha \in K_f([a,b], L(X,Y))$, we define $\tilde{\alpha}_f : [a,b] \to Y$ by

$$\tilde{\alpha}_f(t) = (K) \int_{[a,t]} \alpha(s)\, df(s), \quad t \in [a,b] .$$

If in addition $\alpha \in H_f([a,b], L(X,Y))$, then $\tilde{\alpha}_f(t) = A_f(t) - A_f(a)$ for every $t \in [a,b]$. In an analogous way, given $\alpha : [a,b] \to L(X,Y)$, we define $\tilde{f}^{\alpha} : [a,b] \to Y$ by

$$\tilde{f}^{\alpha}(t) = (K) \int_{[a,t]} d\alpha(s)\, f(s) , \quad t \in [a,b] ,$$

for every $f \in K^{\alpha}([a,b], X)$. If $\alpha(t) = t$, then instead of $K^{\alpha}([a,b], X)$, $R^{\alpha}([a,b], X)$, $H^{\alpha}([a,b], X)$ and $\tilde{f}^{\alpha}$ we write respectively $K([a,b], X)$, $R([a,b], X)$, $H([a,b], X)$ and $\tilde{f}$ (i.e., $\tilde{f}(t) = (K) \int_{[a,t]} f(s)\, ds$, for every $t \in [a,b]$). If moreover $X = \mathbb{R}$, then we write simply $K([a,b])$, $R([a,b])$ and $H([a,b])$ and we have $R([a,b]) \subset K([a,b]) = H([a,b])$.

Let m denote the Lebesgue measure.

Definition 5. A function $f : [a,b] \to X$ satisfies the *Strong Lusin Condition* (we write $f \in SL([a,b], X)$) if given $\varepsilon > 0$ and $B \subset [a,b]$ with $m(B) = 0$, there is a gauge δ of B such that for every δ-fine $d = (\xi_i, t_i) \in TPD_{[a,b]}$ with $\xi_i \in B$ for every i,

$$\sum_{i=1}^{|d|} \|f(t_i) - f(t_{i-1})\| < \varepsilon .$$

If we denote by $AC([a,b], X)$ the space of all absolutely continuous functions from $[a,b]$ to X, then we have $AC([a,b], X) \subset SL([a,b], X) \subset C([a,b], X)$. In $SL([a,b], X)$, we consider the induced supremum norm. When $X = \mathbb{R}$, we write simply $SL([a,b])$.

Given $f \in SL([a,b], X)$ and $\alpha \in H_f([a,b], L(X,Y))$, let $\beta : [a,b] \to L(X,Y)$ be such that $\beta = \alpha$ m-almost everywhere. Then $\beta \in H_f([a,b], L(X,Y))$ and $\tilde{\beta}_f(t) = \tilde{\alpha}_f(t)$, for every $t \in [a,b]$ (see [2]). An analogous result holds when we replace $H_f([a,b], L(X,Y))$ by $K_f([a,b], L(X,Y))$. In this manner we have

Definition 6. Suppose $f \in SL([a,b], X)$. Two functions $\beta, \alpha \in K_f([a,b], L(X,Y))$ are called *equivalent* if and only if $\tilde{\beta}_f = \tilde{\alpha}_f$. Then we denote by $\mathbf{K}_f([a,b], L(X,Y))$ (respectively by $\mathbf{H}_f([a,b], L(X,Y))$ the space of all equivalence classes of functions of $K_f([a,b], L(X,Y))$ (respectively of $H_f([a,b], L(X,Y)))$ equipped with the Alexiewicz norm

$$\|\alpha\|_{A,f} = \sup\left\{\left\|(K)\int_{[a,t]} \alpha(s)\, df(s)\right\|; t \in [a,b]\right\} = \|\tilde{\alpha}_f\|_\infty . \tag{1}$$

2.2.2. *Some properties.*

Lemma 1 (Saks-Henstock Lemma). *Given $f : [a,b] \to X$, let $\alpha \in K_f([a,b], L(X,Y))$ that is, for every $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that*

$$\left\|\sum_{i=1}^{|d|} \alpha(\xi_i)[f(t_i) - f(t_{i-1})] - (K)\int_{[a,b]} \alpha(t)\, df(t)\right\| < \varepsilon ,$$

whenever $d = (\xi_i, t_i) \in TD_{[a,b]}$ is δ-fine. Then for every δ-fine $d' = (\zeta_j, s_j) \in TPD_{[a,b]}$,

$$\left\|\sum_{j=1}^{|d'|} \left\{(K)\int_{[s_{j-1}, s_j]} \alpha(t)\, df(t) - \alpha(\zeta_j)[f(s_j) - f(s_{j-1})]\right\}\right\| < \varepsilon .$$

The proof of Lemma 1 follows standard steps. See for instance [23], Proposition 16. A similar lemma also holds if we replace $K_f([a,b], L(X,Y))$ by $R_f([a,b], L(X,Y))$.

Theorem 1. *If $f \in C([a,b], X)$ and $\alpha \in K_f([a,b], L(X,Y))$, then $\widetilde{\alpha}_f \in C_a([a,b], Y)$.*

For a proof of Theorem 1, see [4].

Theorem 2. *If $f \in SL([a,b], X)$ and $\alpha \in H_f([a,b], L(X,Y))$, then $\widetilde{\alpha}_f \in SL([a,b], Y)$.*

A proof of Theorem 2 can be found in [2], Theorem 7.

Let $F : [a,b] \to X$ be differentiable at $t \in [a,b]$. We denote by $\frac{d}{dt}F(t)$ or by $F'(t)$ its derivative at $t \in [a,b]$. For a proof of the Fundamental Theorem of Calculus below (Theorems 3 and 4), see [2].

Theorem 3. *If $F \in C([a,b],X)$ and there exists the derivative $F'(t) = f(t)$, for every $t \in [a,b]$, then $f \in H([a,b],X)$ and*

$$(K)\int_{[a,t]} f(s)\ ds = F(t) - F(a)\ , \qquad t \in [a,b]\ .$$

Theorem 4. (i) *If $f \in SL([a,b],X)$ and $A \in SL([a,b],Y)$ are both differentiable and $\alpha : [a,b] \to L(X,Y)$ is such that $A'(t) = \alpha(t)\ f'(t)$ for m-almost every $t \in [a,b]$, then $\alpha \in H_f([a,b],L(X,Y))$ and $A = \tilde{\alpha}_f$.*
(ii) *If $f \in SL([a,b],X)$ is differentiable and $\alpha \in H_f([a,b],L(X,Y))$ is bounded, then $\tilde{\alpha}_f \in SL([a,b],Y)$ and there exists the derivative $(\tilde{\alpha}_f)'(t) = \alpha(t)\ f'(t)$ for m-almost every $t \in [a,b]$.*

Corollary 1. *Suppose $f \in SL([a,b],X)$ is differentiable and non-constant on any non-degenerate subinterval of $[a,b]$ and $\alpha \in H_f([a,b],L(X,Y))$ is bounded and such that $\tilde{\alpha}_f = 0$. Then $\alpha = 0$ m-almost everywhere.*

Theorem 5. *Suppose $f \in SL([a,b],X)$ is non-constant on any non-degenerate subinterval of $[a,b]$. Then the mapping*

$$\alpha \in \mathbf{H}_f([a,b],L(X,Y)) \mapsto \widetilde{\alpha}_f \in C_a([a,b],X)$$

is an isometry (i.e., $\|\widetilde{\alpha}_f\|_\infty = \|\alpha\|_{A,f}$) onto a dense subspace of $C_a([a,b],X)$.

Proof. The space $SL_a^{\mathrm{diff}}([a,b],X)$ of all functions of $SL([a,b],X)$ which vanish at $t = a$ and are differentiable in $[a,b]$ is a dense subspace of $C_a([a,b],X)$. Thus by (1) and by Theorem 2 it is sufficient to show that given $\varepsilon > 0$ and $A \in SL_a^{\mathrm{diff}}([a,b],X)$, there exists $\alpha \in \mathbf{H}_f([a,b],L(X,Y))$ such that $\|\widetilde{\alpha}_f - A\|_\infty < \varepsilon$. But, if we take $\alpha : [a,b] \to L(X,Y)$ such that $\alpha(t)\,f'(t) = A'(t)$ for all $t \in [a,b]$ such that $f'(t) \neq 0$, then $\alpha(t)\,f'(t) = A'(t)$ for m-almost every $t \in [a,b]$ and the result follows by the first part of Theorem 4. □

The next result is well-known. It gives the Integration by Parts Formula for the Riemann-Stieltjes integrals. A proof of it can be found in [9] or [4], Theorem 1.5.

