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SOLUTION OF A CAUCHY–JENSEN STABILITY ULAM TYPE PROBLEM

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Abstract. In 1978 P.M. Gruber (Trans. Amer. Math. Soc. 245 (1978), 263–277) imposed the following general problem or Ulam type problem: "Suppose a mathematical object satisfies a certain property approximately. Is it then possible to approximate this objects by objects, satisfying the property exactly?"

The afore-mentioned problem of P. M. Gruber is more general than the following problem imposed by S. M. Ulam in 1940 (Intersci, Publ., Inc., New York 1960): "Give conditions in order for a linear mapping near an approximately linear mapping to exist".

In 1941 D.H. Hyers (Proc. Nat. Acad. Sci., U.S.A. 27 (1941), 411–416) solved a special case of Ulam problem. In 1989 and 1992 we (J. Approx. Th., 57, No. 3 (1989), 268–273; Discuss. Math. 12 (1992), 95–103) solved above Ulam problem.

In this paper we introduce the generalized Cauchy-Jensen functional inequality and solve a stability Ulam type problem for this inequality.

This problem, according to P. M. Gruber, is of particular interest in probability theory and in the case of functional equations of different types.

Definition 1. Let X be a linear space and let Y be a real complete linear space. Then a mapping $J_2 : X \to Y$ is called *Cauchy-Jensen*, if functional equation

$$J_2\left(\frac{x_1+x_2}{2}\right) = \frac{1}{2}[J_2(x_1) + J_2(x_2)] \tag{$*$}$$

holds for all vectors $(x_1, x_2) \in X^2$ with initial condition

$$J_2(0) = 0\,. \tag{$**$}$$

Note that substituting $x_1 = 0$, $x_2 = 2x$ into equation $(*)$ and considering condition $(**)$ one concludes that

$$J_2(x) = 2^{-1}J_2(2x)\,. \tag{F}$$

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Similarly substitution of x with $2x$ into (F) yields

$$J_2(2x) = 2^{-1} J_2(2^2 x)\,. \tag{Fa}$$

Combining (F) with (Fa) one gets that

$$J_2(x) = 2^{-2} J_2(2^2 x)\,. \tag{Fb}$$

Then by induction on $n \in N$ with $x \to 2^{n-1}x$ one proves that the *general identity*

$$J_2(x) = 2^{-n} J_2(2^n x)\,, \tag{Fc}$$

holds for all $x \in X$ and all $n \in N$.

Theorem 1. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is* an approximately Cauchy-Jensen *mapping; that is, a mapping for which there exist constants c, c_0 (independent of x_1, x_2) ≥ 0 such that the Cauchy-Jensen functional inequality*

$$\left\| f\left(\frac{x_1 + x_2}{2}\right) - \frac{1}{2}[f(x_1) + f(x_2)] \right\| \leq c\,, \tag{1}$$

holds for all vectors $(x_1, x_2) \in X^2$ with initial condition

$$\|f(0)\| \leq c_0\,. \tag{1a}$$

Then the limit

$$J_2(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{2}$$

exists *for all $x \in X$ and $J_2 : X \to Y$ is the* unique *Cauchy-Jensen mapping satisfying equation $(*)$ and initial condition $(**)$, such that J_2 is* near *f; that is, inequality*

$$\|f(x) - J_2(x)\| \leq c_1 \qquad (= 2c + c_0)\,, \tag{3}$$

holds for all $x \in X$ with constant c_1 (independent of x) ≥ 0. Moreover, identity

$$J_2(x) = 2^{-n} J_2(2^n x)\,, \tag{3a}$$

holds for all $x \in X$ and all $n \in N$.

Note that from (1a) and (2) one gets

$$\|J_2(0)\| = \lim_{n\to\infty} 2^{-n}\|f(0)\| \leq \left(\lim_{n\to\infty} 2^{-n}\right) c_0 = 0\,, \text{ or}$$
$$\|J_2(0)\| = 0\,, \text{ or } J_2(0) = 0\,, \text{ or } (**)\,.$$

Proof of Existence.

Substitution $x_1 = 0$, $x_2 = 2x$ into (1) yields

$$\|f(x) - 2^{-1}[f(0) + f(2x)]\| \leq c \tag{4}$$

for all $x \in X$.

Inequality (4), triangle inequality and (1a) imply

$$\|f(x) - 2^{-1}f(2x)\| \leq \|f(x) - 2^{-1}[f(0) + f(2x)]\| + 2^{-1}\|f(0)\|, \quad \text{or}$$
$$\|f(x) - 2^{-1}f(2x)\| \leq c + 2^{-1}c_0 = \frac{c_1}{2}(= c_1(1 - 2^{-1})), \tag{5}$$

for all $x \in X$, where $c_1 = 2c + c_0 (\geq 0)$.

Thus substituting x with $2x$ in (5) one gets that

$$\|f(2x) - 2^{-1}f(2^2x)\| \leq \frac{c_1}{2} \quad \text{or}$$
$$\|2^{-1}f(2x) - 2^{-2}f(2^2x)\| \leq \frac{c_1}{2^2}(= c_1 2^{-(2-1)}(1 - 2^{-1})), \tag{5a}$$

holds for all $x \in X$.

Inequalities (5) - (5a) and triangle inequality yield

$$\|f(x) - 2^{-2}f(2^2x)\| \leq \|f(x) - 2^{-1}f(2x)\| + \|2^{-1}f(2x) - 2^{-2}f(2^2x)\|, \quad \text{or}$$
$$\|f(x) - 2^{-2}f(2^2x)\| \leq c_1\left(\frac{1}{2} + \frac{1}{2^2}\right) \; (= c_1(1 - 2^{-2})), \tag{5b}$$

for all $x \in X$.

