ARCHIVUM MATHEMATICUM (BRNO)
Tomus 37 (2001), 185 – 206

ON PROJECTABLE OBJECTS ON FIBRED MANIFOLDS

VASILE CRUCEANU, MARCELA POPESCU AND PAUL POPESCU

Abstract. The aim of this paper is to study the projectable and N-projectable objects (tensors, derivations and linear connections) on the total space E of a fibred manifold ξ, where N is a normalization of ξ.

In this paper we study the projectable and N-projectable objects (tensors, derivations and linear connections) on the total space E of a fibred manifold $\xi = (E, \pi, M)$, where N is a normalization of ξ. Also, with this occasion, we extend, complete and unify certain notions and results concerning the vector bundles and fibred manifolds, presented in the papers quoted in the enclosed references.

1. Definitions and notations

A *fibred manifold* is a triplet $\xi = (E, \pi, M)$, where E and M are differentiable manifolds which are connected and paracompact and $\pi : E \to M$ is a surjective submersion. We say that E is the *total space*, M is the *base* manifold and π is the *canonical projection* of the fibred manifold ξ. In the sequel we identify the fibred manifold with the total space E. All manifolds, maps and objects are assumed to be C^∞ differentiable. For every $x \in M$, the sets $E_x = \pi^{-1}(x)$ are closed submanifolds of E, which are assumed to be connected, too. Let us denote by m the dimension of M and by n the dimension of $E_x, (\forall) x \in M$. We consider on E and M differentiable atlases which are *adapted* to the fibred structures, i.e. in each point $z \in E$ there is a chart (V, ψ) with $\psi(z) = (x^i, y^a)$, $(i, j, k, \ldots = \overline{1, m},\ a, b, c, \ldots = \overline{1, n})$ and in $x = \pi(z) \in M$, a chart (U, φ) so that $U = \pi(V)$ and $\varphi(x) = (x^i)$. The pairs of *natural bases* associated to the local charts on M and on E are (∂_i, d^i) and $(\partial_i, \partial_a; d^i, d^a)$ respectively, where $\partial_i = \dfrac{\partial}{\partial x^i}$, $\partial_a = \dfrac{\partial}{\partial y^a}$, $d^i = dx^i, d^a = dy^a$. The change rules of the adapted charts is given by the following equations:

$$x^{i'} = x^{i'}(x^i), y^{a'} = y^{a'}(x^i, y^a). \tag{1}$$

2000 *Mathematics Subject Classification*: 53B40, 53C07, 53C60.
Key words and phrases: fibred manifold, projectable objects, d-tensor fields.
Received December 13, 1999.

Therefore, the natural bases on M and on E have the following change rules:

$$\begin{aligned} &\partial_i = \tfrac{\partial x^{i'}}{\partial x^i}\partial_{i'}\,,\ d^{i'} = \tfrac{\partial x^{i'}}{\partial x^i} d^i\,,\\ (2)\qquad &\partial_i = \tfrac{\partial x^{i'}}{\partial x^i}\partial_{i'} + \tfrac{\partial y^{a'}}{\partial x^i}\partial_{a'}\,,\ \partial_{a'} = \tfrac{\partial y^a}{\partial y^{a'}}\partial_a\,,\\ &d^{i'} = \tfrac{\partial x^{i'}}{\partial x^i} d^i,\ d^{a'} = \tfrac{\partial y^{a'}}{\partial x^i} d^i + \tfrac{\partial y^{a'}}{\partial y^a} d^a\,. \end{aligned}$$

Thus, the coordinates of the vectors $X = X^i\partial_i \in T_xM$ and $A = A^i\partial_i + A^a\partial_a \in T_zE$ have the following change rules:

$$(3)\qquad X^{i'} = \frac{\partial x^{i'}}{\partial x^i}X^i,\ A^{i'} = \frac{\partial x^{i'}}{\partial x^i}A^i,\ A^{a'} = \frac{\partial y^{a'}}{\partial x^i}A^i + \frac{\partial y^{a'}}{\partial y^a}A^a\,,$$

and the coordinates of the co-vectors $\omega = \omega_i d^i \in T_x^*M$ and $\alpha = \alpha_i d^i + \alpha_a d^a \in T_z^*E$ have the following change rules:

$$(4)\qquad \omega_{i'} = \frac{\partial x^i}{\partial x^{i'}}\omega_i\,,\ \alpha_{i'} = \frac{\partial x^i}{\partial x^{i'}}\alpha_i + \frac{\partial y^a}{\partial x^{i'}}\alpha_a\,,\ \alpha_{a'} = \frac{\partial y^a}{\partial y^{a'}}\alpha_a\,.$$

If we consider $V_zE = \ker T_z\pi$ for every $z = (x,y) \in E$, then we obtain the vertical distribution, therefore the vertical subbundle of TE, denoted by VE. This distribution is tangent to the vertical foliation. If we consider the quotient bundle $WE = TE/VE$, then we obtain the following vector bundles on E exact sequence:

$$(5)\qquad 0 \to VE \xrightarrow{i} TE \xrightarrow{p} WE \to 0\,,$$

where i and p are the canonical injection and the canonical projection respectively.

We have for VE and for WE the local bases (∂_a) and $(\hat{\partial}_i = p(\partial_i))$ respectively. If we put, for every $z \in E$, $V_z^{\perp}E = \{\alpha \in T_z^*E \mid \alpha(A) = 0,\ (\forall)A \in V_zE\}$, we obtain a subbundle of T^*E called the *orthogonal dual* of VE. If we consider $W^{\perp}E = T^*E/V^{\perp}E$ then we obtain a new exact sequence of vector bundles over E:

$$(6)\qquad 0 \to V^{\perp}E \xrightarrow{j} T^*E \xrightarrow{q} W^{\perp}E \to 0\,,$$

where j and q are the canonical injection and the canonical projection respectively. For $V^{\perp}E$ and $W^{\perp}E$ we have the local bases (d^i) and $(\hat{d}^a = q(d^a))$ respectively. If the change rule of the local coordinates is given by (1), then we have the following change rule for the local bases $(\hat{\partial}_i)$ and $(\hat{d}^a)$:

$$(7)\qquad \hat{\partial}_{i'} = \frac{\partial x^i}{\partial x^{i'}}\hat{\partial}_i\,,\ \hat{d}^{a'} = \frac{\partial y^{a'}}{\partial y^a}\hat{d}^a\,.$$

It follows that $(WE)^*$ and $(VE)^*$ are canonically isomorphic with $V^{\perp}E$ and $W^{\perp}E$ respectively. In a similar way WE and $(WE)^*$ are canonically isomorphic with $\pi^{-1}TM$ and $\pi^{-1}T^*M$ respectively.