Theorem 6. *The Riemann-Stieltjes integrals $\int_{[a,b]} d\alpha(t)\,f(t)$ and $\int_{[a,b]} \alpha(t)\ df(t)$ exist and the Integration by Parts Formula*

$$\int_{[a,b]} d\alpha(t)\ f(t) = \alpha(b)\,f(b) - \alpha(a)\,f(a) - \int_{[a,b]} \alpha(t)\ df(t)$$

holds if one of the next conditions are fulfilled:
(i) *$\alpha \in SV([a,b],L(X,Y))$ and $f \in C([a,b],X)$;*
(ii) *$\alpha \in C([a,b],L(X,Y))$ and $f \in BV([a,b],X)$.*

Let W also denote a Banach space. We have

Theorem 7. *If $\alpha \in SV([a,b], L(X,Y))$, $f \in C([a,b], W)$, $\beta \in K_f([a,b], L(W,X))$ and $g(t) = \tilde{\beta}_f(t) = \int_{[a,t]} \beta(s)df(s)$, $t \in [a,b]$, then $\alpha\beta \in K_f([a,b], L(W,Y))$ with*

$$(K)\int_{[a,b]} \alpha(t)\beta(t)\ df(t) = \int_{[a,b]} \alpha(t)\ dg(t)$$

and

$$\left\|(K)\int_{[a,b]} \alpha(t)\beta(t)\ df(t)\right\| \leq [SV(\alpha) + \|\alpha(a)\|]\ \|\beta\|_{A,f}\ .$$

A proof of Theorem 7 can be found in [3], Corollary 20.

Lemma 2 (Straddle Lemma)**.** *Suppose $f, F : [a,b] \to X$ are such that $F'(\xi) = f(\xi)$, for all $\xi \in [a,b]$. Then given $\varepsilon > 0$, there exists $\delta(\xi) > 0$ such that*

$$\|F(t) - F(s) - f(\xi)(t-s)\| < \varepsilon(t-s)\ ,$$

whenever $\xi - \delta(\xi) < s < \xi < t < \xi + \delta(\xi)$.

For a proof of Lemma 2, see [12], 3.4 or [6].

By $E([a,b], L(X,Y))$ we mean the space of all step functions from $[a,b]$ to $L(X,Y)$, that is, $\alpha \in E$ if and only if α is bounded, there is a division $d = (t_i) \in D_{[a,b]}$ and there are numbers $\alpha_1, \alpha_2, \ldots, \alpha_{|d|}$ such that $\alpha(t) = \sum_{i=1}^{|d|} \alpha_i \chi_{[t_{i-1},t_i[}(t)$, $t \in [a,b[$, where χ_A denotes the characteristic function of $A \subset [a,b]$.

Theorem 8. *Let $f : [a,b] \to X$ be differentiable and non-constant on any non-degenerate subinterval of $[a,b]$. Then the spaces $C([a,b], L(X,Y))$ and $E([a,b], L(X,Y))$ are dense in $K_f([a,b], L(X,Y))$ in the norm $\|\cdot\|_{A,f}$.*

Proof. It is enough to show that the space $PC([a,b], L(X,Y))$ of piecewise continuous functions from $[a,b]$ to $L(X,Y)$ is dense in $K_f([a,b], L(X,Y))$ in the norm $\|\cdot\|_{A,f}$. Thus given $\alpha \in K_f([a,b], L(X,Y))$ and $\varepsilon > 0$, we want to find a function $\beta \in PC([a,b], L(X,Y))$ such that $\|\beta - \alpha\|_{A,f} < \varepsilon$, or equivalently,

$$\left\|(K)\int_{[a,t]} \beta(s)\, df(s) - (K)\int_{[a,t]} \alpha(s)\, df(s)\right\| < \varepsilon\,, \quad t \in [a,b]\ . \tag{2}$$

Let $\alpha \in K_f([a,b], L(X,Y))$. By Theorem 1, $\widetilde{\alpha}_f \in C_a([a,b], Y)$. Let $C_a^{(1)}([a,b], Y)$ be the subspace of $C_a([a,b], Y)$ of functions which are differentiable with continuous derivative. Hence given $\varepsilon > 0$, there is a function $h \in C_a^{(1)}([a,b], Y)$ such that

$$\|h - \widetilde{\alpha}_f\|_\infty < \varepsilon\,. \tag{3}$$

Let $\beta : [a,b] \to L(X,Y)$ be defined by $\beta(t)x = h'(t)$, for all $x \in X$ such that $x \neq 0$, and by $\beta(t)0 = 0$. In particular, $\beta(t)f'(t) = h'(t)$ whenever $f'(t) \neq 0$. Therefore $\beta(t)f'(t) = h'(t)$ for m-almost every $t \in [a,b]$, since $f : [a,b] \to X$

is differentiable and non-constant on any non-degenerate subinterval of $[a,b]$. It follows then that the Bochner-Lebesgue integral $(L)\int_{[a,t]}\beta(s)\,f'(s)\,ds$ exists and

$$(L)\int_{[a,t]}\beta(s)\,f'(s)\,ds=\int_{[a,t]}h'(s)\,ds=h(t)\,,\quad t\in[a,b]\,, \tag{4}$$

where we applied the Fundamental Theorem of Calculus for the Riemann integral in order to obtain the last equality. Then equations (3) and (4) imply

$$\left\|(L)\int_{[a,t]}\beta(s)\,f'(s)\,ds-(K)\int_{[a,t]}\alpha(s)\,df(s)\right\|<\varepsilon\,,\quad t\in[a,b]\,. \tag{5}$$

Hence in view of (2), if we prove that the Kurzweil vector integral $(K)\int_{[a,b]}\beta(s)df(s)$ exists and

$$(K)\int_{[a,t]}\beta(s)\,df(s)=(L)\int_{[a,t]}\beta(s)\,f'(s)\,ds\,,\quad t\in[a,b]\,, \tag{6}$$

then the proof is complete.

Because the Bochner-Lebesgue integral is a special case of the Henstock integral (see [20], [16] and [13]), we will write $(K)\int_{[a,b]}\beta(s)\,f'(s)\,ds$ instead of $(L)\int_{[a,b]}\beta(s)\,f'(s)\,ds$. Let δ_1 be the gauge of $[a,b]$ from the definition of $(K)\int_{[a,b]}\beta(s)\,f'(s)\,ds$. Take $t\in[a,b]$ and for every $\xi\in[a,t]$, let $\delta_2(\xi)>0$ be such that if $\xi-\delta_2(\xi)<s<\xi<u<\xi+\delta_2(\xi)$, then

$$\|f(t)-f(s)-f'(\xi)(u-s)\|<\varepsilon(u-s) \tag{7}$$

(see Lemma 2). We now define a gauge δ of $[a,t]$ by $\delta(\xi)=\min\{\delta_1(\xi),\delta_2(\xi)\}$, for every $\xi\in[a,t]$. Hence for every δ-fine $d=(\xi_i,t_i)\in TD_{[a,t]}$ we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{|d|}\beta(\xi_i)\left[f(t_i)-f(t_{i-1})\right]-(K)\int_{[a,t]}\beta(s)\,f'(s)\,ds\right\|\\
&\quad\le\left\|\sum_{i=1}^{|d|}\beta(\xi_i)\left[f(t_i)-f(t_{i-1})\right]-\sum_{i=1}^{|d|}\beta(\xi_i)\,f'(t_i)(t_i-t_{i-1})\right\|\\
&\qquad+\left\|\sum_{i=1}^{|d|}\beta(\xi_i)\,f'(t_i)(t_i-t_{i-1})-(K)\int_{[a,t]}\beta(s)\,f'(s)\,ds\right\|\\
&\quad<\|\beta\|\sum_{i=1}^{|d|}\|f(t_i)-f(t_{i-1})-f'(t_i)(t_i-t_{i-1})\|+\varepsilon\\
&\quad<\|\beta\|\sum_{i=1}^{|d|}\varepsilon(t_i-t_{i-1})+\varepsilon=\|\beta\|\,\varepsilon(t-a)+\varepsilon\,,
\end{aligned}$$

by (7) and by the Kurzweil integrability of $\beta(\cdot)\,f'(\cdot)$. Now (2) follows from (5) and (6). $\square$

3. Pre-requisites for the main results

Suppose $\alpha \in SV([a,b], L(X,Y))$ and $f \in C([a,b],X)$. Then the Riemann-Stieltjes integral $\int_{[a,b]} d\alpha(t) f(t)$ exists (Theorem 6). Furthermore, if we define

$$F_\alpha(f) = \int_{[a,b]} d\alpha(t) f(t) ,$$

then $F_\alpha \in L(C([a,b],X),Y)$ and $\|F_\alpha\| \leq SV(\alpha)$ (see [9], Theorem I.4.6).