Similarly by induction on $n \in N$ with $x \to 2^{n-1}x$ in (5) one concludes that

$$\|f(2^{n-1}x) - 2^{-1}f(2^nx)\| \leq c_1(1 - 2^{-1}), \quad \text{or}$$
$$\|2^{-(n-1)}f(2^{n-1}x) - 2^{-n}f(2^nx)\| \leq c_1 2^{-(n-1)}(1 - 2^{-1}), \tag{6}$$

holds for all $x \in X$.

By induction hypothesis on $n \in N$ inequality

$$\|f(x) - 2^{-(n-1)}f(2^{n-1}x)\| \leq c_1(1 - 2^{-(n-1)}) \tag{6a}$$

holds for all $x \in X$.

Then inequalities (6) - (6a) and triangle inequality yield that

$$\|f(x) - 2^{-n}f(2^nx)\| \leq \|f(x) - 2^{-(n-1)}f(2^{n-1}x)\|$$
$$+\|2^{-(n-1)}f(2^{n-1}x) - 2^{-n}f(2^nx)\|, \quad \text{or}$$
$$\|f(x) - 2^{-n}f(2^nx)\| \leq c_1[(1 - 2^{-(n-1)}) + 2^{-(n-1)}(1 - 2^{-1})], \quad \text{or}$$

the general inequality:

$$\|f(x) - 2^{-n} f(2^n x)\| \le c_1(1 - 2^{-n}) \tag{6b}$$

holds for all $x \in X$ and all $n \in N$, with $c_1 = 2c + c_0 (\ge 0)$.

Claim that the sequence

$$\{2^{-n} f(2^n x)\}$$

converges.

Note that from the general inequality (6b) and the *completeness* of Y, one proves that the afore-mentioned sequence is a *Cauchy sequence*.

In fact, if $i > j > 0$, then

$$\|2^{-i} f(2^i x) - 2^{-j} f(2^j x)\| = 2^{-j} \|2^{-(i-j)} f(2^i x) - f(2^j x)\|\,, \tag{7}$$

holds for all $x \in X$ and all $i,\, j \in N$.

Setting $h = 2^j x$ in (7) and employing inequality (6b) one gets

$$\|2^{-i} f(2^i x) - 2^{-j} f(2^j x)\| = 2^{-j} \|2^{-(i-j)} f(2^{i-j} h) - f(h)\|\,, \quad \text{or}$$

$$\|2^{-i} f(2^i x) - 2^{-j} f(2^j x)\| \le 2^{-j} c_1 (1 - 2^{-(i-j)})\,, \quad \text{or}$$

$$\|2^{-i} f(2^i x) - 2^{-j} f(2^j x)\| \le c_1 (2^{-j} - 2^{-i}) < c_1 2^{-j}\,, \quad \text{or}$$

$$\lim_{j \to \infty} \|2^{-i} f(2^i x) - 2^{-j} f(2^j x)\| = 0\,, \tag{7a}$$

completing the proof that the sequence $\{2^{-n} f(2^n x)\}$ converges.

Hence mapping $J_2 = J_2(x)$ is *well-defined* via formula

$$J_2(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)\,, \tag{8}$$

for all $x \in X$ and all $n \in N$. This means that limit (2) (or (8)) *exists* for all $x \in X$.

In addition *claim* that J_2 satisfies functional equation $(*)$.
In fact, it is clear from (1) and (8) that

$$2^{-n} \left\| f\left(\frac{2^n x_1 + 2^n x_2}{2}\right) - \frac{1}{2}[f(2^n x_1) + f(2^n x_2)] \right\| \le 2^{-n} c\,,$$

for all $x_1, x_2 \in X$ and all $n \in N$.

Therefore

$$\left\| \lim_{n\to\infty} 2^{-n} f\left(2^n \frac{x_1 + x_2}{2}\right) - \frac{1}{2}\left[\lim_{n\to\infty} 2^{-n} f(2^n x_1) + \lim_{n\to\infty} 2^{-n} f(2^n x_2)\right] \right\|$$

$$\le \left(\lim_{n\to\infty} 2^{-n}\right) c = 0\,, \quad \text{or}$$

$$\left\| J_2\left(\frac{x_1 + x_2}{2}\right) - \frac{1}{2}[J_2(x_1) + J_2(x_2)] \right\| = 0\,, \quad \text{or}$$

J_2 satisfies the functional equation $(*)$ for all $(x_1, x_2) \in X^2$.

Thus J_2 is a *Cauchy-Jensen mapping*.

It is clear now from general inequality (6b), $n \to \infty$, and formula (8) that inequality (3) holds in X, completing *the existence proof* of this Theorem 1.

Proof of Uniqueness. Let $J_2' : X \to Y$ be another Cauchy-Jensen mapping satisfying functional equation $(*)$ and initial condition $(**)$, such that inequality

$$\|f(x) - J_2'(x)\| \leq c_1 \tag{3'}$$

holds for all $x \in X$ with constant c_1 (independent of $x \in X$) ≥ 0.

If there exists a Cauchy-Jensen mapping $J_2 : X \to Y$ satisfying equation $(*)$ and initial condition $(**)$, then

$$J_2(x) = J_2'(x)\,, \tag{9}$$

holds for all $x \in X$.

