ARCHIVUM MATHEMATICUM (BRNO)
Tomus 37 (2001), 329 – 332

RANDOM FIXED POINTS OF INCREASING COMPACT RANDOM MAPS

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Abstract. Let (Ω, Σ) be a measurable space, (E, P) be an ordered separable Banach space and let $[a, b]$ be a nonempty order interval in E. It is shown that if $f : \Omega \times [a, b] \to E$ is an increasing compact random map such that $a \le f(\omega, a)$ and $f(\omega, b) \le b$ for each $\omega \in \Omega$ then f possesses a minimal random fixed point α and a maximal random fixed point β.

1. Introduction

Špaček [13] and Hans [5,6] initiated the study of random fixed point theorems for random contraction mappings on Polish spaces. Subsequently Bharucha-Reid [4] has given sufficient conditions for a stochastic analogue of Schauder's fixed point theorem for a random operator. Itoh [7] introduced random condensing operators and considerably improved the known results. Recently Sehgal and Waters [12], Papageorgiou [10], Beg et al [1, 2], Tan and Yuan [14], Lishan [9] and many other authors have studied the fixed points of random maps. In this paper we shall consider stochastic version of a very interesting theorem regarding minimal fixed points of increasing compact maps defined on ordered Banach spaces.

2. Ordered Banach Spaces

Let E be a real Banach space. A cone P of E induces an ordering $\le$ by setting $x \le y$ if and only if $y - x \in P$. By an *ordered Banach space*, denoted by (E, P), we mean a Banach space E together with an ordering $\le$ induced by a cone P, the positive cone of E. The norm of an ordered Banach space E is called *monotone* if $0 \le x \le y$ implies $||x|| \le ||y||$ and *semi-monotone* if there exists a constant r such that $0 \le x \le y$ implies $||x|| \le r||y||$. The positive cone is called *normal* if the norm is semi-monotone. The *order interval* $[x, y]$ is defined by

$$[x, y] = \{z \in E : x \le z \le y\} = (x + P) \cap (y - P)\,.$$

2000 *Mathematics Subject Classification*: 47H07, 47H40, 47H10, 60H25.
Key words and phrases: random fixed point, random map, measurable space, ordered Banach space.
Received July 10, 2000.

We now state a characterization of normal cones for subsequent use in Section 4 (for proofs see [8, 11, 15]).

Theorem 2.1. *Let (E, P) be an ordered Banach space. Then the following statements are equivalent:*

(i) *P is normal;*

(ii) *every order interval is bounded;*

(iii) *there exists an equivalent monotone norm.*

3. Random Maps

Let (Ω, Σ) be a measurable space (Σ = sigma algebra) and K a nonempty subset of a metric space M. A mapping $\xi : \Omega \to M$ is *measurable* if and only if $\xi^{-1}(U) \in \Sigma$ for each open subset U of M. The mapping $f : \Omega \times K \to M$ is a *random map* if and only if for each fixed $x \in K$, the mapping $f(., x) : \Omega \to M$ is measurable. We denote by $f^n(\omega, x)$ the n-th iterate $f(\omega, f(\omega, \dots f(\omega, x) \dots)))$ of f.

Definition 3.1. Let X be a nonempty subset of a Banach space E and $f : \Omega \times X \to E$ be a random map. Then f is called *compact* if $f(\omega, .)$ is continuous and $cl\{f(\omega, x) : x \in X\}$ is compact for each $\omega \in \Omega$. The random map f is called *completely continuous* if f is compact on bounded subsets of X.

For more details and other related results we refer to [3, 4].

4. Random Fixed Points

Definition 4.1. Let X be a nonempty subset of an ordered Banach space E and $f : \Omega \times X \to E$ be a random map. A measurable mapping $\xi : \Omega \to E$ is a *random fixed point* of the random map f if and only if $f(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$. A random fixed point ξ of f is called *minimal (maximal)* random fixed point if every random fixed point η of f satisfies $\xi(\omega) \leq \eta(\omega)$ $(\eta(\omega) \leq \xi(\omega))$ for each $\omega \in \Omega$.

Definition 4.2. Let (E, P) be an ordered Banach space and X be a nonempty subset of E. A random map $f : \Omega \times X \to E$ is called *increasing* if $x \leq y$ implies $f(\omega, x) \leq f(\omega, y)$ for each $\omega \in \Omega$.

Theorem 4.3. *Let (E, P) be an ordered separable Banach space and let $[a, b]$ be a nonempty order interval in E. Suppose $f : \Omega \times [a, b] \to E$ is an increasing compact random map such that $a \leq f(\omega, a)$ and $f(\omega, b) \leq b$ for each $\omega \in \Omega$. Then f possesses a minimal random fixed point α and a maximal random fixed point β.*

Proof. Since f is increasing with $a \leq f(\omega, a)$ and $f(\omega, b) \leq b$ for each $\omega \in \Omega$. It follows that f maps $\Omega \times [a, b]$ into $[a, b]$. Hence the sequence $\{f^n(\omega, a)\}$ is well-defined and it is increasing and relatively compact. This implies the convergence of the whole sequence $\{f^n(\omega, a)\}$ towards its only limit point $\alpha(\omega)$. Since X is separable therefore α is measurable. As f is continuous,

$$\alpha(\omega) = \lim_{n\to\infty} f^n(\omega, a) = f(\omega, \lim_{n\to\infty} f^n(\omega, a)) = f(\omega, \alpha(\omega)),$$

for each $\omega \in \Omega$. If ξ is an arbitrary random fixed point of f, then by replacing b by $\xi(\omega)$ in the above argument, it follows that $\alpha(\omega) \in [a, \xi(\omega)]$. Hence α is the minimal random fixed point of f. The assertion concerning the maximal random fixed point β follows by an analogous argument. $\square$

Corollary 4.4. *Let (E, P) be an ordered separable Banach space with normal positive cone, and let $f : \Omega \times P \to E$ be a completely continuous increasing map. The f has a minimal random fixed point if and only if f has a random fixed point at all i.e. if and only if there exists a measurable $\beta : \Omega \to P$ such that $f(\omega, \beta(\omega)) \leq \beta(\omega)$ for every $\omega \in \Omega$.*

Proof. The proof follows from the Theorems 2.1 and 4.3 and the fact that $f(\omega, 0) \geq 0$. $\square$

Remark 4.5. We do not assert the existence of a maximal random fixed point in P. The existence of a random fixed point in the order interval $[0, b]$ is an immediate consequence of Schauder's random fixed point theorem. For many applications it is of great importance that there exists a minimal random fixed point. It should be observed that minimal random fixed point can be computed interatively since $\alpha(\omega) = \lim_{n\to\infty} f^n(\omega, 0(\omega))$.

Acknowledgement. This work is supported by the Kuwait University research grant number SM - 03/00.

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