If $\alpha \in SV([a,b], L(X,Y))$, then for every $t \in [a,b[$ there is one and only one element $\alpha(t\dot{+}) \in L(X,Y'')$ such that for every $x \in X$ and every $y' \in Y'$,

$$\lim_{\rho \downarrow 0} \langle \alpha(t+\rho) x, y' \rangle = \langle \alpha(t\dot{+}) x, y' \rangle$$

where $\rho > 0$ (see [8], Corollary after I.3.6). Thus given $\alpha \in SV([a,b], L(X,Y))$, we define $\alpha^+(t) = \alpha(t\dot{+})$, whenever $a < t < b$ and $\alpha^+(a) = \alpha(a)$. Then the function α^+ is of bounded semi-variation and we write $\alpha^+ \in SV^+(]a,b], L(X,Y''))$. If in addition $\alpha^+(b) = 0$, then we write $\alpha^+ \in SV_b^+(]a,b], L(X,Y''))$. Moreover,

$$\int_{[a,b]} d\alpha^+(t) f(t) = \int_{[a,b]} d\alpha(t) f(t)$$

for every $f \in C([a,b],X)$ and $\|F_\alpha\| = SV(\alpha^+)$ ([8], Corollary 3.9).

For a proof of the next result, see [8], Corollary I.3.9.

Theorem 9. *For every $F \in L(C([a,b],X),Y)$ there exists one and only one function $\alpha \in SV_b^+(]a,b], L(X,Y''))$ such that $F = F_\alpha$, where $F_\alpha(f) = \int_{[a,b]} d\alpha(t) f(t)$.*

In what follows we present three results (with proofs) due to Hönig. Such results were borrowed from [11]. The next result completes Theorem 9 and characterizes the range of the mapping $F \mapsto \alpha_F$.

Suppose $\alpha \in SV_b^+(]a,b], L(X,Y''))$ is such that $\alpha(a) \in L(X,Y)$ and $\int_{[a,t]} \alpha(s)\, x\, ds \in Y$ for all $t \in [a,b]$ and $x \in X$. Then we have

Theorem 10 (Hönig). *The mapping*

$$\alpha \mapsto F_\alpha \in L(C([a,b],X),Y) ,$$

where $F_\alpha(f) = \int_{[a,b]} d\alpha(t) f(t)$, is an isometry (i.e., $\|F_\alpha\| = SV(\alpha)$) of the first Banach space onto the second. We have $\int_{[a,t]} \alpha(s)\, x\, ds = -F_\alpha(g_{t,x})$ and $\alpha(a) x = -F_\alpha(\chi_{[a,b]} x)$, where for $t \in [a,b]$ and $x \in X$ we define

$$g_{t,x}(s) = \begin{cases} (s-a)\, x, & \text{if} \quad a \leq s \leq t \\ (t-a)\, x, & \text{if} \quad t \leq s \leq b \,. \end{cases}$$

Proof. For $F \in L(C([a,b],X),Y)$, let α be the corresponding element by Theorem 9. We will prove that $\alpha \in SV_b^+(]a,b], L(X,Y''))$ with $\alpha(a) \in L(X,Y)$ and $\int_{[a,t]} \alpha(s)\, x\, ds \in Y$ for all $t \in [a,b]$ and $x \in X$.

Since $g_{t,x} \in C([a,b], X)$, then $F(g_{t,x}) \in Y$. But

$$F(g_{t,x}) = \int_{[a,b]} d\alpha(s)\ g_{t,x}(s) = -\int_{[a,b]} \alpha(s)\ dg_{t,x}(s) = -\int_{[a,t]} \alpha(s)\, x\, ds\,,$$

where we applied Theorem 6 with $\alpha(b) = 0$ and $g_{t,x}(a) = 0$. Analogously, $F(\chi_{[a,b]}x) = -\alpha(a)\, x \in Y$.

On the other hand, the functions $g_{t,x}$ and $\chi_{[a,b]}x$, $t \in [a,b]$ and $x \in X$, form a total subset of $C([a,b], X)$. Because the space of functions $\alpha \in SV_b^+(]a,b], L(X,Y''))$ which satisfy $\alpha(a) \in L(X,Y)$ and $\int_{[a,t]} \alpha(s)\, x\, ds \in Y$, for all $t \in [a,b]$ and $x \in X$, is a closed subspace of $SV_b^+(]a,b], L(X,Y''))$, it follows that the isometry is onto. The proof is complete. ☐

Let $SV_b^+([a,b], L(X,Y''))$ denote the space of all functions $\alpha \in SV_b^+(]a,b], L(X,Y''))$ such that $\alpha(a\dot{+}) = \alpha(a)$. Suppose $\alpha \in SV_b^+([a,b], L(X,Y''))$ with $\int_{[a,t]} \alpha(s)\, x\, ds \in Y$, for all $t \in [a,b]$ and $x \in X$. Then we have

Theorem 11 (Hönig). *The mapping*

$$\alpha \mapsto F_\alpha \in L(C_a([a,b], X), Y)$$

is an isometry of the first Banach space onto the second.

Proof. At first, we prove that the mapping is one-to-one. If α is such that $F_\alpha(f) = 0$, for every $f \in C_a([a,b], X)$, then for all $t \in [a,b]$, $x \in X$ and $y \in Y$, we have

$$0 = \langle F_\alpha(g_{t,x}), y\rangle = -\int_{[a,t]} \langle \alpha(s)\, x, y\rangle\ ds\,.$$

Hence $\alpha = 0$, since by hypothesis $\alpha(s\dot{+}) = \alpha(s)$ for every $s \in [a,b[$.

Now we prove that the mapping is onto. If $\alpha \in SV_b^+(]a,b], L(X,Y''))$ is such that $\int_{[a,t]} \alpha(s)\, x\, ds \in Y$ for all $t \in [a,b]$ and $x \in X$, then we define $\alpha_a(a) = \alpha(a\dot{+})$ and $\alpha_a(t) = \alpha(t)$, $a < t \leq b$. Because $\alpha_a(a\dot{+}) = \alpha_a(a)$ and $F_{\alpha_a}(f) = F_\alpha(f)$ for every $f \in C_a([a,b], X)$, it follows that $\alpha_a \in SV_b^+([a,b], L(X,Y''))$ and $\int_{[a,t]} \alpha(s)\, x\, ds \in Y$ for all $t \in [a,b]$ and $x \in X$.

The isometry follows from Theorem 10. ☐

Given a function $\alpha : (t,s) \in [c,d] \times [a,b] \mapsto \alpha(t,s) \in L(X,Y'')$, we write $\alpha^t(s) = \alpha_s(t) = \alpha(t,s)$. Suppose

- $\widetilde{C}^\sigma$: for every $s \in [a,b]$, the function

$$h_{\alpha,x,s} : t \in [c,d] \mapsto h_{\alpha,x,s}(t) = \int_{[a,s]} \alpha(t,\rho)\, x\, d\rho$$

  is continuous for every $x \in X$, and
- $\left(SV_b^+\right)^u$: for every $t \in [c,d]$, $\alpha^t \in SV_b^+([a,b], L(X,Y''))$ and

$$SV^u(\alpha) = \sup\left\{SV\left(\alpha^t\right); t \in [a,b]\right\} < \infty\,.$$

In this case, we write $\alpha \in \tilde{C}^{\sigma} \times \left(SV_b^+\right)^u \left([c,d] \times [a,b], L\left(X, Y''\right)\right)$. When we consider functions of bounded variation instead of functions of bounded semi-variation, we write $\alpha \in \tilde{C}^{\sigma} \times \left(BV_b^+\right)^u \left([c,d] \times [a,b], L\left(X, Y''\right)\right)$.