To prove the afore-mentioned *uniqueness* employ (Fc) for J_2', as well, so that

$$J_2'(x) = 2^{-n} J_2'(2^n x)\,. \tag{Fc'}$$

holds for all $x \in X$ and all $n \in N$.

Moreover triangle inequality and (3) imply that

$$\begin{gathered}\|J_2(2^n x) - J_2'(2^n x)\| \leq \|J_2(2^n x) - f(2^n x)\| + \|f(2^n x) - J_2'(2^n x)\|\,, \text{ or} \\ \|J_2(2^n x) - J_2'(2^n x)\| \leq c_1 + c_1 = 2c_1\,,\end{gathered} \tag{10}$$

for all $x \in X$ and all $n \in N$.

Then from (Fc), (Fc)$'$ and (10) one proves that

$$\begin{gathered}\|J_2(x) - J_2'(x)\| = \|2^{-n} J_2(2^n x) - 2^{-n} J_2'(2^n x)\|\,, \quad \text{or} \\ \|J_2(x) - J_2'(x)\| \leq 2^{1-n} c_1\,,\end{gathered} \tag{10a}$$

holds for all $x \in X$ and all $n \in N$.

Therefore from inequality (10a), and $n \to \infty$, one gets that

$$\begin{gathered}\lim_{n\to\infty} \|J_2(x) - J_2'(x)\| \leq \left(\lim_{n\to\infty} 2^{1-n}\right) c_1 = 0\,, \quad \text{or} \\ \|J_2(x) - J_2'(x)\| = 0\,, \quad \text{or} \\ J_2(x) = J_2'(x)\,,\end{gathered}$$

for all $x \in X$, completing the proof of *uniqueness* and thus the *stability* of Theorem 1.

Definition 2. Let X be a linear space and let Y be a real complete linear space. Then a mapping $J_p : X \to Y$ is called *Cauchy-Jensen*, if functional equation

$$J_p\left(\frac{x_1 + x_2 + \cdots + x_p}{p}\right) = \frac{1}{p}\left[J_p(x_1) + J_p(x_2) + \ldots J_p(x_p)\right] \tag{[*]}$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$ with initial condition

$$([**]) \qquad J_p(0) = 0\,.$$

Note that substituting $x_1 = x_2 = \cdots = x_{p-1} = 0$, $x_p = px$ into equation ([*]) and considering condition ([**]) one concludes that

$$([\mathrm{F}]) \qquad J_p(x) = p^{-1} J_p(px)\,.$$

Similarly substitution of x with px into ([F]) yields

$$([\mathrm{Fa}]) \qquad J_p(px) = p^{-1} J_p(p^2 x)\,.$$

Combining ([F]) with ([Fa]) one gets that

$$([\mathrm{Fb}]) \qquad J_p(x) = p^{-2} J_p(p^2 x)\,.$$

Then by induction on $n \in N$ with $x \to p^{n-1}x$ one proves that the general identity

$$([\mathrm{Fc}]) \qquad J_p(x) = p^{-n} J_p(p^n x)\,,$$

holds for all $x \in X$ and all $n \in N$.

Theorem 2. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is* an approximately Cauchy-Jensen *mapping; that is, a mapping for which there exist constants c, c_0 (independent of $x_1, x_2, \ldots, x_p$) ≥ 0 such that the Cauchy-Jensen functional inequality*

$$(11) \qquad \left\| f\left(\frac{x_1 + x_2 + \cdots + x_p}{p}\right) - \frac{1}{p}\left[f(x_1) + f(x_2) + \cdots + f(x_p)\right]\right\| \leq c\,,$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, $= 2, 3, \ldots$ with initial condition

$$(11\mathrm{a}) \qquad \|f(0)\| \leq c_0\,.$$

Then the limit

$$(12) \qquad J_p(x) = \lim_{n\to\infty} p^{-n} f(p^n x)$$

exists *for all $x \in X$ and $J_p; X \to Y$ is the* unique *Cauchy-Jensen mapping satisfying equation ([*]) and initial condition ([**]), such that J_p is* near *f; that is, inequality*

$$(13) \qquad \|f(x) - J_p(x)\| \leq \frac{c_1}{p-1}\,,$$

holds for all $x \in X$ *with constant* c_1 *(independent of* x*)* ≥ 0 *such that* $c_1 = pc + (p-1)c_0$*. Moreover, identity*

$$J_p(x) = p^{-n} J_p(p^n x)\,, \tag{13a}$$

holds for all $x \in X$*, all* $n \in N$ *and* $p = 2, 3 \ldots$

To prove this theorem it is enough to show that the following *new general inequality*

$$\|f(x) - p^{-n} f(p^n x)\| \leq \frac{c_1}{p-1}(1 - p^{-n}) \tag{14}$$

holds for all $x \in X$, all $n \in N$ and $p = 2, 3, \ldots$ with $c_1 = pc + (p-1)c_0$, $p = 2, 3, \ldots$

In fact, substitution $x_1 = x_2 = \cdots = x_{p-1} = 0$, $x_p = px$ into (11) yields

$$\|f(x) - p^{-1}[(p-1)\, f(0) + f(px)]\| \leq c \tag{14a}$$

for all $x \in X$ and $p = 2, 3, \ldots$

Inequality (14a), triangle inequality and (11a) imply

$$\|f(x) - p^{-1} f(px)\| \leq \|f(x) - p^{-1}[(p-1)\, f(0) + f(px)]\| + p^{-1}(p-1)\|f(0)\|\,, \text{ or}$$

$$\|f(x) - p^{-1} f(px)\| \leq c + p^{-1}(p-1)c_0 = \frac{c_1}{p} \left(= \frac{c_1}{p-1}(1 - p^{-1}) \right)\,, \tag{15}$$

for all $x \in X$, where $c_1 = pc + (p-1)c_0 (\geq 0)$ and $p = 2, 3, \ldots$

Thus with $x \to px$ in (15) one gets that

$$\|f(px) - p^{-1} f(p^2 x)\| \leq \frac{c_1}{p}\,, \quad \text{or}$$

$$\|p^{-1} f(px) - p^{-2} f(p^2 x)\| \leq \frac{c_1}{p^2} \left(= \frac{c_1}{p-1} p^{-(2-1)}(1 - p^{-1}) \right)\,, \tag{15a}$$

holds for all $x \in X$.