A *distinguished tensor field* (or *d-tensor*) of type $\binom{p\ r}{q\ s}$ on the total space E of the fibred manifold ξ is a section in the vector bundle $\otimes^p WE \otimes^r VE \otimes^q V^{\perp}E \otimes^s W^{\perp}E$ on the base E. The local expression of a d-tensor field T of type $\binom{p\ r}{q\ s}$ is:

$$(8)\qquad T = T^{i_1\cdots i_p a_1\cdots a_r}_{j_1\cdots j_q b_1\cdots b_s}\hat{\partial}_{i_1}\otimes\cdots\otimes\partial_{a_1}\otimes\cdots\otimes d^{j_1}\otimes\cdots\otimes\hat{d}^{b_1}\otimes\cdots\otimes\hat{d}^{b_s}\,.$$

Let $\mathcal{F}(M)$ be the ring of the real functions, $\mathcal{T}^p_q(M)$ be the $\mathcal{F}(M)$-module of tensor fields of type (p,q) and $\mathcal{T}(M)$ the two-graded $\mathcal{F}(M)$-algebra of tensor fields on M. We denote as $\overline{\mathcal{T}}^{p,r}_{q,s}$ and $\overline{\mathcal{T}}(E)$ the $\mathcal{F}(E)$-module of d-tensor fields of type $\binom{p\ r}{q\ s}$ and the four-graded $\mathcal{F}(E)$-algebra of d-tensor fields on E respectively. Four one-graded algebras and six two-graded algebras are more important subalgebras of $\overline{\mathcal{T}}(E)$, namely the subalgebras generated by the d-tensor fields of type $\binom{p\ 0}{0\ 0}$, $\binom{0\ r}{0\ 0}$, $\binom{0\ 0}{q\ 0}$, $\binom{0\ 0}{0\ s}$; $\binom{p\ 0}{q\ 0}$, $\binom{0\ r}{0\ s}$, $\binom{p\ r}{0\ 0}$, $\binom{0\ r}{q\ 0}$, $\binom{p\ 0}{0\ s}$, $\binom{0\ 0}{q\ s}$, $p,q,r,s \in \mathbb{N}$. Among these we remark two subalgebras:

• The subalgebra of $\mathcal{T}(E)$ of two-graded d-tensors of type $\binom{0\ r}{q\ 0}$, $q,r \in \mathbb{N}$, which consists of sections $E \to \otimes^r VE \otimes^q V^\perp E$. We call such sections as *vertical-horizontal* or *semi-basic* tensor fields.

• The subalgebra of $\mathcal{T}(E)$ of two-graded d-tensors of type $\binom{p\ 0}{q\ 0}$, $p,q \in \mathbb{N}$, which consists of sections $E \to \otimes^p WE \otimes^q V^\perp E$. The local coordinates of these sections follow the same change rule as the local coordinates of tensors of type (p,q) on M.

The *d-lift* (or $WV^\perp$*-lift*) of the tensor field $t \in \mathcal{T}^p_q(M)$, given in local coordinates by $t = t^{i_1\cdots i_p}_{j_1\cdots j_q}\partial_{i_1} \otimes \cdots \otimes d^{j_1} \otimes \cdots \otimes d^{j_q}$, is the d-tensor field $t^d \in \overline{\mathcal{T}}^{p,0}_{q,0}(E)$ given locally by

$$t^d = t^{i_1\cdots i_p}_{j_1\cdots j_q} \circ \pi \hat{\partial}_{i_1} \otimes \cdots \otimes \hat{\partial}_{i_p} \otimes d^{j_1} \otimes \cdots \otimes d^{j_q}\,. \tag{9}$$

Notice that the d-lift is an $\mathcal{F}(M)$-linear application of $\mathcal{T}^p_q(M)$ in $\overline{\mathcal{T}}^{p,0}_{q,0}(E)$.

The W*-lift* of a tensor field $t \in \mathcal{T}^p_0(M) = \mathcal{T}^p(M)$ is $t^d \in \overline{\mathcal{T}}^{p,0}_{0,0}(E)$ and the $V^\perp$*-lift* of a tensor field $t \in \mathcal{T}^0_q(M) = \mathcal{T}_q(M)$ is $t^d \in \overline{\mathcal{T}}^{0,0}_{q,0}(E)$.

A *vertical vector field* is a section in the vector bundle VE. A vertical vector field has a local form $A = A^a(x^i, y^b)\partial_a$. Since the vertical distribution is integrable, for every $A, B \in VT^1(E) = \overline{\mathcal{T}}^{1,0}_{0,0}(E)$, we have $[A,B] \in VT^1(E)$. Thus $VT^1(E)$ is an $\mathcal{F}(E)$-submodule and a Lie $\mathcal{F}(M)$-subalgebra of $\mathcal{T}^1(E)$.

A *horizontal 1-form* on E is a section in the vector bundle $V^\perp E$. It has the local expression $\alpha = \alpha_i(x^j, y^a)d^i$ and $\alpha(A) = 0$, for every vertical vector field A.

The *horizontal lift* of an 1-form $\omega \in \mathcal{T}_1(M)$ is the 1-form on E given by $\omega^h = T^*\pi(\omega)$. If $\omega(x) = \omega_i(x^j)d^i$, then $\omega^h(x) = \omega_i(x^j)d^i$. Particularly $(d^i)^h = d^i$.

2. Induced derivations and derivation laws in the algebra of d-tensor fields

Generally, if D is a derivation in the algebra $\mathcal{T}(E)$, then it does not induce a derivation in algebra $\overline{\mathcal{T}}(E)$, since it does not respect always the graduation.

Let D be a derivation on $\mathcal{T}(E)$ which has the local expression

$$\begin{gathered} D(x^i) = D^i\,, \quad D(y^a) = D^a\,, \\ D(\partial_j) = D^k_j\partial_k + D^c_j\partial_c\,, \quad D(\partial_b) = D^k_b\partial_k + D^c_b\partial_c\,. \end{gathered} \tag{10}$$

Proposition 2.1. *A derivation D in algebra $\mathcal{T}(E)$ induces a derivation in algebra $\overline{\mathcal{T}}(E)$ iff it satisfies one of the following equivalent conditions:*

1. *in the local expression* (10), $D^k_b = 0$;

2. *D carries every vertical vector field in a vertical vector field;*
3. $p \circ D \circ i = 0$.

Proof. Indeed, the four components of $\overline{\mathcal{T}}(E)$ must be invariant by D, particularly the component of the type $\binom{0\ 1}{0\ 0}$. Conversely, if the vector subbundle VE is invariant by D, then its dual orthogonal is also invariant, since

$$D\alpha(A) = D(\alpha(A)) - \alpha(DA),\ \alpha \in \mathcal{T}_1(E),\ A \in \mathcal{T}^1(E).$$

Defining $\hat{D}\hat{A} = \widehat{DA}$ and $\hat{D}\hat{\alpha} = \widehat{D\alpha}$, we obtain two derivations denoted as $\hat{D}$ on WE and on $W^{\perp}E$ respectively and consequently a derivation $\hat{D}$ in algebra $\overline{\mathcal{T}}(E)$. The conditions 2) and 3) are obviously equivalent to 1). □

Let now $\mathcal{D}$ be a linear connection on E. Taking into account that $\mathcal{D}_A$ is a derivation in the algebra $\mathcal{T}(E)$ for every $A \in \mathcal{T}^1(E)$, and that the local expression of $\mathcal{D}$ in local coordinates is

$$\begin{array}{c}\mathcal{D}_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k + \Gamma^c_{ij}\partial_c,\ \mathcal{D}_{\partial_a}\partial_j = \Gamma^k_{aj}\partial_k + \Gamma^c_{aj}\partial_c\,, \\ \mathcal{D}_{\partial_i}\partial_b = \Gamma^k_{ib}\partial_k + \Gamma^c_{ib}\partial_c,\ \mathcal{D}_{\partial a}\partial_b = \Gamma^k_{ab}\partial_k + \Gamma^c_{ab}\partial_c\end{array} \tag{11}$$

from Proposition 2.1, we obtain:

Proposition 2.2. *A linear connection $\mathcal{D}$ on E induces a derivation law in the four graded algebra $\overline{\mathcal{T}}(E)$ iff it satisfies one of the following equivalent conditions:*

1. *using local coordinates, we have $\Gamma^k_{ib} = 0$, $\Gamma^k_{ab} = 0$;*
2. *$\mathcal{D}_A B \in V\mathcal{T}^1(E)$ for every $A \in \mathcal{T}^1(E)$ and $B \in V\mathcal{T}^1(E)$;*
3. *$p \circ \mathcal{D}_A \circ i = 0$, $(\forall) A \in \mathcal{T}^1(E)$.*

3. Projectable objects on the total space E

A function $\tilde{f} \in \mathcal{F}(E)$ is called *projectable* if there is a function $f \in \mathcal{F}(M)$ so that $\tilde{f} = f \circ \pi$.