Suppose $\alpha \in \widetilde{C}^{\sigma} \times \left(SV_b^+\right)^u \left([a,b] \times [a,b], L\left(X, Y''\right)\right)$ is such that $\int_{[a,t]} \alpha\left(t, \sigma\right) x d\sigma \in Y$, for every $t \in [a,b]$ and every $x \in X$. By $\int_{[a,b]} d_s \alpha\left(t,s\right) f\left(s\right)$ we denote the Riemann-Stieltjes integral approximated by sums of the form

$$\sum_{i=1}^{|d|} \left[\alpha\left(t, s_i\right) - \alpha\left(t, s_{i-1}\right)\right] f\left(\xi_i\right) ,$$

where $t \in [a,b]$ is given, δ is an appropriate constant gauge of $[a,b]$ and $d = \left(s_i, \xi_i\right) \in TD_{[a,b]}$ is δ-fine. Under these assumptions we have

Theorem 12 (Hönig). *The mapping*

$$\alpha \mapsto F_{\alpha} \in L\left(C_a\left([a,b], X\right), C\left([c,d], Y\right)\right) ,$$

where $F_{\alpha}\left(f\right)\left(t\right) = \int_{[a,b]} d_s \alpha\left(t,s\right) f\left(s\right)$, $c \leq t \leq d$, *is an isometry (i.e.,* $\|F_{\alpha}\| = SV^u\left(\alpha\right)$*) of the first Banach space onto the second. We have* $\int_{[a,s]} \alpha\left(t,\sigma\right) x d\sigma = -F_{\alpha}\left(g_{s,x}\right)\left(t\right)$, $a \leq s \leq b$, *and* $\alpha\left(t,a\right) x = -F_{\alpha}\left(\chi_{[a,b]} x\right)\left(t\right)$.

Proof. It follows the steps of [9], Theorem I.5.10 and the remark that follows it (see [9], p. 49-52). The properties of F_{α} follow from the Theorem of Helly (see, for instance, [9]). Reciprocally, given F for every $t \in [c,d]$, it follows by Theorem 11 that the continuous mapping $f \in C_a\left([a,b], X\right) \mapsto F\left(f\right)\left(t\right) \in Y$ can be represented by an $\alpha^t \in SV_b^+\left([a,b], L\left(X, Y''\right)\right)$ with $\int_{[a,s]} \alpha\left(t,\sigma\right) x d\sigma \in Y$, for all $s \in [a,b]$ and $x \in X$. □

Given $g \in SL\left([a,b]\right)$, we denote by $\widetilde{C}_g^{\sigma} \times \left(BV_c^+\right)^u \left([a,b] \times [a,b], L(\mathbb{R})\right)$ the set of all functions $\alpha : [a,b] \times [a,b] \to L\left(\mathbb{R}, \mathbb{R}''\right) \cong L\left(\mathbb{R}\right)$ (we write $\alpha\left(t,s\right) = \alpha^t\left(s\right) = \alpha_s\left(t\right)$) such that

- $\widetilde{C}_g^{\sigma}$: for every $s \in [a,b]$ and every $x \in \mathbb{R}$, the function

$$t \in [a,b] \mapsto \int_{[a,s]} \alpha\left(t,\sigma\right) x \, dg\left(\sigma\right) \in \mathbb{R}$$

  is continuous;
- $\left(BV_b^+\right)^u$: for every $t \in [a,b]$, $\alpha^t \in BV_b^+\left([a,b], L(\mathbb{R})\right)$ and

$$V^u = \sup\left\{V\left(K^t\right); t \in [a,b]\right\} < \infty .$$

When $g\left(t\right) = t$, we write simply $\alpha \in \widetilde{C}^{\sigma} \times \left(BV_c^+\right)^u \left([a,b] \times [a,b], L(\mathbb{R})\right)$.

Suppose $\alpha \in \widetilde{C}_g^{\sigma} \times \left(BV_b^+\right)^u \left([a,b] \times [a,b], L(\mathbb{R})\right)$, where $g \in SL\left([a,b]\right)$ if differentiable and non-constant on any non-degenerate subinterval of $[a,b]$. Under these circumstances we have

Theorem 13. *The mapping*

$$\alpha \mapsto H_{\alpha,g} \in L\left(\mathbf{K}_g\left([a,b], L\left(\mathbb{R}\right)\right), C\left([a,b]\right)\right) ,$$

where $H_{\alpha,g}(f)(t) = (K)\int_{[a,b]} \alpha(t,s) f(s)\, dg(s)$ *for all* $t \in [a,b]$, *is an isometry (i.e.,* $\|H_{\alpha,g}\| = V^u(\alpha)$*) onto. We have*

$$H_{\alpha,g}\left(\chi_{[a,s]} f\right)(t) = (K)\int_{[a,s]} \alpha(t,\sigma) f(\sigma)\, dg(\sigma) \ ,$$

for every $t, s \in [a,b]$ *and every* $f \in \mathbf{K}_g([a,b], L(\mathbb{R}))$.

Proof. At first we will prove that the mapping is one-to-one. Let $t \in [a,b]$. Given $\rho \in [a,b]$ and $x \in L(\mathbb{R})$, we define a function $f_{\rho,x} : [a,b] \to L(\mathbb{R})$ by $f_{\rho,x}(s) = x$, if $s \in \,]a,\rho]$, and by $f_{\rho,x}(s) = 0$ otherwise. Then $f_{\rho,x} \in R_g([a,b], L(\mathbb{R}))$ (see Theorem 6) and $H_{\alpha,g}(f)(t) = (K)\int_{[a,\rho]} \alpha(t,s)\, x\, dg(s)$. Since for every $x \in L(\mathbb{R})$, $\alpha(t,\cdot) : [a,b] \to L(\mathbb{R})$ is such that $\alpha(t,\cdot)\, x \in R_g([a,b], L(\mathbb{R})) \subset K_g([a,b], L(\mathbb{R})) = H_g([a,b], L(\mathbb{R}))$, then the derivative

$$\frac{d}{d\rho}\left((K)\int_{[a,\rho]} \alpha(t,s)\, x\, dg(s)\right) = \alpha(t,\rho)\, x g'(\rho) \ ,$$

exists for m-almost every $\rho \in [a,b]$ by Theorem 4. If $\alpha^t(\cdot)$ is not identically zero, then there exist $\overline{\rho} \in [a,b]$ and $\overline{x} \in L(\mathbb{R})$ such that $\alpha^t(\overline{\rho})\, \overline{x} \neq 0$. Besides, we can suppose, without loss of generality, that $g'(\overline{\rho}) \neq 0$. Since the Kurzweil-Stieltjes integral and, in particular, the Riemann-Stieltjes integral is invariant on sets of m-measure zero (see [2]), then $\frac{d}{d\overline{\rho}}\left(\int_{[a,\overline{\rho}]} \alpha(t,s)\, \overline{x}\, dg(s)\right) = \alpha(t,\overline{\rho})\, \overline{x} g'(\overline{\rho}) \neq 0$ which implies $H_{\alpha,g}(f_{\overline{\rho},\overline{x}})(t) = \int_{[a,\overline{\rho}]} \alpha(t,s)\, \overline{x}\, dg(s)$ is non-constant and therefore different from zero. Thus we proved that the mapping is one-to-one.

By Theorem 7, $\|H_{\alpha,g}\| \leq V^u(\alpha)$.