Inequalities (15) - (15a), and triangle inequality yield

$$\|f(x) - p^{-2} f(p^2 x)\| \leq \|f(x) - p^{-1} f(px)\| + \|p^{-1} f(px) - p^{-2} f(p^2 x)\|\,, \text{ or}$$

$$\|f(x) - p^{-2} f(p^2 x)\| \leq c_1 \left(\frac{1}{p} + \frac{1}{p^2} \right) \left(= \frac{c_1}{p-1}(1 - p^{-2}) \right)\,, \tag{15b}$$

for all $x \in X$.

Similarly by induction on $n \to N$ with $x \to p^{n-1}x$ in (15) one concludes that

$$\|f(p^{n-1}x) - p^{-1} f(p^n x)\| \leq \frac{c_1}{p-1}(1 - p^{-1})\,, \text{ or}$$

$$\|p^{-(n-1)} f(p^{n-1}x) - p^{-n} f(p^n x)\| \leq \frac{c_1}{p-1} p^{-(n-1)}(1 - p^{-1})\,, \tag{16}$$

holds for all $x \in X$, $p = 2, 3, \ldots$

But by induction hypothesis on $n \in N$ inequality

$$\|f(x) - p^{-(n-1)} f(p^{n-1}x)\| \le \frac{c_1}{p-1}(1 - p^{-(n-1)}) \tag{16a}$$

holds for all $x \in X$.

Then inequalities (16) - (16a) and triangle inequality yield

$$\|f(x) - p^{-n} f(p^n x)\| \le \|f(x) - p^{-(n-1)} f(p^{n-1}x)\| + \|p^{-(n-1)} f(p^{n-1}x) - p^{-n} f(p^n x)\|, \text{ or}$$

$$\|f(x) - p^{-n} f(p^n x)\| \le \frac{c_1}{p-1}\left[(1 - p^{-(n-1)}) + p^{-(n-1)}(1 - p^{-1})\right], \text{ or}$$

the new general inequality:

$$\|f(x) - p^{-n} f(p^n x)\| \le \frac{c_1}{p-1}(1 - p^{-n})$$

completing the proof of inequality (14).
The rest of the proof of Theorem 2 is omitted as similar to that one of Theorem 1.

Definition 3. Let X be a linear space and let Y be a real complete linear space. Then a mapping $J_p : X \to Y$ is called *generalized Cauchy-Jensen*, if functional equation

$$J_p\left(\sum_{i=1}^{p} a_i x_i\right) = \sum_{i=1}^{p} a_i J_p(x_i) \tag{a}$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, $p = 2, 3, \ldots$, and all fixed real numbers a_i, $i = 1, 2, \ldots, p$ with *a-condition*: $a = (a_1, a_2, \ldots, a_p)$,

$$\sum_{i=1}^{p} a_i = l \ge 0\,, \tag{b}$$

and initial condition

$$J_p(0) = 0\,. \tag{c}$$

Note that substituting $x_1 = x_2 = \cdots = x_{p-1} = 0$, $x_p = a_p^{-1} x : 0 < a_p < 1$ into equation (a) and considering conditions (b) - (c) one concludes that

$$J_p(x) = a_p J_p(a_p^{-1} x)\,. \tag{G}$$

Similarly substitution of x with $a_p^{-1}x$ into (G) yields

$$J_p(a_p^{-1}x) = a_p J_p(a_p^{-2}x)\,. \tag{Ga}$$

Combining (G) with (Ga) one gets that

$$J_p(x) = a_p^2 J_p(a_p^{-2}x)\,. \tag{Gb}$$

Then by induction on $n \in N$ with $x \to a_p^{-(n-1)}x$ one proves that the generalized functional identity

$$J_p(x) = a_p^n J_p(a_p^{-n}x)\,, \tag{Gc}$$

holds for all $x \in X$, all $n \in N$ and all fixed reals $a_p : 0 < a_p < 1$, $p = 2, 3, \ldots$

Theorem 3. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is* an approximately generalized Cauchy-Jensen *mapping; that is, a mapping for which there exist constants c_0, c (independent of of $x_1, x_2, \ldots, x_p$)≥ 0 such that the generalized Cauchy-Jensen inequality*

$$\left\| f\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i f(x_i)\right\| \leq c\,, \tag{11$'$}$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, $p = 2, 3, \ldots$ and all fixed reals a_i $(i = 1, 2, \ldots, p) : 0 < a_p < 1$ with a-condition: $a = (a_1, a_2, \ldots, a_p)$, $\sum_{i=1}^{p} a_i = 1$ and initial condition