Using local adapted coordinates (x^i, y^a), it follows that a projectable function depends only on the coordinates (x^i). Since the set $\widetilde{\mathcal{F}}(E)$ of projectable functions is endowed with a real subalgebra structure of $\mathcal{F}(E)$, it is isomorphic with the real algebra $\mathcal{F}(M)$.

A vector field $A \in \mathcal{T}^1(E)$ is called *projectable* if for every projectable function $\tilde{f}$, the function $A(\tilde{f})$ is projectable.

Proposition 3.1. *A vector field $A \in \mathcal{T}^1(E)$ is projectable iff it satisfies one of the following equivalent conditions:*

1. *in the local expression $A = A^i\partial_i + A^a\partial_a$, the functions A^i are projectable;*
2. *there is $X \in \mathcal{T}^1(M)$ so that $T\pi(A) = X$;*
3. *for every $B^v \in V\mathcal{T}^1(E)$ it follows that $[A, B^v] \in V\mathcal{T}^1(E)$.*

Notice that condition *3)* is equivalent with $p(\mathcal{L}_{B^v}A) = 0$, $(\forall)B^v \in V\mathcal{T}^1(E)$, where $\mathcal{L}_{B^v}$ is the Lie derivation with respect of B^v. It follows that the set $\mathcal{P}^1(E)$ of projectable fields is a submodule of $\mathcal{T}^1(E)$ over $\mathcal{F}(M)$, it has $V\mathcal{T}^1(E)$ as a submodule over $\mathcal{F}(M)$ and it is a Lie subalgebra of $\mathcal{T}^1(E)$ which has $V\mathcal{T}^1(E)$ as an ideal.

A differential 1-form $\alpha \in \mathcal{T}_1(E)$ is called *projectable* if for every projectable vector field A the function $\alpha(A)$ is projectable.

Proposition 3.2. *A differential* 1*-form* $\alpha \in \mathcal{T}_1(E)$ *is projectable iff it satisfies one of the following equivalent conditions:*

1. *in the local expression* $\alpha = \alpha_i d^i + \alpha_a d^a$*, the functions* α_i *are projectable and* $\alpha_a = 0$*;*
2. *there is a differential* 1*-form* $\omega \in \mathcal{T}_1(M)$ *so that* $\alpha = T^*\pi(\omega)$*;*
3. $d\alpha(A, B^v) = 0$*,* $(\forall)A \in \mathcal{P}^1(E)$*,* $B^v \in V\mathcal{T}^1(E)$*,* $\alpha \in \mathcal{T}^1(E)$*.*

A tensor field $T \in \mathcal{T}_v^u(E)$, $u, v \geq 0$, is *projectable* if for every projectable 1-forms $\alpha_1, \ldots, \alpha_u$ and projectable vector fields $A_1, \ldots, A_v$, the function $T(\alpha_1, \ldots ,\alpha_u, A_1, \ldots, A_v)$ is projectable.

Using Propositions 3.1 and 3.2, the following result holds:

Proposition 3.3. *A tensor field* $T \in \mathcal{T}_v^u(E)$*,* $u, v \geq 0$*, is projectable iff it satisfies one of the following equivalent conditions:*

1. *in the local expression* $T = T^{i_1 \cdots i_p a_1 \cdots a_r}_{j_1 \cdots j_q b_1 \cdots b_s} \partial_{i_1} \otimes \cdots \partial_{a_1} \otimes \cdots d^{j_1} \otimes \cdots d^{b_1} \otimes \cdots \otimes d^{b_s}$*,* $p + r = u$*,* $q + s = v$*, the local functions* $T^{i_1 \cdots i_u}_{j_1 \cdots j_v}$ *are projectable and* $T^{i_1 \cdots i_u}_{j_1 \cdots j_{v-1} b_v} = 0, \ldots , T^{i_1 \cdots i_u}_{b_1 \cdots b_v} = 0$*;*
2. *there is a tensor field* $t \in \mathcal{T}_v^u(M)$ *such that*

$$T\left(\alpha_1, \ldots , \alpha_u, A_1, \ldots , A_v\right) = t\left(\omega_1, \ldots , \omega_u, X_1, \ldots , X_v\right) \circ \pi\,, \tag{12}$$

$(\forall)\omega_i \in \mathcal{T}_1(M)$*,* $X_j \in \mathcal{T}^1(M)$*,* $\alpha_i \in \mathcal{P}_1(E)$*,* $A_j \in \mathcal{P}^1(E)$*, so that* $\alpha_i = T^*\omega_i$ *and* $T\pi(A_j) = X_j$*,* $i = 1, \ldots , u$*,* $j = 1, \ldots , v$*.*

Particularly, a tensor field $T \in \mathcal{T}_0^u(E)$ is projectable iff the local components $T^{i_1 \cdots i_u}$ are projectable.

As immediate properties of projectable tensor fields we can state:

Proposition 3.4. *The set* $\mathcal{P}(E)$ *of projectable tensor fields on* E *is a subalgebra of* $\mathcal{T}(E)$ *over* $\mathcal{F}(M)$*. The map which associate with every projectable tensor field* $T \in \mathcal{P}(E)$ *its projection* $t \in \mathcal{T}(M)$ *is an* $\mathcal{F}(M)$*-morphism of algebras; restricted to* $\mathcal{P}^1(E)$*, it takes values in* $\mathcal{T}^1(M)$ *and it is a Lie algebra morphism.*

A derivation D of algebra $\mathcal{D}er\,\mathcal{T}(E)$ is called *projectable* if for every projectable vector field $A \in \mathcal{P}^1(E)$, the vector field DA is also projectable.

Proposition 3.5. *A derivation* D *of algebra* $\mathcal{D}er\,\mathcal{T}(E)$ *is projectable iff it satisfies one of the following equivalent conditions:*

1. *in the local expression of* D*, the functions* D^i *and* D^k_j *are projectable and* $D^k_b = 0$*;*
2. *there is a derivation* $\tilde{D} \in \mathcal{D}er\,\mathcal{T}(M)$ *so that*

$$T\pi(DA) = \tilde{D}X$$

$(\forall)A \in \mathcal{P}^1(E)$ *and* $X \in \mathcal{T}^1(M)$ *which satisfies the condition* $T\pi(A) = X$*.*

Remarks.

1. According to Proposition 2.1, a projectable derivation induces a derivation in algebra $\overline{\mathcal{T}}(E)$.

2. The set of projectable derivations is an $\mathcal{F}(M)$-submodule and a real Lie subalgebra of $\mathcal{D}er\,\mathcal{T}(E)$, homomorphic with the Lie algebra $\mathcal{D}er\,\mathcal{T}(M)$.

3. The Lie derivation $\mathcal{L}_A$ is projectable iff A is projectable.

4. The derivation defined by a tensor $S \in \mathcal{T}_1^1(E)$ is projectable iff S is projectable.

A linear connection $\mathcal{D}$ on E is called *projectable* if for every projectable vector field A the derivation $D = \mathcal{D}_A$ is projectable.