Given $H \in L(\mathbf{K}_g([a,b], L(\mathbb{R})), C([a,b]))$ and $f \in \mathbf{K}_g([a,b], L(\mathbb{R}))$, let $\widehat{H}\left(\widetilde{f}_g\right)(t) = -H(f)(t)$, for every $t \in [a,b]$. Since $\mathbf{K}_g([a,b], L(\mathbb{R})) = \mathbf{H}_g([a,b], L(\mathbb{R}))$, it follows by Theorem 5 that there exists a unique continuous extension of $\widehat{H}$ to $C_a([a,b])$ which we also denote by $\widehat{H}$. Hence $\widehat{H} \in L(C_a([a,b]), C([a,b]))$. Besides, if α represents $\widehat{H}$, then by Theorem 12 we have $\widehat{H}\left(\widetilde{f}_g\right)(t) = \int_{[a,b]} d_s\alpha(t,s)\, \widetilde{f}_g(s)$, for every $t \in [a,b]$, and $\|\widehat{H}\| = V^u(\alpha)$. Moreover, given $t \in [a,b]$, we have

$$\begin{aligned} H(f)(t) &= -\widehat{H}\left(\widetilde{f}_g\right)(t) = -\int_{[a,b]} d_s\alpha(t,s)\ \widetilde{f}_g(s) \\ &= \int_{[a,b]} \alpha(t,s)\ d\widetilde{f}_g(s) = (K)\int_{[a,b]} \alpha(t,s) f(s)\, dg(s) \ , \end{aligned}$$

where we applied Theorems 6 and 7 respectively in order to obtain the last two equalities. Since by definition $\|f\|_{A,g} = \|\widetilde{f}_g\|_\infty$, it follows that $\|H\| = \|\widehat{H}\| = V^u(\alpha)$. ☐

Lemma 3. *If* $f \in \mathbf{K}_g([a,b], L(\mathbb{R}))$ *and* $\alpha \in \widetilde{C}_g^\sigma \times \left(BV_b^+\right)^u([a,b]\times[a,b], L(\mathbb{R}))$, *where* $g \in SL([a,b])$ *is differentiable and non-constant on any non-degenerate* *subinterval of* $[a,b]$*, then the function*

$$t \in [a,b] \mapsto (K) \int_{[a,t]} \alpha(t,s) f(s)\, dg(s) \in \mathbb{R}$$

is continuous.

Proof. $E([a,b], L(\mathbb{R}))$ is dense in $\mathbf{K}_g([a,b], L(\mathbb{R}))$ in the norm $\|\cdot\|_{A,g}$ by Theorem 8. Thus it is sufficient to prove the result for every step function $f : [a,b] \to L(\mathbb{R})$. We will prove then that for every $x \in L(\mathbb{R})$, the function

$$t \in [a,b] \mapsto \int_{[a,t]} \alpha(t,s)\, x\, dg(s) \in \mathbb{R} \tag{8}$$

is continuous.

Given $\varepsilon > 0$ and $\rho > 0$, we have

$$\int_{[a,t+\rho]} \alpha(t+\rho,s)\, x\, dg(s) = \int_{[a,t]} \alpha(t+\rho,s)\, x\, dg(s) + \int_{[t,t+\rho]} \alpha(t+\rho,s)\, x\, dg(s) \ ,$$

where $\int_{[a,t]} \alpha(t+\rho,s)\, x\, dg(s)$ tends to $\int_{[a,t]} \alpha(t,s)\, x\, dg(s)$ as $\rho \to 0$, by the hypothesis $\widetilde{C}_g^{\sigma}$ for α. Let δ be a gauge of $[a,b]$ from the definition of

$$(K) \int_{[a,b]} \alpha(t,s)\, x\, dg(s) = \int_{[a,b]} \alpha(t,s)\, x\, dg(s)$$

and suppose $\rho < \delta(t)$. Hence $(t, [t, t+\rho])$ is a tagged partial division of $[a,b]$ which is δ-fine and, by Lemma 1, we have

$$\begin{aligned} &\left| \int_{[t,t+\rho]} \alpha(t+\rho,s)\, x\, dg(s) \right| \\ &\quad \leq \left| \int_{[t,t+\rho]} \alpha(t+\rho,s)\, x\, dg(s) - \alpha(t+\rho,t)\, x\, [g(t+\rho) - g(t)] \right| \\ &\qquad + |\alpha(t+\rho,t)\, x\, [g(t+\rho)-g(t)]| < \varepsilon + |\alpha(t+\rho,t)\, x\, [g(t+\rho)-g(t)]| \ . \end{aligned}$$

Now, since $\alpha^t(b) = 0$ for every $t \in [a,b]$, it follows that

$$\begin{aligned} &|\alpha(t+\rho,t)\, x\, [g(t+\rho)-g(t)]| = |[\alpha(t+\rho,t) - \alpha(t+\rho,b)]\, x\, [g(t+\rho)-g(t)]| \\ &\quad \leq V\left(\alpha^{t+\rho}\right) \|x\|\, |g(t+\rho)-g(t)| \leq V^u(\alpha)\, \|x\|\, |g(t+\rho)-g(t)| \ . \end{aligned}$$

Therefore, from the continuity of g, it follows that the function in (8) is right continuous.

In an analogous way we prove the left continuity of (8). □

Let $f\mid_B$ denote the restriction of a function $f : [a,b] \to X$ to a set $B \subset [a,b]$.

Definition 7. An operator $H \in L(\mathbf{K}([a,b], X), C([a,b], Y))$ is *causal* if given $f \in \mathbf{K}([a,b], X)$ and $t \in [a,b]$, then $f\mid_{[a,t]} = 0$ implies $H(f)\mid_{[a,t]} = 0$. Given $g \in SL([a,b])$ differentiable and non-constant on any non-degenerate subinterval of $[a,b]$, an operator $H \in L(\mathbf{K}_g([a,b], L(\mathbb{R})), C([a,b]))$ will also be called *causal* if given $f \in \mathbf{K}([a,b], L(\mathbb{R}))$ and $t \in [a,b]$, then $f\mid_{[a,t]} = 0$ implies $H(f)\mid_{[a,t]} = 0$.

Given $g \in SL([a,b])$ differentiable and non-constant on any non-degenerate subinterval of $[a,b]$, suppose $\alpha \in \widetilde{C}_g^\sigma \times \left(BV_b^+\right)^u([a,b]\times[a,b], L(\mathbb{R}))$ with $\alpha(t,s) = 0$ for m-almost every $s > t$. Under these circumstances the next result holds.

Theorem 14. *The mapping*

$$\alpha \mapsto H_{\alpha,g} \in L\left(\mathbf{K}_g([a,b], L(\mathbb{R})), C([a,b])\right),$$

where $H_{\alpha,g}(f)(t) = (K)\int_{[a,b]} \alpha(t,s) f(s)\, dg(s)$ for every $t \in [a,b]$, is an isometry (i.e., $\|H_{\alpha,g}\| = V^u(\alpha)$) onto the space of causal operators.

Proof. In view of Theorem 13, it is enough to show that the mapping is onto. If $\alpha \in \widetilde{C}_g^\sigma \times \left(BV_b^+\right)^u([a,b]\times[a,b], L(\mathbb{R}))$ and $\alpha(t,s) = 0$ for m-almost every $s > t$, then $H_{\alpha,g}$ is causal by Lemma 3. Reciprocally, let $H \in L(\mathbf{K}_g([a,b], L(\mathbb{R})), C([a,b]))$ be causal. By Theorem 13, there exists one and only one $\alpha \in \widetilde{C}_g^\sigma \times (BV_b^+)^u([a,b]\times[a,b] \to L(\mathbb{R}))$ such that $H = H_{\alpha,g}$.

From the causality of $H = H_{\alpha,g}$, it follows that $0 = H_{\alpha,g}\left(\chi_{[t,b]}x\right)(t) = \int_{[t,b]} \alpha(t,\sigma)\, x\, dg(\sigma)$, for every $t \in [a,b]$ and every $x \in L(\mathbb{R})$. Besides, $\alpha^t(\cdot)\, x \in R_g([a,b], L(\mathbb{R})) \subset K_g([a,b], L(\mathbb{R})) = H_g([a,b], L(\mathbb{R}))$ and, therefore, $\alpha(t,s) = 0$ for m-almost every $s > t$, by the Corollary after Theorem 4. $\square$

4. The main results

The next two results give the analogue of Hönig's result for the vector case mentioned in the abstract. They were borrowed from [1].