$$\|f(0)\| \leq c_0\,. \tag{11a$'$}$$

Then the limit

$$J_p(x) = \lim_{n\to\infty} a_p^n\, f(a_p^{-n}x)\,, \tag{12$'$}$$

exists *for all $x \in X$ and $J_p : X \to Y$ is the* unique *generalized Cauchy-Jensen mapping satisfying equation* (a), *a-condition: $\sum\limits_{i=1}^{p} a_i = 1$, $p = 2, 3, \ldots$ and initial condition* (c), *such that J_p is* near *f; that is, inequality*

$$\|f(x) - J_p(x)\| \leq \frac{a_p}{1-a_p} c_1 \left(= \frac{c + (1-a_p)c_0}{1-a_p}\right) \tag{13$'$}$$

holds for all $x \in X$ with constant c_1 (independent of x) $\geq 0 : c_1 = \frac{c+(1-a_p)c_0}{a_p}$. Moreover, identity

$$J_p(x) = a_p^n J_p(a_p^{-n}x)\,, \tag{13a$'$}$$

holds for all $x \in X$, *all* $n \in N$ *and all fixed reals* $a_p : 0 < a_p < 1$, $p = 2, 3, \ldots$ *with* $\sum_{i=1}^{p} a_i = 1$.

To prove this theorem it is enough to show that the following *generalized inequality*

$$\|f(x) - a_p^n\, f(a_p^{-n} x)\| \leq \frac{a_p}{1 - a_p} c_1 (1 - a_p^n) \tag{14$'$}$$

holds for all $x \in X$, all $n \in N$ and fixed reals $a_p : 0 < a_p < 1$, $p = 2, 3, \ldots$ Substitution $x_1 = x_2 = \cdots = x_{p-1}$=0, $x_p = a_p^{-1} x$ into inequality (11′) yields

$$\|f(x) - [(1 - a_p)\, f(0) + a_p\, f(a_p^{-1} x)]\| \leq c \tag{14a$'$}$$

for all $x \in X$.

Inequality (14a′), triangle inequality and initial condition (11a′) imply

$$\|f(x) - a_p\, f(a_p^{-1} x)\| \leq \|f(x) - [(1 - a_p)\, f(0) + a_p\, f(a_p^{-1} x)]\| + (1 - a_p)\|f(0)\,, \text{ or}$$

$$\|f(x) - a_p\, f(a_p^{-1} x)\| \leq c + (1 - a_p) c_0\,, \text{ or}$$

$$\|f(x) - a_p\, f(a_p^{-1} x)\| \leq c + (1 - a_p) c_0 = \frac{a_p}{1 - a_p} c_1 (1 - a_p)\,, \tag{15$'$}$$

for all $x \in X$ with $c_1 = \frac{c + (1 - a_p) c_0}{a_p}$.

Thus with $x \to a_p^{-1} x$ in (15′) one gets

$$\|f(a_p^{-1} x) - a_p\, f(a_p^{-2} x)\| \leq \frac{a_p}{1 - a_p} c_1 (1 - a_p)\,, \text{ or}$$

$$\|a_p\, f(a_p^{-1} x) - a_p^2\, f(a_p^{-2} x)\| \leq \frac{a_p}{1 - a_p} c_1\, a_p (1 - a_p)\,, \tag{15a$'$}$$

for all $x \in X$.

Inequalities (15′) - (15a′), and triangle inequality yield

$$\|f(x) - a_p^2\, f(a_p^{-2} x)\| \leq \|f(x) - a_p\, f(a_p^{-1} x)\|$$
$$+\|a_p\, f(a_p^{-1} x) - a_p^2\, f(a_p^{-2} x)\|\,, \text{ or}$$

$$\|f(x) - a_p^2\, f(a_p^{-2} x)\| \leq \frac{a_p}{1 - a_p} c_1 (1 - a_p)\,, \tag{15b$'$}$$

for all $x \in X$.

Similarly by induction on $n \in N$ with $x \to a_p^{-(n-1)} x$ in (15′) one concludes that the required inequality (14′) holds.

The rest of the proof of Theorem 3 is omitted as similar to that one of Theorem 2.

General Theorem 4. *Let X be a normed linear space and let Y be a real complete normed linear space. Assume in addition that $f : X \to Y$ is* an approximately generalized Cauchy-Jensen *mapping; that is, a mapping for which there exist constants c_0, c (independent of $x_1, x_2, \ldots, x_p$)≥ 0 such that the generalized Cauchy-Jensen inequality*

$$\left\| f\left(\sum_{i=1}^{p} a_i x_i\right) - \sum_{i=1}^{p} a_i\, f(x_i) \right\| \leq c\,, \tag{11''}$$

holds for all vectors $(x_1, x_2, \ldots, x_p) \in X^p$, $p = 2, 3, \ldots,$ and all fixed reals a_i $(i = 1, 2, \ldots, p)$ with generalized a-condition*: $a = (a_1, a_2, \ldots, a_p)$, $\sum_{i=1}^{p} a_i = l \geq 0$, and initial condition*

$$\|f(0)\| \leq c_0 = \begin{cases} c\,, & \text{if } 0 < a_p < 1 : l = 0 \\ c\,, & \text{if } a_p > 1 : l = 0 \\ \frac{c}{1-l}\,, & \text{if } 0 < l < 1 \\ c_0\,, & \text{if } 0 < a_p < 1 : l = 1 \\ c_0\,, & \text{if } a_p > 1 : l = 1 \\ \frac{c}{l-1}\,, & \text{if } l > 1 \end{cases} \tag{11a''}$$