Proposition 3.6. *A linear connection $\mathcal{D}$ on E is projectable iff it satisfies one of the following equivalent conditions:*

1. *the local functions Γ^i_{jk} are projectable and $\Gamma^k_{aj} = \Gamma^k_{ib} = \Gamma^k_{ab} = 0$;*
2. *for every projectable vector fields $A, B \in \mathcal{P}^1(E)$, the vector field $\mathcal{D}_A B$ is also projectable;*
3. *there is a connection $\widetilde{\mathcal{D}}$ on M (called the* projected connection*) so that*

$$T\pi\left(\mathcal{D}_A B\right) = \widetilde{\mathcal{D}}_{T\pi(A)} T\pi(B)\,,\ (\forall)\, A, B \in \mathcal{P}^1(E)\,. \tag{13}$$

Remark. According to Proposition 2.2, a projectable connection induces a derivation law in the four graded algebra $\overline{\mathcal{T}}(E)$.

Proposition 3.7. *The torsion $\mathcal{T}$ of a projectable connection $\mathcal{D}$ on E is a projectable tensor field and its projection on M is the torsion $\widetilde{\mathcal{T}}$ of the projected connection $\widetilde{\mathcal{D}}$:*

$$T\pi\left(\mathcal{T}\left(A, B\right)\right) = \widetilde{\mathcal{T}}\left(T\pi(A), T\pi(B)\right)\,,\ (\forall)\, A, B \in \mathcal{P}^1(E)\,. \tag{14}$$

Since the bracket of two projectable vector fields on E is also a projectable vector field, the following result holds:

Proposition 3.8. *The curvature $\mathcal{R}$ of a projectable connection $\mathcal{D}$ on E is a projectable tensor field and its projection on M is the curvature $\widetilde{\mathcal{R}}$ of the projected connection $\widetilde{\mathcal{D}}$:*

$$\begin{gathered} T\pi\left(\mathcal{R}\left(A, B\right) C\right) = \widetilde{\mathcal{R}}\left(T\pi(A), T\pi(B)\right) T\pi(C)\,, \\ (\forall)\, A, B, C \in \mathcal{P}^1(E)\,. \end{gathered} \tag{15}$$

Remark. The set of projectable linear connections on E is an affine $\mathcal{F}(M)$-submodule [4] of the module of linear connections on E, which is homomorphic with the $\mathcal{F}(M)$-module of linear connections on M.

4. Normalizations of the vertical foliation

Since the total space E has a vertical foliation VE, we can consider for its study a normalization of this foliation, i.e. a distribution HE on E which is supplementary to VE. The distribution HE is called the *horizontal distribution*. We denote also by HE the *horizontal subbundle*. A such normalization can be defined by a right or a left splitting of the exact sequences (5) or (6).

A *right splitting* of the exact sequence (5) is a map $N : WE \to TE$ which satisfies the conditions that N is an E-morphism of vector bundles and $p \circ N = I_{WE}$. Denoting as $HE = N(WE)$, it is a subbundle of TE which is supplementary to VE, thus we obtain a normalization on E with HE the suitable horizontal bundle. In local coordinates, we can consider

$$\delta_i = N\left(\hat{\partial}_i\right) = \partial_i - N_i^a \partial_a \,, \; i = 1, \dots, n \,, \tag{16}$$

which is a local basis (δ_i) of the sections of HE. The change rule when the local coordinates change is:

$$\delta_{i'} = \frac{\partial x^i}{\partial x^{i'}} \delta_i \,, \; i, i' = 1, \dots, n \,. \tag{17}$$

It follows that the change rule for the coefficients $\{N_i^a(x, y)\}$ of the normalization is:

$$N_{i'}^{a'} = \frac{\partial x^i}{\partial x^{i'}} \left(N_i^a \frac{\partial y^{a'}}{\partial y^a} - \frac{\partial y^{a'}}{\partial x^i} \right) . \tag{18}$$

Conversely, if we assume that on the domain of every local chart on E, adapted to the fibred structure on E, the local functions $N_i^a(x, y)$ are given, such that the change rule (18) on the intersection of two domains holds, then the map N given by relation (16) is a normalization of E. The normalization N gives an embedding of WE in TE and a decomposition of TE in the direct Whitney sum

$$TE = HE \oplus VE \,. \tag{19}$$

Denoting as H and V the horizontal and vertical projectors and as F the almost product structure canonically associated with the normalization, we have:

$$H = N \circ p, \quad V = I_{TE} - H \,, \quad F = H - V \,. \tag{20}$$

Locally, for a vector field $A = A^i \partial_i + A^a \partial_a$ on E, from (16) we obtain:

$$\begin{aligned} A &= A^i \delta_i + \left(A^a + N_i^a A^i\right) \partial_a \,, \; HA = A^i \delta_i = A^i \partial_i - N_i^a A^i \partial_a \,, \\ VA &= \left(A^a + N_i^a A^i\right) \partial_a \,. \end{aligned} \tag{21}$$

A *horizontal* vector field on E with respect to the normalization N is a section of the horizontal subbundle. Locally, a horizontal vector field has the form $A = A^i(x, y)\delta_i$. The relations (16) and (21) imply that $A \in \mathcal{T}^1(E)$ is projectable iff HA is projectable. As for $A = A^i(x)\delta_i$ its projection is $X = A^i \partial_i$, it follows that the restriction of $T\pi$ to the $\mathcal{F}(M)$-module $H\mathcal{P}^1(E)$ of horizontal projectable vector fields is an $\mathcal{F}(M)$-isomorphism.

The *horizontal lift* of the vector field $X \in \mathcal{T}^1(M)$ is the horizontal projectable vector field A which projects on X. We denote $A = X^h$. Using local coordinates, if $X = X^i \partial_i$, then $X^h = X^i(x)\delta_i$.

A *left splitting* of the exact sequence (5) is a map $\tilde{N} : TE \to VE$ which satisfies the conditions that $\tilde{N}$ is a morphism of vector bundles over the base E and $\tilde{N} \circ i = I_{VE}$.

Proposition 4.1. *If N is a normalization on E given by a right splitting of the exact sequence* (5), *then the formula $\tilde{N}(A)_z = (VA)_z$, $(\forall)A \in \mathcal{T}^1(E)$ defines a left splitting of the exact sequence* (5). *Locally, it means:*

$$\tilde{N}(\partial_i) = -N_i^a \partial_a\,,\ \tilde{N}(\partial_a) = \partial_a\,,\ i = 1,\dots,m\,,\ a = 1,\dots,n\,. \tag{22}$$

A *right splitting* of the exact sequence (6) is a map $N^\perp : W^\perp E \to T^*E$ which satisfies the conditions that $N^\perp$ is a morphism of vector bundles over the base E and $q \circ N^\perp = I_{W^\perp E}$.

Proposition 4.2. *If N is a normalization on E given by a right splitting of the exact sequence* (5), *then the formula*

$$N^\perp(\hat{\omega})\left(N\left(\hat{X}\right)\right) = 0\,,\ (\forall)\hat{\omega} \in \mathcal{X}\left(W^\perp E\right)\,,\ \hat{X} \in \mathcal{X}(WE) \tag{23}$$

defines a right splitting of the exact sequence (6).