Suppose $g \in SL([a,b])$ is differentiable and non-constant on any non-degenerate subinterval of $[a,b]$ and $\alpha \in \widetilde{C}_g^\sigma \times \left(BV_b^+\right)^u([a,b]\times[a,b], L(\mathbb{R}))$ with $\alpha(t,s) = 0$, for m-almost every $s > t$. Then we have

Theorem 15. *For every $f \in \mathbf{K}_g([a,b], L(\mathbb{R}))$, the linear integral equation of Volterra-Kurzweil-Henstock-Stieltjes*

$$x(t) - (K)\int_{[a,t]} \alpha(t,s)\, x(s)\, dg(s) = f(t), \qquad t \in [a,b]. \tag{9}$$

admits one and only one solution $x_f \in \mathbf{K}_g([a,b], L(\mathbb{R}))$. Moreover, the bijection $f \mapsto x_f$ is causal and can be written as

$$x_f(t) = f(t) - (K)\int_{[a,t]} \phi(t,s)\, f(s)\, dg(s), \qquad t \in [a,b],$$

where $\phi \in \widetilde{C}_g^\sigma \times \left(BV_b^+\right)^u([a,b]\times[a,b], L(\mathbb{R}))$ with $\phi(t,s) = 0$, for m-almost every $s > t$.

Theorem 16. *Suppose the conditions of Theorem 15 are satisfied. Then the Neumann series $I - H_{\phi,g} = I + H_{\alpha,g} + \left(H_{\alpha,g}\right)^2 + \left(H_{\alpha,g}\right)^3 + \ldots$ converges in $L\left(\mathbf{K}_g([a,b], L(\mathbb{R}))\right)$, where $H_{\alpha,g}(f)(t) = (K)\int_{[a,b]} \alpha(t,s) f(s)\, dg(s)$ for every $t \in [a,b]$.*

In order to prove Theorems 15 and 16, we need some lemmas.

We denote by $C^\sigma \times (SV)^u([a,b]\times[a,b], L(X,Y))$ the set of all functions $K : [a,b]\times[a,b] \to L(X,Y)$ (we write $K(t,s) = K^t(s) = K_s(t)$) such that

- C^σ : for every $s\in[a,b]$ and every $x\in X$, the function
$$K(\cdot,s)\,x : t\in[a,b] \mapsto K_s(t)\,x \in Y$$
is continuous;
- $(SV)^u$: for every $t\in[a,b]$, $K^t \in SV([a,b], L(X,Y))$ and
$$SV^u = \sup\left\{SV\left(K^t\right); t\in[a,b]\right\} < \infty\,.$$

If in addition $K^t \in SV_a([a,b], L(X,Y))$ for every $t\in[a,b]$, then we write $K \in C^\sigma \times (SV_a)^u([a,b]\times[a,b], L(X,Y))$.

Suppose $K \in C^\sigma \times (SV)^u([a,b]\times[a,b], L(X'))$ and $y\in C([a,b], L(\mathbb{R},X))$. By $\int_{[a,b]} d_sK(t,s)\;y(s)$ we mean the Riemann-Stieltjes integral approximated by sums of the form $\sum_{i=1}^{|d|}[K(t,s_i) - K(t,s_{i-1})]\,y(\xi_i)$, where $t\in[a,b]$ is given, δ is an appropriate constant gauge of $[a,b]$ and $d = (s_i,\xi_i) \in TD_{[a,b]}$ is δ-fine.

Consider $g\in SL([a,b])$ differentiable and non-constant on any non-degenerate subinterval of $[a,b]$ and $\alpha \in \widetilde{C}_g^\sigma \times \left(BV_b^+\right)^u([a,b]\times[a,b], X')$. Under these conditions we have

Lemma 4. *If $K_g : [a,b]\times[a,b] \to X'$ is defined by*
$$K_g(t,s)\,x = \int_{[a,s]} \alpha(t,\sigma)\,x\,dg(\sigma)$$
for every $x\in L(\mathbb{R},X)$, then $K_g \in C^\sigma \times (SV_a)^u([a,b]\times[a,b], X')$ and

$$\int_{[a,b]} d_sK_g(t,s)\;y(s) = (K)\int_{[a,b]} \alpha(t,s)\,y(s)\;dg(s)\;, \tag{10}$$

for every $y\in C([a,b], L(\mathbb{R},X))$ and every $t\in[a,b]$.

Proof. For every $s\in[a,b]$, the function $\alpha_s : [a,b]\to X'$ satisfies property $\widetilde{C}_g^\sigma$. Thus $K_g(\cdot,s) : [a,b]\to X'$ satisfies property C^σ. Let $\varepsilon>0$, $t\in[a,b]$ and $d=(s_i)$ be a division of $[a,b]$. We may suppose, without loss of generality, that $d = (\xi_i, s_i) \in TD_{[a,b]}$ is δ-fine, where δ is the gauge of $[a,b]$ from the definition of $(K)\int_{[a,b]} dh(t)$, for $h = |g|$. Hence for $x_i\in L(\mathbb{R},X)$ with $\|x_i\|\le 1$ for every i, we have

$$\begin{aligned}\Big|\sum_i [K_g\;(t,s_i) - K_g(t,s_{i-1})]\,x_i\Big| &= \Big|\sum_i \int_{[s_{i-1},s_i]} \alpha(t,\sigma)\,x_i\,dg(\sigma)\Big| \\ &\le \sum_i \|\alpha^t\|_\infty \|x_i\| \int_{[s_{i-1},s_i]} dh(\sigma) \le \|\alpha^t\|_\infty (K)\int_{[a,b]} dh(\sigma) \\ &\le V\left(\alpha^t\right)|g(b)-g(a)|\;,\end{aligned}$$

where $h = |g|$. Therefore $K_g(\cdot, s) \in SV([a,b], L(X, \mathbb{R})) = BV([a,b], X')$, for every $t \in [a,b]$. Now it is easy to see that $K_g \in C^{\sigma} \times (SV_a)^u([a,b] \times [a,b], X')$.

By Theorem 6, the Riemann-Stieltjes integral $\int_{[a,b]} d_s K_g(t,s)\, y(s)$ exists for every $y \in C([a,b], L(\mathbb{R}, X))$ and every $t \in [a,b]$. Thus it remains to prove that equation (10) holds for every $y \in C([a,b], L(\mathbb{R}, X))$ and every $t \in [a,b]$. As a matter of fact, it is enough to show that (10) holds whenever y is a step function. In this manner, we will prove that given $t \in [a,b]$ and $x \in L(\mathbb{R}, X)$, we have

$$\frac{d}{ds}(K_g(t,s)\, x) = \frac{d}{ds}\Big((K)\int_{[a,s]} \alpha(t,\sigma)\, x\, dg(\sigma)\Big) = \alpha(t,s)\, x g'(s),$$

for m-almost every $s \in [a,b]$. By [2], Theorem 9, we have

$$\int_{[a,b]} \frac{d}{ds}(K_g(t,s)\, x)\; ds = \int_{[a,b]} d_s K_g(t,s)\; x.$$

Then, from the invariance of the integral over sets of m-measure zero, it will follow that

$$\int_{[a,b]} d_s K_g(t,s)\; x = \int_{[a,b]} \alpha(t,s)\, x\, g'(s)\; ds. \tag{11}$$

We will also prove that

$$\int_{[a,b]} \alpha(t,s)\, x\, dg(s) = \int_{[a,b]} \alpha(t,s)\, x\, g'(s)\; ds. \tag{12}$$

Thus the result will follow from (11) and (12).