Then the limit

$$J_p x = \lim_{n\to\infty} \begin{cases} a_p^n\, f(a_p^{-n} x), & \text{if } 0 < a_p < 1 : l = 0 \\ a_p^{-n}\, f(a_p^n x)\,, & \text{if } a_p > 1 : l = 0 \\ l^n\, f(l^{-n} x)\,, & \text{if } 0 < l < 1 \\ a_p^n\, f(a_p^{-n} x)\,, & \text{if } 0 < a_p < 1 : l = 1 \\ a_p^{-n}\, f(a_p^n x)\,, & \text{if } a_p > 1 : l = 1 \\ l^{-n}\, f(l^n x)\,, & \text{if } l > 1 \end{cases} \tag{12''}$$

exists *for all $x \in X$ and $J_p : X \to Y$ is the* unique *generalized Cauchy-Jensen mapping satisfying equation* (a)*, generalized a-condition: $\sum_{i=1}^{p} a_i = 1 \geq 0$, $p = 2, 3, \ldots$ ad initial condition* (c)*, such that J_p is* near f*; that is, inequality*

$$\|f(x) - J_p(x)\| \leq \begin{cases} \frac{1+a_p}{1-a_p} c\,, & \text{if } 0 < a_p < 1 : l = 0 \\ \frac{a_p+1}{a_p-1} c\,, & \text{if } a_p > 1 : l = 0 \\ \frac{1}{1-l} c\,, & \text{if } 0 < l < 1 \\ \frac{c+(1-a_p)c_0}{1-a_p}, & \text{if } 0 < a_p < 1 : l = 1 \\ \frac{c+(a_p-1)c_0}{a_p-1}\,, & \text{if } a_p > 1 : l = 1 \\ \frac{1}{l-1} c\,, & \text{if } l > 1 \end{cases} \tag{13''}$$

holds for all $x \in X$. Moreover, identity

$$
(13\mathrm{a}'') \qquad J_p(x) = \begin{cases} a_p^n J_p(a_p^{-n} x)\,, & \text{if } 0 < a_p < 1 : l = 0 \\ a_p^{-n} J_p(a_p^n x)\,, & \text{if } a_p > 1 : l = 0 \\ l^n J_p(l^{-n} x)\,, & \text{if } 0 < l < 1 \\ a_p^n J_p(a_p^{-n} x)\,, & \text{if } 0 < a_p < 1 : l = 1 \\ a_p^{-n} J_p(a_p^n x)\,, & \text{if } a_p > 1 : l = 1 \\ l^{-n} J_p(l^n x)\,, & \text{if } l > 1 \end{cases}
$$

holds for all $x \in X$, all $n \in N$, $p = 2, 3, \ldots$, and all fixed real vectors $a = (a_1, a_2, \ldots, a_p)$, $p = 2, 3, \ldots$ with $\sum_{i=1}^{p} a_i = l \geq 0$.

Proof. To prove this theorem it is enough to show that the following *bounds* on $\|f(0)\|$, *functional equations* on $J_p(x)$ and *functional inequalities* on $\|f(x) - (\cdot)\, f(\cdot x)\|$ hold in X.

First claim that the following *bounds* hold for $x = 0 \in X$:

(B_1) $\|f(0)\| \leq c\,,$ if $l = 0$

(B_2) $\|f(0)\| \leq \frac{c}{1-l}\,,$ if $0 < l < 1$

(B_3) $\|f(0)\| \leq \frac{c}{l-1}\,,$ if $l > 1$

In fact:

Substituting $x_1 = x_2 = \cdots = x_p = x$ in Cauchy-Jensen inequality ($11''$), and considering the special a-condition $\sum_{i=1}^{p} a_i = l = 0$ one gets that $\|f(0)\| \leq c$, completing the proof of (B_1).

Similarly substituting $x_1 = 0$, $i = 1, 2, \ldots, p$ in ($11''$), and considering $0 < \sum_{i=1}^{p} a_i = l < 1$ one gets

$$
\begin{gathered}
\|f(0) - l f(0)\| \leq c\,, \text{ or} \\
\|f(0)\| \leq \frac{c}{1-l}\,, \text{ if } 0 < l < 1,
\end{gathered}
$$

and thus (B_2) holds.

Also substituting $x_i = 0$, $i = 1, 2, \ldots, p$ in ($11''$), and considering $\sum_{i=1}^{p} a_i = l > 1$ one gets that

$$
\|f(0)\| \leq \frac{c}{l-1}\,, \qquad \text{if } l > 1
$$

holds. Therefore (B_3) is true.

Second claim that the following *functional equations* hold for all $x \in X$:

(F_1) $J_p(x) = a_p^n J_p(a_p^{-n} x)$, if $0 < a_p < 1 : l = 0$, or $l = 1$.

(F_2) $J_p(x) = a_p^{-n} J_p(a_p^n x)$, if $a_p > 1 : l = 0$, or $l = 1$.

(F_3) $J_p(x) = l^n J_p(l^{-n} x)$, if $0 < l < 1$.

(F_4) $J_p(x) = l^{-n} J_p(l^n x)$, if $l > 1$.

In fact, the proof of equation (F_1) has been established via functional identities (G) - (Ga) - (Gb) - (Gc).

Substitution $x_1 = x_2 = \cdots = x_{p-1} = 0, x_p = x$ in equation (a) and considering conditions (b) - (c), one concludes that

$$J_p(x) = a_p^{-1} J_p(a_p x)\,, \qquad a_p > 1\,. \tag{H}$$

Then substitution of x with $a_p x(: a_p > 1)$, in (H) yields

$$J_p(a_p x) = a_p^{-1}\, J_p(a_p^2 x)\,. \tag{Ha}$$

Combining (H) with (Ha) one gets that

$$J_p(x) = a_p^{-2} J_p(a_p^2 x)\,. \tag{Hb}$$

Then by induction on $n \in N$ with $x \to a_p^{n-1} x$ one proves that the formula

$$J_p(x) = a_p^{-n} J_p(a_p^n x)\,, \quad a_p > 1\,, \tag{Hc}$$

holds for all $x \in X$ and all $n \in N$, completing the proof of equation (F_2).