We denote as $H^\perp E = N^\perp\left(W^\perp E\right)$. Notice that it is a subbundle of T^*E, supplementary to $V^\perp E$ and which is the orthogonal dual of HE. In local coordinates, the formula

$$\delta^a = N^\perp\left(\hat{d}^a\right) = d^a + N_i^a d^i\,,\ a = 1,\dots,n\,, \tag{24}$$

defines a local base (δ^a) on $H^\perp E$, which on the intersection of the domains of two charts change according the rule:

$$\delta^{a'} = \frac{\partial y^{a'}}{\partial y^a}\delta^a\,. \tag{25}$$

The normalization $N^\perp$ gives a direct sum decomposition of T^*E as:

$$T^*E = V^\perp E \oplus H^\perp E\,. \tag{26}$$

Denoting as $V^\perp$, $H^\perp$ the vertical and horizontal projectors respectively and as $F^\perp$ the almost product structure defined by the normalization $N^\perp$ on T^*E, we obtain:

$$H^\perp = N^\perp \circ q\,,\ V^\perp = I_{T^*E} - H^\perp\,,\ F^\perp = H^\perp - V^\perp\,. \tag{27}$$

A *vertical* 1-*form* on E is a section of $H^\perp E$. A vertical 1-form has a local expression $\alpha = \alpha_a(x,y)\delta^a$ and every horizontal vector field belongs to its kernel.

Remark. The normalization N can be defined by one of the tensor fields V, H, F or $V^\perp$, $H^\perp$, $F^\perp$ providing suitable conditions, or by the condition that for every function $f \in \mathcal{F}(E)$ the local functions $\dfrac{\delta f}{\delta x^i} = \dfrac{\partial f}{\partial x^i} - N_i^a \dfrac{\partial f}{\partial y^a}$ be the coordinates of a d-field of co–vectors (or a d-1-form).

Proposition 4.3. *If N is a normalization on E, then the formula*

$$\tilde{N}^\perp(\alpha)_z = (H^\perp \alpha)_z\,,\ (\forall)\,\alpha \in \mathcal{T}_1(E) \tag{28}$$

defines a left splitting of the exact sequence (6).

Using a local coordinates, we have:

$$\tilde{N}^{\perp}\left(d^{i}\right)=d^{i}\,,\quad \tilde{N}^{\perp}\left(d^{a}\right)=-N_{i}^{a}d^{i}\,. \tag{29}$$

Notice that it is easy to see that any one of the four splittings N, $\tilde{N}$, $N^{\perp}$, $\tilde{N}^{\perp}$ gives uniquely the other three and thus it defines a normalization on E.

From the above remarks it results that the following systems of local sections (δ_i, ∂_a) and (d^i, δ^a), $i = 1, \ldots, m$, $a = 1, \ldots, n$ are local dual bases adapted to the normalization N and to the adapted coordinates on E and M. In the sequel we call these bases as *natural N-adapted bases.*

Using a normalization N and natural local N-adapted bases, the study of fibred manifolds may be considerably improved. The structure equation of the normalization N are:

$$\begin{gathered}
[\delta_i,\delta_j]=-\left(\delta_i N_j^c-\delta_j N_i^c\right)\partial_c\,,\quad [\delta_i,\partial_b]=\partial_b N_i^c\partial_c\,,\quad [\partial_a,\partial_b]=0\,,\\
d\left(d^i\right)=0\,,\quad d\left(\delta^a\right)\left(\delta_i,\delta_j\right)=\delta_i N_j^a-\delta_j N_i^a\,,\\
d\left(\delta^a\right)\left(\delta_i,\partial_b\right)=-\partial_b N_i^a\,,\quad d\left(\delta^a\right)\left(\partial_b,\partial_c\right)=0\,.
\end{gathered} \tag{30}$$

Considering the Nijenhuis tensors N_H, N_V and N_F of H, $V$and F respectively we obtain:

$$\begin{gathered}
N_H\left(X^h,Y^h\right)=V\left[X^h,Y^h\right]\,,\quad N_H\left(X^h,B^v\right)=0\,,\\
N_H\left(A^v,B^v\right)=0\,,\quad N_V=N_H\,,\ N_F=4N_H\,.
\end{gathered} \tag{31}$$

Using local coordinates we have:

$$N_H\left(\delta_i,\delta_j\right)=V\left[\delta_i,\delta_j\right]=-\left(\delta_i N_j^c-\delta_j N_i^c\right)\partial_c\,,\quad N_H\left(\delta_i,\partial_b\right)=0,\ N_H\left(\partial_a,\partial_b\right)=0\,.$$

It follows the result:

Proposition 4.4. *The horizontal distribution associated to the normalization N is integrable iff $N_H = 0$.*

Coming back to the normalization N on E, in the case of vector bundles [7, 11] the tensor field $\Omega = -N_H$ is called the curvature tensor of the nonlinear connection N, since in the case when N is defined by a linear connection on M, it involves in its expression the curvature tensor of the linear connection and it vanishes simultaneously with the curvature. It was used in the paper [3] in the study of the curvature of an infinitesimal connection on a principal bundle. The equations (31) implies the useful relation:

$$\left[X^h,Y^h\right]=[X,Y]^h-\Omega\left(X^h,Y^h\right) \tag{32}$$

and thus the following result holds:

Proposition 4.5. *The horizontal lift $h:\mathcal{T}^1(M)\to\mathcal{T}^1(E)$ is a morphism of Lie algebras iff $\Omega = 0$.*

5. N-DECOMPOSABLE TENSOR FIELDS

An N-*decomposable* tensor field (or N-*tensor*) of type $\binom{p\ r}{q\ s}$ on the total space E, according to the normalization N, is a section in the vector bundle:

$$\otimes^p HE \otimes^r VE \otimes^q V^\perp E \otimes^p H^\perp E\,. \tag{33}$$

Using local coordinates and local N-adapted bases, a such N-tensor has the local expression:

$$\tilde{T}(z) = \tilde{T}^{i_1\ldots i_p a_1\ldots a_r}_{j_1\ldots j_q b_1\ldots b_s}(z)\delta_{i_1}\otimes\cdots\otimes\partial_{a_1}\otimes\cdots d^{j_1}\otimes\cdots\delta^{b_1}\otimes\cdots\otimes\delta^{b_s}\,. \tag{34}$$

We denote as $\mathcal{T}^{p,r}_{q,s}(E,N)$ and $\mathcal{T}(E,N)$ the $\mathcal{F}(E)$-module of N-decomposable tensor fields of type $\binom{p\ r}{q\ s}$ and the corresponding four-graded $\mathcal{F}(E)$-algebra. Considering a tensor field $\tilde{T} \in \mathcal{T}^{p,r}_{q,s}(E,N)$ as a multi-linear map $\tilde{T} : \mathcal{T}_1(E)^{p+r} \times \mathcal{T}^1(E)^{q+s} \to \mathcal{F}(E)$, we obtain:

Proposition 5.1. *A tensor field* $\tilde{T} \in \mathcal{T}^{p+r}_{q+s}(E)$ *is* N*-decomposable of type* $\binom{p\ r}{q\ s}$ *iff*

$$\tilde{T} = \tilde{T}\circ\left(\left(H^\perp\right)^p \times \left(V^\perp\right)^r \times H^q \times V^s\right)\,, \textit{i.e.} \tag{35}$$

$$\tilde{T}\left(\alpha_1,\ldots\beta_1,\ldots,X_1,\ldots,Y_1,\ldots\right) = \tilde{T}\left(H^\perp\alpha_1,\ldots,V^\perp\beta_1,\ldots,HX_1,\ldots,VY_1,\ldots\right)\,.$$

Every tensor field $T \in \mathcal{T}^u_v(E)$ can be decomposed as a sum of 2^{u+v} tensor fields N-decomposable of type $\binom{p\ r}{q\ s}$ with $p+r=u$ and $q+s=v$ and thus for every $u,v\in\mathbb{N}$ we have:

$$\mathcal{T}^u_v(E) = \bigoplus_{\substack{p+r=u\\ q+s=v}} \mathcal{T}^{p,r}_{q,s}(E,N)\,, \tag{36}$$

thus the two-graded algebra $\mathcal{T}(E)$ can be replaced by the four-graded algebra $\mathcal{T}(E,N)$.