Given $\varepsilon > 0$, $t \in [a,b]$ and $x \in L(\mathbb{R}, X)$, let δ_1 and δ_2 be constant gauges from the definitions of $\int_{[a,b]} \alpha(t,s)\, x\, dg(s)$ and $\int_{[a,b]} \alpha(t,s)\, x g'(s)\; ds$ respectively. For every $\xi \in [a,b]$, let $\delta_3(\xi) > 0$ be such that if $\xi - \delta_3(\xi) < s < \xi < v < \xi + \delta_3(\xi)$, then

$$|g(v) - g(s) - g'(\xi)(v - s)| < \varepsilon (v - s) \tag{13}$$

(see Lemma 2). Define a gauge δ of $[a,b]$ by $\delta(\xi) = \min\{\delta_j(\xi)\,; j = 1,2,3\}$. Then for every δ-fine $d = (\xi_i, s_i) \in TD_{[a,b]}$, we have

$$\begin{aligned}
&\Big| \int_{[a,b]} \alpha(t,s)\, x\, dg(s) - \int_{[a,b]} \alpha(t,s)\, x g'(s)\; ds \Big| \\
&\quad \le \Big| \int_{[a,b]} \alpha(t,s)\, x\, dg(s) - \sum_i \alpha(t,\xi_i)\, x\, [g(s_i) - g(s_{i-1})] \Big| \\
&\qquad + \Big| \sum_i \alpha(t,\xi_i)\, x\, [g(s_i) - g(s_{i-1})] - \sum_i \alpha(t,\xi_i)\, x g'(\xi_i)(s_i - s_{i-1}) \Big| \\
&\qquad + \Big| \sum_i \alpha(t,\xi_i)\, x g'(\xi_i)(s_i - s_{i-1}) - \int_{[a,b]} \alpha(t,s)\, x g'(s)\; ds \Big| \\
&\quad < \varepsilon + \left\|\alpha^t\right\|_\infty \|x\| \sum_i |g(s_i) - g(s_{i-1}) - g'(\xi_i)(s_i - s_{i-1})| + \varepsilon \\
&\quad < 2\varepsilon + \left\|\alpha^t\right\|_\infty \|x\|\, \varepsilon (b - a)\;,
\end{aligned}$$

by the integrability of $\alpha^t(\cdot)\, x$ with respect to g, by the integrability of $\alpha^t(\cdot)\, x\, g'(\cdot)$ and by (13). $\square$

Suppose $K \in C^\sigma \times (SV)^u([a,b]\times[a,b], L(X))$ and there exists a division (s_i) of $[a,b]$ such that for every i, $\sup\left\{SV_{[s_{i-1},t]}(K^t); t \in [s_{i-1}, s_i]\right\} < 1$, where $SV_{[s_{i-1},t]}(K^t)$ denotes the semi-variation of $K^t(\cdot)$ on $[s_{i-1}, t]$. Under these conditions we have

Lemma 5. *The following assertions are equivalent.*
(i) *For every $t \in [a,b]$, the operator $I - K(t+,t)$ is invertible, where I is the identity in $L(X)$ and for every $x \in X$, $K(t+,t) = K(t,t)\,x$, where $K(t+,t) = \lim_{\rho\downarrow 0} K(t+\rho, t)\, x$;*
(ii) *For every $h \in C([a,b], X)$, the equation*

$$y(t) - \int_{[a,t]} d_s K(t,s)\; y(s) = h(t)\;, \qquad t \in [a,b]\;,$$

admits one and only one solution $y_h \in C([a,b], X)$ and the operator $h \mapsto y_h$ is causal.

For a proof of Lemma 5, see [10], Theorems 3.8 and 3.4. For a proof of the next lemma, see [10], Theorem 3.9

Lemma 6 (Arbex). *Suppose $K \in C^\sigma \times (SV)^u([a,b]\times[a,b], L(X))$, with $K(t,s) = 0$ for $s > t$, and there is a division (s_i) of $[a,b]$ such that for every i,*

$$\sup\left\{SV_{[s_{i-1},t]}\left(K^t\right); t \in [s_{i-1}, s_i]\right\} < 1\,.$$

Then K has resolvent given by the Neumann series.

Lemma 7 (Hönig). *Let E be a normed space and F be a Banach space such that $F \subset E$ with continuous immersion. Suppose that $H \in L(E,F)$ is such that for every $f \in E$, the equation $x - Hx = f$ admits one and only one solution $x_f \in E$. Then $f \in E \mapsto x_f \in E$ is a bicontinuous mapping and, if the Neumann series $I + H + H^2 + H^3 + \ldots = (I-H)^{-1}$ converges in $L(F)$, then it also converges in $L(E)$.*

Proof. At first we will prove that for every $g \in F$, the equation $y - Hy = g$ has one and only one solution $y_g \in F$. Because $F \subset E$, the equation $y - Hy = g$, with $g \in F$, has one and only one solution $y_g \in E$ by hypothesis. But since $HE \subset F$, then $Hy_g \in F$ and therefore $y_g = Hy_g + g \in F$. On the other hand, the mapping $y \in F \mapsto g = y - Hy \in F$ is a continuous bijection. Hence, by the Closed Graph Theorem, its inverse $g \in F \mapsto y_g \in F$ is continuous.

Now we will prove that the mapping $g \in F \mapsto y_g \in F$ is bicontinuous. The equation $x - Hx = f$, with $x, f \in E$, is equivalent to the equation $y - Hy = g$, with $g = Hf$ and $y = x - f$. The mapping $f \in E \mapsto g = Hf \in E$ is continuous. By the previous paragraph, the mapping $Hf \in F \mapsto y_{Hf} \in F$ is also continuous. Since the mapping $y_{Hf} \in F \mapsto y_{Hf} \in E$ is continuous, it follows that the composed mapping $f \in E \mapsto y_{Hf} \in E$ is also continuous and so is the mapping $f \in E \mapsto y_{Hf} + f \in E$. The result follows from $y_{Hf} = x_f - f$, or else, $x_f = y_{Hf} + f$.

Finally, suppose $(I-H)^{-1}=I+H+H^2+H^3+\dots$ is convergent in $L(F)$. Since $H\in L(E,F)$ and the immersion $F\hookrightarrow E$ is continuous, it follows that the series is also convergent in $L(E)$. It is immediate that if a Neumann series is convergent in some $L(Z)$, with Z not necessarily complete, then it converges to $(I-H)^{-1}$. □

Now we are able to prove Theorems 15 and 16.

Proof of Theorem 15. Let $y=x-f$ and $h(t)=(K)\int_{[a,t]}\alpha(t,s)\,f(s)\,dg(s)\,,t\in[a,b]$. Then both functions h and y are continuous by Theorem 14, since $h=H_{\alpha,g}\left(\chi_{[a,t]}f\right),\,y=H_{\alpha,g}\left(\chi_{[a,t]}x\right)$ and $H_{\alpha,g}\in L\left(\mathbf{K}_g\left([a,b]\,,L(\mathbb{R})\right),C([a,b])\right)$. Hence equation (9) is equivalent to equation

$$y(t)-(K)\int_{[a,t]}\alpha(t,s)\;y(s)\,dg(s)=h(t)\,,\qquad t\in[a,b]\;. \tag{14}$$

Let $K_g:[a,b]\times[a,b]\to L\left(L(\mathbb{R}),\mathbb{R}\right)$ be such that

$$K_g(t,s)\,x=\int_{[a,s]}\alpha(t,\sigma)\,xdg(\sigma)\;,$$

for each $x\in L(\mathbb{R})$. By Lemma 4,

$$\int_{[a,t]}d_sK(t,s)\;y(s)=(K)\int_{[a,t]}\alpha(t,s)\;y(s)\,dg(s)$$

and therefore equation (14) is equivalent to

$$y(t)-\int_{[a,t]}d_sK(t,s)\;y(s)=h(t)\;,\qquad t\in[a,b]\;. \tag{15}$$

Now we will prove that given $h\in C([a,b])$, equation (15) admits one and only one solution $y_h\in C([a,b])$ and the operator $h\mapsto y_h$ is causal.

By Lemma 4, the function $K_g:[a,b]\times[a,b]\to L\left(L(\mathbb{R}),\mathbb{R}\right)\cong L(\mathbb{R})$ belongs to $C^\sigma\times(SV_a)^u\left([a,b]\times[a,b]\,,L(\mathbb{R})\right)=C^\sigma\times(BV_a)^u\left([a,b]\times[a,b]\,,L(\mathbb{R})\right)$. In view of Lemma 5, it is sufficient to show that the following conditions are satisfied:

(i) there is a division (s_i) of $[a,b]$ such that for every i,

$$\sup_{t\in[s_{i-1},s_i]}\left\{SV_{[s_{i-1},t]}\left(K_g(t,\cdot)\right)\right\}<1;$$

(ii) for each $t\in[a,b[$, the operator $I-K_g(t+,t)$ is invertible.