Also substitution $x_1 = x_2 = \cdots = x_p = l^{-1} x$, $0 < l < 1$, in (a) and considering (b) - (c) one concludes that

$$J_p(x) = l\, J_p(l^{-1} x)\,, \quad 0 < l < 1\,. \tag{17}$$

Then substitution of x with $l^{-1} x$, $0 < l < 1$, in (17) yields

$$J_p(l^{-1} x) = l\, J_p(l^{-2} x)\,. \tag{17a}$$

Combining (17) with (17a) one gets that

$$J_p(x) = l^2\, J_p(l^{-2} x)\,. \tag{17b}$$

Then by induction on $n \in N$ with $x \to l^{-(n-1)} x$ one proves that the formula

$$J_p(x) = l^n\, J_p(l^{-n} x)\,, \quad 0 < l < 1\,, \tag{17c}$$

holds for all $x \in X$ and all $n \in N$.

Thus the proof of equation (F_3) is complete.

Finally substitution $x_1 = x_2 = \cdots = x_p = x$, $l > 1$, in (a) and considering (b) - (c) one gets that

$$J_p(lx) = l\,J_p(x)\,, \text{ or}$$
$$J_p(x) = l^{-1}J_p(lx)\,, \quad l > 1\,. \tag{18}$$

Then substitution of x with lx, $l > 1$, in (18) yields

$$J_p(lx) = l^{-1}J_p(l^2x)\,. \tag{18a}$$

Combining (18) with (18a) one concludes that

$$J_p(x) = l^{-2}J_p(l^2x)\,. \tag{18b}$$

Then by induction on $n \in N$ with $x \to l^{n-1}x$ one proves that the formula

$$J_p(x) = l^{-n}J_p(l^nx)\,, \quad l > 1\,, \tag{18c}$$

holds for all $x \in X$ and all $n \in N$, completing the proof of equation (F_4).

Third claim that the following *functional inequalities* hold for all $x \in X$:

(I_1) $\|f(x) - a_p^n f(a_p^{-n}x)\| \le \frac{1+a_p}{1-a_p}c(1-a_p^n)$, if $0 < a_p < 1 : l = 0$

(I_2) $\|f(x) - a_p^{-n} f(a_p^{n}x)\| \le \frac{a_p+1}{a_p-1}c(1-a_p^{-n})$, if $a_p > 1 : l = 0$

(I_3) $\|f(x) - l^n f(l^{-n}x)\| \le \frac{1}{1-l}c(1-l^n)$, if $0 < l < 1$

(I_4) $\|f(x) - a_p^n f(a_p^{-n}x)\| \le \frac{c+(1-a_p)c_0}{1-a_p}(1-a_p^n)$, if $0 < a_p < 1 : l = 1$

(I_5) $\|f(x) - a_p^{-n} f(a_p^{n}x)\| \le \frac{c+(a_p-1)c_0}{a_p-1}(1-a_p^{-n})$, if $a_p > 1 : l = 1$

(I_6) $\|f(x) - l^{-n} f(l^{n}x)\| \le \frac{1}{l-1}c(1-l^{-n})$, if $l > 1$.

Note that from above inequalities (I_i) ($i = 1, 2, 3, 4, 5, 6,$) and $n \to \infty$ one gets inequalities ($13''$).

In fact, substitution $x_1 = x_2 = \cdots = x_{p-1} = 0$, $x_p = a_p^{-1}x$, $0 < a_p < 1$, into ($11'$) yields

$$\left\|f(x) - \left[(-a_p)\,f(0) + a_p f(a_p^{-1}x)\right]\right\| \le c\,, \text{ or}$$
$$\left\|f(x) - a_p f(a_p^{-1}x)\right\| \le c + a_p c = (1+a_p)c\,, \text{ or}$$
$$\left\|f(x) - a_p f(a_p^{-1}x)\right\| \le \frac{1+a_p}{1-a_p}c(1-a_p)\,.$$

Therefore, by induction on $n \in N$ one concludes

$$\|f(x) - a_p^n f(a_p^{-n}x)\| \le \frac{1+a_p}{1-a_p}c(1-a_p^n)\,, \quad 0 < a_p < 1 : l = 0\,.$$

Thus the proof of inequality (I_1) is complete.

Also substitution $x_1 = x_2 = \cdots = x_{p-1} = 0$, $x_p = x$, $a_p > 1$ into ($11'$) yields

$$\|f(a_p x) - [-a_p f(0) + a_p f(x)]\| \le c\,, \text{ or}$$
$$\left\| \left[f(x) - a_p^{-1} f(a_p x)\right] - [-f(0)] \right\| \le \frac{c}{a_p}\,, \quad 0 < a_p < 1\,, \text{ or}$$
$$\left\| f(x) - a_p^{-1} f(a_p x) \right\| \le \frac{c}{a_p} + c = \frac{a_p + 1}{a_p} c\,, \text{ or}$$
$$\left\| f(x) - a_p^{-1} f(a_p x) \right\| \le \frac{a_p + 1}{a_p - 1} c(1 - a_p^{-1})\,.$$

Therefore, by induction on $n \in N$ with $x \to a_p^{n-1} x$:

$$\left\| f(x) - a_p^{-n} f(a_p^n x) \right\| \le \frac{a_p + 1}{a_p - 1} c(1 - a_p^{-n})\,, \quad a_p > 1 : l = 0\,.$$

Hence the proof of inequality (I_2) is complete.