Given a normalization N on E, the N-*lift* with respect to N of a d-tensor field T of type $\binom{p\ r}{q\ s}$ given by relation (8) is the N-decomposable tensor field $\tilde{T}$ which have the same type, given by relation (34), where

$$\tilde{T}^{i_1\ldots i_p a_1\ldots a_r}_{j_1\ldots j_q b_1\ldots b_s} = T^{i_1\ldots i_p a_1\ldots a_r}_{j_1\ldots j_q b_1\ldots b_s}\,. \tag{37}$$

The N-lift is an $\mathcal{F}(E)$-isomorphism of the tensor algebras $\overline{\mathcal{T}}(E)$ and $\mathcal{T}(E,N)$. Notice that there are some distinguished subalgebras of $\mathcal{T}(E,N)$, which are very important ones:

The one-graded subalgebras generated by the N-tensor fields of types $\binom{p\ 0}{0\ 0}$, $\binom{0\ r}{0\ 0}$, $\binom{0\ 0}{q\ 0}$ and $\binom{0\ 0}{0\ s}$ respectively; where $p,q,r,s\in\mathbb{N}$;

The two-graded subalgebras generated by the N-tensor fields of types $\binom{p\ 0}{q\ 0}$, $\binom{0\ r}{0\ s}$, $\binom{0\ r}{q\ 0}$ and $\binom{p\ 0}{0\ s}$;

The two-graded contravariant subalgebra, generated by the N-tensor fields of types $\binom{p\ r}{0\ 0}$ and the two-graded covariant subalgebra, generated by the N-tensor fields of types $\binom{0\ 0}{q\ s}$.

We can give suitable names to these subalgebras. For example, the subalgebra generated by N-decomposable tensor fields of type $\binom{p\ 0}{q\ 0}$ can be called the *horizontal subalgebra*. A horizontal N-tensor field has the local form $T^{i_1\dots i_p}_{j_1\dots j_q}\delta_{i_1}\otimes\cdots\delta_{i_p}\otimes d^{j_1}\otimes\cdots\otimes d^{j_q}$. For an arbitrary tensor $T\in\mathcal{T}^p_q(E)$, the tensor $HT = T\circ(H^\perp)^p\times H^q\in H\mathcal{T}^p_q(E,N)$ is called the *horizontal projection* of T.

Given a normalization N on E, the *N-horizontal lift* of a tensor field $t\in\mathcal{T}^p_q(M)$ is the tensor field $t^h\in H\mathcal{T}^p_q(E,N)$ given by

$$t^h\left(\omega_1^h,\dots,\omega_p^h,X_1^h,\dots,X_q^h\right)=t\left(\omega_1,\dots,\omega_p,X_1,\dots,X_q\right)\circ\pi\,, \tag{38}$$

$(\forall)\omega_i\in\mathcal{T}_1(M)\,,\ X_j\in\mathcal{T}^1(M)$.

An immediate consequence of the above Definition is:

Proposition 5.2. *The N-horizontal lift $h:\mathcal{T}(M)\to H\mathcal{T}(E,N)$ is an $\mathcal{F}(M)$-morphism of two-graded tensor algebras and $h=N\circ d$.*

6. Derivations and derivation laws in the algebra of N-decomposable tensor fields

Let N be a normalization on E and $D=(D_0,D_1)$ be a derivation in the tensor algebra of E, given in local N-adapted bases by:

$$\begin{aligned} D_0\left(x^i\right)=D^i\,,\quad D_0\left(y^a\right)=D^a\,,\\ D_1\left(\delta_j\right)=D_j^k\delta_k+D_j^c\partial_c\,,\quad D_1\left(\partial_b\right)=D_b^k\delta_k+D_b^c\partial_c\,. \end{aligned} \tag{39}$$

Given a normalization N on E, an *N-horizontal derivation* is a derivation on E which preserves the horizontal distribution (i.e. it sends a horizontal vector field in a horizontal vector field).

Proposition 6.1. *A derivation D is horizontal iff it satisfies one of the following equivalent conditions:*

(1) *Using the local form* (39) *one has $D_j^a=0$;* (2) *$V\circ D\circ H=0$.*

Proof. If the derivation D is horizontal then $D_j^a=0$. The condition $D_j^a=0$ implies that $V\circ D\circ H=0$, and it implies to its turn that the derivation D is horizontal. □

It is easy to see that an N-horizontal derivation D, restricted to the horizontal bundle HE by the conditions:

$$\tilde{D}_0=D_0\,,\ \tilde{D}_1\left(\delta_j\right)=D_j^k\delta_k\,,$$

defines a derivation $\tilde{D}$ on HE.

Remark. An N-horizontal derivation D induces also a derivation in the vector subbundle $H^\perp E$ and thus a derivation in the subalgebra of N-tensors of type $\binom{p\ 0}{0\ s}$, $p,s\in\mathbb{N}$.

Considering H and V as endomorphisms of the tensor algebra $\mathcal{T}(E)$, we obtain:

Proposition 6.2. *If D is a derivation on E, then the formula $\tilde{D}=H\circ D\circ H$ defines by restriction a derivation $\tilde{D}$ on HE.*

In fact we have $\tilde{D}(fA) = H \circ D \circ H(fA) = H(D(f \cdot HA)) = H(Df \cdot HA + f \cdot D\,HA) = \tilde{D}f \cdot HA + f\tilde{D}A$ for every $A \in \mathcal{T}^1(E)$, i.e. we have an *Otsuki quasi-derivation* on E. When restricted to HE we obtain a derivation on HE, since in this case $\tilde{D}(fA) = \tilde{D}f \cdot A + f\tilde{D}A$, $(\forall)A \in H\mathcal{T}^1(E)$.

The *horizontal lift* of a derivation D on the base manifold M is the derivation D^h on HE given by

$$D_0^h = (D_0)^h \ , \quad D_1^h \left(X^h\right) = (D_1 X)^h \ , \tag{40}$$

where $(D_0)^h$ is the N-horizontal lift of the vector field $D_0 \in \mathcal{T}^1(M)$. Using local coordinates, if we denote as $D(x^i) = D^i$ and $D(\partial_j) = D_j^k \partial_k$, one has:

(41)
$$D_0^h \left(x^i\right) = D^i\left(x\right) \ , \quad D_0^h\left(y^a\right) = -N_i^a D^i\left(x\right) \ , \quad D_1^h\left(\partial_j\right) = D_j^k\left(x\right)\partial_k \ .$$

Given a normalization N on E, an *N-vertical derivation* is a derivation on E which preserves the vertical distribution.

In the same manner as Proposition 6.1, we can prove:

Proposition 6.3. *A derivation D is N-vertical iff it satisfies one of the following equivalent conditions:*

(1) *Using the local form* (39) *one has* $D_a^k = 0$; (2) $H \circ D \circ V = 0$.

Remark. The N-vertical derivation D induces also, by restriction, a derivation in the vertical subbundle VE. It induces a derivation in the subbundle $V^\perp E$ and so a derivation in the subalgebra of N-tensors of type $\binom{0\ r}{q\ 0}$, $q, r \in \mathbb{N}$, i.e. the *semi-basic tensors*.

Proposition 6.4. *If D is a derivation on E, then the formula $\tilde{\tilde{D}} = V \circ D \circ V$ defines, by restriction, a derivation on VE.*

A derivation on E is *N-decomposable* (or *N-derivation*) if it preserves the algebra of N-decomposable tensor fields.