Proof of (ii). Since $g\in SL([a,b])$, given $\varepsilon>0$ with $\varepsilon<1/2V^u(\alpha)$ and $t\in[a,b]$, there exists $\delta(t)>0$ such that $|g(t+\rho)-g(t)|<\varepsilon$, whenever $0<\rho<\delta(t)$. Hence for every $x\in L(\mathbb{R})$, we have

$$\begin{aligned}|K_g(t+\rho,t)\,x| &= \left|\int_{[t,t+\rho]}\alpha(t+\rho,\sigma)\,x\,dg(\sigma)\right|\\ &\le \|\alpha^{t+\rho}\|_\infty\,\|x\|\int_{[t,t+\rho]}d\,|g|(\sigma)\\ &\le V^u(\alpha)\,\|x\|\,|g(t+\rho)-g(t)|\end{aligned}$$

which tends to zero as $\rho \to 0$. Thus $K_g(t+,t) = 0$ and therefore $I - K_g(t+,t)$ is invertible.

Proof of (i). Let δ be a gauge of $[a,b]$ defined as above and let $d = (\xi_i, s_i) \in TD_{[a,b]}$ be δ-fine. Given $i \in \{1, 2, \ldots, |d|\}$ and $t \in [s_{i-1}, s_i]$, let $d' = (r^j)$ be a division of $[s_{i-1}, t]$. Then for $x_j \in L(\mathbb{R})$ with $\|x_j\| \leq 1$, we have

$$\begin{aligned}\Big|\sum_{j=1}^{|d'|}\left[K_g\left(t, r^j\right) - K_g\left(t, r^{j-1}\right)\right] x_j\Big| &= \Big|\sum_{j=1}^{|d'|}\int_{[r^{j-1}, r^j]} \alpha(t,\sigma)\, x_j\, dg(\sigma)\Big| \\ &\leq \|\alpha^t\|_\infty \int_{[s_{i-1},t]} d|g|(\sigma) \\ &\leq V^u(\alpha)\,|g(t+\rho) - g(t)| < V^u(\alpha)\,\varepsilon < \frac{1}{2}\,.\end{aligned}$$

Hence $\sup\limits_{t \in [s_{i-1}, s_i]} \left\{SV_{[s_{i-1},t]}\left(K_g(t,\cdot)\right)\right\} < 1$ and we have (i). □

Proof of Theorem 16. By Lemma 6 the assertion about the Neumann series is true for the resolvent of equation (15) in $L(C([a,b]))$. By Lemma 7, the same applies to the resolvent of equation (15) in $L(\mathbf{K}_g([a,b]))$. Also by Lemma 6, the operator F_{K_g} given by

$$F_{K_g} y(t) = \int_{[a,t]} d_s K_g(t,s)\; y(s)\ ,$$

for every $t \in [a,b]$, is causal as well as $\left(F_{K_g}\right)^n$ and therefore $\left(I - F_{K_g}\right)^{-1}$. From the fact that $F_{K_g} = H_{\alpha,g}$ (Lemma 4), it follows that $(I - H_{\phi,g}) = (I - H_{\alpha,g})^{-1}$ is causal. □

In what follows we make some final comments and give examples.

All results in which differentiability is required could be weakened to differentiability m-almost everywhere.

Since the spaces $L(\mathbb{R})$ and $\mathbb{R}$ are isomorphic, we can consider real-valued function in Theorems 15 and 16.

If in Theorem 15 we do not assume that $\alpha(t,s) = 0$, for m-almost every $s > t$, then similar conclusions (not regarding causality) can be obtained by applying Theorem 13 instead of Theorem 14. We give next an application in this direction.

Example. Let $g \in SL([a,b]) \cap BV([a,b])$ be differentiable and non-constant on any non-degenerate subinterval of $[a,b]$. If $f : [a,b] \to \mathbb{R}$ is a regulated function, that is, f has only discontinuities of the first kind (consult [9] for the definition and main properties), then the Kurzweil-Henstock integral $(K)\int_{[a,b]} f(t)\,dg(t)$ exists (see [23], Theorem 15). Now, let $\alpha \in BV_b^+([a,b])$ and consider the initial value problem

$$\begin{cases} \frac{d}{dt}x(t) = \alpha(t)\,x(t)\,\frac{d}{dt}g(t) + \frac{d}{dt}f(t)\,, \\ x(a) = f(a)\ . \end{cases} \tag{16}$$

Integrating (16) in the sense of the Kurzweil-Henstock integral we obtain the Volterra-Stieltjes integral equation

$$x(t) - (K)\int_{[a,t]} \alpha(s)\,x(s)\;dg(s) = f(t), \qquad t \in [a,b],$$

which admits a unique solution $x_f \in \mathbf{K}_g([a,b])$ given by

$$x_f(t) = f(t) - (K)\int_{[a,t]} \phi(t,s)\,f(s)\;dg(s), \qquad t \in [a,b],$$

where $\phi \in \widetilde{C}^{\sigma}_{g} \times \left(BV_b^{+}\right)^{u}([a,b]\times[a,b],\mathbb{R})$. Such a solution is a solution in the sense of Henstock (called Henstock solution - see [22]) of problem 16.

When $g(t) = t$ in Theorems 15 and 16, we obtain Hönig's result specialized for the real-valued case which is:

Corollary 2. *Suppose* $\alpha \in \widetilde{C}^{\sigma} \times \left(BV_b^{+}\right)^{u}([a,b]\times[a,b],L(\mathbb{R}))$ *with* $\alpha(t,s) = 0$ *for m-almost every* $s > t$. *Then for every* $f \in \mathbf{K}([a,b],L(\mathbb{R}))$*, the linear integral equation of Volterra-Kurzweil-Henstock*

$$x(t) - (K)\int_{[a,t]} \alpha(t,s)\,x(s)\;ds = f(t)\;, \qquad t \in [a,b]\;.$$

admits one and only one solution $x_f \in \mathbf{K}([a,b],L(\mathbb{R}))$. *The bijection* $f \to x_f$ *is causal and can be written as*

$$x_f(t) = f(t) - (K)\int_{[a,t]} \phi(t,s)\,f(s)\,ds, \qquad t \in [a,b],$$

where $\phi \in \widetilde{C}^{\sigma} \times \left(BV_b^{+}\right)^{u}([a,b]\times[a,b],L(\mathbb{R}))$ *and* $\phi(t,s) = 0$ *for m-almost every* $s > t$. *Moreover, the Neumann series* $I - H_{\phi} = I + H_{\alpha} + (H_{\alpha})^2 + (H_{\alpha})^3 + \dots$ *converges in* $L(\mathbf{K}([a,b],L(\mathbb{R})))$*, where* $H_{\alpha}(f)(t) = (K)\int_{[a,b]} \alpha(t,s)\,f(s)\,ds$ *for every* $t \in [a,b]$.

An application of Corollary 2 is given next.

Example. When we consider only continuously differentiable functions, the initial value problem

$$(17) \qquad \begin{cases} \frac{d^2}{dt}x(t) + A(t)\,\frac{d}{dt}x(t) + B(t)\,x(t) = h(t)\;, \\ x(a) = c_1\,, \\ \frac{d}{dt}x(a) = c_2\;. \end{cases}$$

can be represented by the linear integral equation of the second kind

$$(18) \qquad x(t) - \int_{[a,t]} \alpha(t,s)\,x(s)\,ds = f(t)\;, \qquad t \in [a,b]\;,$$

where

$$f(t) = \int_{[a,t]} (t-s)\,h(s)\;ds + (t-a)\left[A(a)\,c_1 + c_2\right] + c_1$$

and

$$\alpha(t,s) = (s-t)\left[B(s) - \frac{d}{ds}A(s)\right] - A(s)\ ,$$

the integral being that of Riemann. If however we take $h \in \mathbf{K}([a,b])$ (say, $[a,b] = [0,1]$, $h = \frac{d}{dt}H$ and $H(t) = t^2 \sin(1/t^2)$, for $t \in]0,1]$), then $f \in \mathbf{K}([a,b])$ since for every $t \in [a,b]$, the function $s \mapsto t-s$ belongs to $BV([a,b])$ (see Theorem 7). Suppose $A(t)$, $\frac{d}{dt}A(t)$ and $B(t)$ are such that $\alpha \in \widetilde{C}^{\sigma} \times \left(BV_b^+\right)([a,b]\times[a,b],\mathbb{R})$ and $\alpha(t,s) = 0$, for m-almost every $s > t$. Then by Corollary 2 one can obtain an explicit solution of equation (18) by means of Neumann series method. And this solution is the unique Henstock solution of the initial value problem (17).

For other examples, including an application of the Neumann series method of iterated kernels, consult [4].

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