Then substitution $x_1 = x_2 = \cdots = x_p = l^{-1} x$, $0 < l < 1$ into ($11'$) yields

$$\left\| f(x) - l f(l^{-1} x) \right\| \le c = \frac{c}{1 - l}(1 - l)\,.$$

Thus induction on $n \in N$ with $x \to l^{-(n-1)} x$ yields

$$\left\| f(x) - l^n f(l^{-n} x) \right\| \le \frac{c}{1 - l}(1 - l^n)\,, \quad 0 < l < 1\,.$$

Therefore, the proof if inequality (I_3) is complete.

The proof of inequality (I_4) has been completely established by means of (14) - (14a$'$), and ($15'$) - (15a$'$) - (15b$'$).

Moreover, substitution $x_1 = x_2 = \cdots = x_{p-1} = 0$, $x_p = x$, $a_p > 1$, into ($11'$) yields

$$\|f(a_p x) - [(1 - a_p)\, f(0) + a_p f(x)]\| \le c\,, \text{ or}$$
$$\left\| \left[f(x) - a_p^{-1} f(a_p x)\right] - \left[\frac{1 - a_p}{a_p} f(0)\right] \right\| \le \frac{c}{a_p}\,, \text{ or}$$
$$\left\| f(x) - a_p^{-1} f(a_p x) \right\| \le \frac{c}{a_p} + \frac{a_p - 1}{a_p} c_0\,, \text{ or}$$
$$\|f(x) - a_p^{-1} f(a_p x)\| \le \frac{c + (a_p - 1) c_0}{a_p - 1} (1 - a_p^{-1})\,, \quad a_p > 1 : l = 1\,.$$

Thus induction on $n \in N$ with $x \to a_p^{n-1} x$ implies

$$\|f(x) - a_p^{-n} f(a_p^n x)\| \le \frac{c + (a_p - 1) c_0}{a_p - 1} (1 - a_p^{-n})\,, \quad a_p > 1 : l = 1\,.$$

Thus the proof of inequality (I_5) is complete.

Finally, substitution $x_1 = x_2 = \cdots = x_p = x$, $l > 1$, into (11′) yields

$$|f(lx) - lf(x)\| \le c\,, \quad \text{or} \quad \|f(x) - l^{-1}f(lx)\| \le \frac{c}{l} = \frac{1}{l-1}c(1 - l^{-1})\,.$$

Therefore induction on $n \in N$ with $x \to l^{n-1}x$ yields

$$\|f(x) - l^{-n}f(l^n x)\| \le \frac{1}{l-1}c(1 - l^{-n})\,, \quad l > 1\,.$$

Therefore the proof of inequality (I_6) is complete.

The rest of the proof of Theorem 4 is omitted as similar to the proof of Theorem 2.

Examples.

1. Let $f : R \to R$ be a real function, such that $f(x) = x + k$ with k a real constant: $|k| \le c_0$ and a-condition $\left(a_1 = \frac{2}{5}, a_2 = \frac{3}{5}\right) : a_1 + a_2 = l = 1$. Then the limit

$$J_p(x) = \lim_{n\to\infty} a_2^n f(a_2^{-n}x) = \lim_{n\to\infty}\left(\frac{3}{5}\right)^n\left[\left(\frac{5}{3}\right)^n x + k\right] = x\,,$$

exists for all $x \in X$ and $J_2 : R \to R$ is the unique Cauchy-Jensen mapping satisfying inequality

$$|f(x) - J_2(x)| = |(x+k) - x| = |k| \le c_0 < \frac{5}{2}c + c_0 = \frac{5c + 2c_0}{2}\left(= \frac{3}{2}c_1\right)\,.$$

It is clear that f satisfies Cauchy-Jensen inequality (11′) and initial condition (11a′), as well.

2. Let $a = (a_1, a_2, \ldots, a_p)$ be such that

$$a_1 = \frac{1}{\frac{p(p+1)}{2}}, a_2 = \frac{2}{\frac{p(p+1)}{2}}, \ldots, a_p = \frac{p}{\frac{p(p+1)}{2}}\left(= \frac{2}{p+1}\right)\,,$$

$l = a_1 + a_2 + \cdots + a_p = \frac{1+2+\cdots+p}{\frac{p(p+1)}{2}} = 1$, and let $f : R \to R$ be a real function, such that $f(x) = x + k$ with k a real constant: $|k| \le c_0$.

Then the limit

$$J_p(x) = \lim_{n\to\infty} a_p^n f(a_p^{-n}x) = \lim_{n\to\infty}\left(\frac{2}{p+1}\right)^n\left[\left(\frac{p+1}{2}\right)^n x + k\right] = x\,,\ p = 2, 3, \ldots,$$

exists for all $x \in X$ and $J_p : R \to R$ is the *unique* generalized Cauchy-Jenssen mapping satisfying inequality

$$\begin{aligned}|f(x) - J_p(x)| &= |(x+k) - x| = |k| \le c_0 \\ &< \frac{p+1}{p-1}c + c_0 = \frac{(p+1)c + (p-1)c_0}{p-1}\left(= \frac{2}{p-1}c_1\right)\,.\end{aligned}$$

It is clear that f satisfies inequality (11′) and condition (11a′), as well.

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