Proposition 6.5. *A derivation D on E is an N-derivation iff it satisfies one of the following equivalent conditions:* (1) *Using the local form* (39) *one has* $D_j^c = 0$, $D_a^k = 0$; (2) $DH = 0$; (3) $DV = 0$; (4) $DF = 0$; (5) *D is in the same time an N-horizontal and an N-vertical derivation;* (6) *D induces, by restriction, the derivations $\tilde{D}$ on HE and $\tilde{\tilde{D}}$ on VE respectively, such that:*

$$D_0 = \tilde{D}_0 = \tilde{\tilde{D}}_0 \, , \quad D_1 = \tilde{D}_1 \circ H + \tilde{\tilde{D}}_1 \circ V \, . \tag{42}$$

Proof. From (39) it follows that $Dd^i = -D_j^i d^j - D_b^i \delta^b, \quad D\delta^a = -D_j^a d^j - D_b^a \delta^a$. Since D preserves the four-graded algebra $\mathcal{T}(E, N)$ of N-decomposable tensor fields, it preserves the tensors of type $\binom{1\ 0}{0\ 0}$, $\binom{0\ 1}{0\ 0}$, $\binom{0\ 0}{1\ 0}$ and $\binom{0\ 0}{0\ 1}$ respectively, thus it satisfies condition 1). Conversely, if D satisfies the condition 1), then using the local form of a N-decomposable tensor field, it follows that D preserves the algebra $\mathcal{T}(E, N)$. The other conditions are obviously equivalent with 1). $\square$

Proposition 6.6. *If $\tilde{D}$ is a derivation on M and D^v is a derivation in the vertical subbundle VE, such that $\tilde{D}_0^h = D_0^v$, then the formulas:*

$$D_0 = \tilde{D}_0^h = D_0^v\,, \quad D_1 X^h = \left(\tilde{D}_1 X\right)^h\,, \quad D_1 A^v = D_1^v A^v\,, \qquad (\forall) X \in \mathcal{T}^1(M), A^v \in V\mathcal{T}^1(E)\,, \tag{43}$$

define an N-derivation on E.

Proposition 6.7. *If D is a derivation on E and F is the almost product structure associated with the normalization N, then the formula*

$$\bar{D} = \frac{1}{2}\left(D + F \circ D \circ F\right) \tag{44}$$

defines an N-decomposable derivation on E.

Proof. It is easy to see that $D^c = F \circ D \circ F$ is also a derivation on E, called in [4] as the conjugate derivation of D with respect with F. By a direct computation one get to $\bar{D}F = 0$. It can be easy proved that:

$$\bar{D} = H \circ D \circ H + V \circ D \circ V = \tilde{D} \circ H + \tilde{\tilde{D}} \circ V\,, \tag{45}$$

where $\tilde{D}$ and $\tilde{\tilde{D}}$ are the derivations induced by D on HE and VE respectively. Then one use 6) of Proposition 6.5. □

The derivation given by one of the relations (44) or (45) is called the *N-derivation associated* with D.

Proposition 6.8. *The set of N-derivations in algebra $\mathcal{T}(E)$ is given by the derivations D which have the form:*

$$D = \Phi_F\left(\overset{\circ}{D}\right) + \Omega_F\left(\tau\right)\,, \tag{46}$$

where $\overset{\circ}{D}$ is an arbitrary, but otherwise fixed derivation of the algebra $\mathcal{T}(E)$, τ is an arbitrary tensor field from $\mathcal{T}_1^1(E)$, and Φ_F, Ω_F are operators in $\mathcal{D}er\mathcal{T}(E)$ and $\mathcal{T}_1^1(E)$ respectively, defined by the formulas:

$$\Phi_F\left(D\right) = \frac{1}{2}\left(D + F \circ D \circ F\right)\,, \quad \Omega_F\left(\tau\right) = \frac{1}{2}(\tau + F \circ \tau \circ F)\,. \tag{47}$$

Proof. Let D and $\bar{D}$ be two derivations in algebra $\mathcal{T}(E)$. If we put $\bar{D} = D + \tau$, it follows that $\tau \in \mathcal{T}_1^1(E)$. Then we have $\bar{D}F = DF - 2F \circ \Omega^*(\tau)$, where Ω^* is the operator on $\mathcal{T}_1^1(E)$ given by $\Omega^*(\tau) = \frac{1}{2}(\tau - F \circ \tau \circ F)$. Thus if $DF = 0$, then $\bar{D}F = 0$ iff $\Omega^*(\tau) = 0$. So, in order to get to an arbitrary $\bar{D}$ it suffices to consider an N-derivation and an arbitrary tensor field $\tau \in \ker \Omega^*$. In conclusion, in order to obtain all the N-derivations on E we consider an arbitrary derivation $\overset{\circ}{D}$ on E, we fix it and we consider also its associated N-derivation $D = \frac{1}{2}\left(\overset{\circ}{D} + F \circ \overset{\circ}{D} \circ F\right) = \Phi_F\left(\overset{\circ}{D}\right)$. Since Ω and Ω^* are supplementary projectors on $\mathcal{T}_1^1(E)$ (as it can be easy proved), we have $\ker \Omega^* = im\Omega$ and thus $\tau \in \ker \Omega$ iff there is $\tau' \in \mathcal{T}_1^1(E)$ such that $\tau = \Omega(\tau')$. Thus omitting the accent for τ' we obtain for an arbitrary derivation on E the general form (46). □

Consider now a normalization N and a connection $\mathcal{D}$ on E. In local coordinates which are N-adapted, we denote:

$$\begin{aligned} \mathcal{D}_{\delta_i}\delta_j &= \Gamma^k_{ij}\delta_k + \Gamma^c_{ij}\partial_c,\ \mathcal{D}_{\partial_a}\delta_j = \Gamma^k_{aj}\delta_k + \Gamma^c_{aj}\partial_c\,, \\ \mathcal{D}_{\delta_i}\partial_b &= \Gamma^k_{ib}\delta_k + \Gamma^c_{ib}\partial_c,\ \mathcal{D}_{\partial_a}\partial_b = \Gamma^k_{ab}\delta_k + \Gamma^c_{ab}\partial_c\,. \end{aligned} \tag{48}$$

Let us take as derivation $D = \mathcal{D}_A$, with $A \in \mathcal{T}^1(E)$.

Given a normalization N, an N-*horizontal connection* $\mathcal{D}$ on E is a linear connection on E which preserves by parallelism the horizontal distribution.

Proposition 6.9. *A linear connection $\mathcal{D}$ on E is an N-horizontal connection iff it satisfies one of the following equivalent conditions:*

1. *using the local expressions* (48), *we have $\Gamma^c_{ij} = 0$, $\Gamma^c_{aj} = 0$;*
2. *$V \circ \mathcal{D}_A \circ H = 0$, $(\forall)A \in \mathcal{T}^1(E)$.*

An N-horizontal connection induces, by restriction, a linear connection $\widetilde{\mathcal{D}}$ on the horizontal subbundle. It induces also a linear connection in the vector subbundle $H^{\perp}E$ and thus a derivation law in the subalgebra of N-tensor fields of type $\binom{p\ 0}{0\ s}$, with $p, s \in \mathbb{N}$. Using local coordinates, we have:

$$\begin{aligned} \mathcal{D}_{\delta_i}\delta_j = \Gamma^k_{ij}\delta_k\,,\quad \mathcal{D}_{\partial_a}\delta_j = \Gamma^k_{aj}\delta_k\,, \\ \mathcal{D}_{\delta_i}\delta^b = -\Gamma^b_{ic}\delta^c\,,\quad \mathcal{D}_{\partial_a}\delta^b = -\Gamma^b_{ac}\delta^c\,. \end{aligned} \tag{49}$$

Given a normalization N, the N-*horizontal lift* of a linear connection ∇ on the base manifold M is the connection ∇^h on the horizontal vector subbundle HE, given by:

$$\begin{aligned} \nabla^h_{X^h}Y^h = (\nabla_X Y)^h\,,\quad \nabla^h_{A^v}Y^h = 0\,, \\ (\forall)X, Y \in \mathcal{T}^1(M)\,,\quad A^v \in V\mathcal{T}^1(E)\,. \end{aligned} \tag{50}$$

Using local coordinates, adapted to the normalization N, ∇^h has the form:

$$\nabla^h_{\delta_i}\delta_j = \Gamma^k_{ij}(x)\delta_k\,,\quad \nabla^h_{\partial_a}\delta_j = 0\,. \tag{51}$$

Proposition 6.10. *If $\mathcal{D}$ is a linear connection on E, then the formula $\widetilde{\mathcal{D}}_A = H \circ \mathcal{D}_A \circ H$ defines, by restriction, a linear connection on HE.*

In fact $(H, \widetilde{\mathcal{D}})$ is an Otsuki quasi-connection on E, since for $A, B \in \mathcal{T}^1(E)$ and $f \in \mathcal{F}(E)$ we have $\widetilde{\mathcal{D}}_A(fB) = A(f)\cdot HB + f\widetilde{\mathcal{D}}_A B$.

Given a normalization N on E, an N-*vertical connection* $\mathcal{D}$ on E is a linear connection on E which preserves by parallelism the vertical distribution.

Proposition 6.11. *A linear connection $\mathcal{D}$ on E is an N-vertical connection iff it satisfies one of the following equivalent conditions:*

1. *using the local expressions* (48), *we have $\Gamma^k_{ib} = 0$, $\Gamma^k_{ab} = 0$;*
2. *$H \circ \mathcal{D}_A \circ V = 0$, $(\forall)A \in \mathcal{T}^1(E)$.*

An N-vertical connection induces, by restriction, a linear connection $\widetilde{\widetilde{\mathcal{D}}}$ on the vertical subbundle. It induces also a linear connection in the vector subbundle

$V^{\perp}E$, thus a derivation law in the subalgebra of N-tensors of type $\binom{0\ r}{q\ 0}$, $q, r \in \mathbb{N}$. Using local coordinates, we have:

$$\begin{array}{ll}\mathcal{D}_{\delta_i}\partial_b = \Gamma^c_{ib}\partial_c\,, & \mathcal{D}_{\partial_a}\partial_b = \Gamma^c_{ab}\partial_c\,,\\ \mathcal{D}_{\delta_i}d^j = -\Gamma^j_{ik}d^k\,, & \mathcal{D}_{\partial_a}d^j = -\Gamma^j_{ak}d^k\,.\end{array} \tag{52}$$

According to [13], a *linear R-connection* (or a *Wong quasi-connection*) on the vertical bundle VE is a map $D : \Gamma(VE) \times \Gamma(VE) \to \Gamma(VE)$, $D(A,B) \overset{not.}{=} D^v_A B$, which enjoys the properties that it is $\mathcal{F}(E)$-linear in the first argument, additive in the second argument and $D_A(fB) = A(f)\cdot B + f\cdot D_A B$, $(\forall)A, B \in \Gamma(VE)$, $f \in \mathcal{F}(E)$. Since restricted to each leaf of the vertical foliation it induces a linear connection on that leaf, we call an R-linear connection on VE as a *connection on E along the vertical leaves.*

If N is a normalization on the fibred manifold E, the *N-vertical lift* of a linear connection D along the vertical leaves (given locally by $D^v_{s_a}s_b = \Gamma^c_{ab}(x,y)s_c$) is the linear connection D^v on the vertical subbundle VE, given by:

$$\begin{array}{c}D^v_{X^h}B^v = \left[X^h, B^v\right]\,,\ \ D^v_{A^v}B^v = D_{A^v}B^v\,,\\ (\forall)\,X \in \mathcal{T}^1(M)\,,\ \ A^v, B^v \in V\mathcal{T}^1(E)\,.\end{array} \tag{53}$$

Using N-adapted local coordinates, D^v has the form:

$$D^v_{\delta_i}\partial_b = \partial_b N^c_i \partial_c\,,\ \ D^v_{\partial_a}\partial_b = \Gamma^c_{ab}\partial_c\,. \tag{54}$$

Proposition 6.12. *If $\mathcal{D}$ is a connection on E, then the formula $\tilde{\tilde{\mathcal{D}}}_A = V\circ\mathcal{D}_A\circ V$ defines, by restriction, a linear connection on VE.*

A linear connection $\mathcal{D}$ on E is called *N-decomposable* (or *N-connection*) if it induces a derivation law in algebra $\mathcal{T}(E,N)$ of N-decomposable tensor fields.

Proposition 6.13. *A linear connection $\mathcal{D}$ on E is N-decomposable iff it satisfies one of the following equivalent conditions:* (1) *Using the local form* (48) *one has $\Gamma^c_{ij} = 0$, $\Gamma^c_{aj} = 0$, $\Gamma^k_{ib} = 0$; $\Gamma^k_{ab} = 0$;* (2) *$\mathcal{D}H = 0$;* (3) *$\mathcal{D}V = 0$;* (4) *$\mathcal{D}F = 0$;* (5) *$\mathcal{D}$ is in the same time an N-horizontal and an N-vertical connection;* (6) *There are the linear connections $\tilde{\mathcal{D}}$ on HE and $\tilde{\tilde{\mathcal{D}}}$ on VE respectively, such that: $\mathcal{D} = \tilde{\mathcal{D}}_1\circ H + \tilde{\tilde{\mathcal{D}}}\circ V$.*

Proposition 6.14. *Consider a fibred manifold E, ∇ a linear connection on the base M and D a linear connection on E along the vertical leaves.*

Then the formulas:

$$\begin{array}{c}\mathcal{D}_{X^h}Y^h = \left(\nabla_X Y\right)^h\,,\ \ \mathcal{D}_{A^v}Y^h = 0\,,\\ \mathcal{D}_{X^h}B^v = \left[X^h, B^v\right]\,,\ \ \mathcal{D}_{A^v}B^v = D^v_{A^v}B^v\end{array} \tag{55}$$

*$(\forall)X, Y \in \mathcal{T}^1(M)$, $A^v, B^v \in V\mathcal{T}^1(E)$, defines an N-connection on E, which we call the N-*lift *or the* diagonal lift *of the pair (∇, D).*

Proposition 6.15. *If $\mathcal{D}$ is a linear connection on E, then the formula*

$$\overline{\mathcal{D}}_A = \frac{1}{2}\left(\mathcal{D}_A + F\circ\mathcal{D}_A\circ F\right)\,,\ \ (\forall)A \in \mathcal{T}^1(E)\,, \tag{56}$$

defines an N-connection on E and the following formula also holds:

$$\overline{\mathcal{D}}_A = H \circ \mathcal{D}_A \circ H + V \circ \mathcal{D}_A \circ V = \widetilde{\mathcal{D}}_A \circ H + \widetilde{\widetilde{\mathcal{D}}}_A \circ V\,, \quad (\forall)\, A \in \mathcal{T}^1(E)\,. \tag{57}$$

We call $\overline{\mathcal{D}}$ *the* N-connection associated with $\mathcal{D}$.

Taking $D = \mathcal{D}_A$ in Proposition 6.8 we obtain:

Proposition 6.16. *The set of N-connections (or N-decomposable connections) on E is given by the linear connections $\mathcal{D}$ which have the form:*

$$\mathcal{D} = \Phi_F\left(\overset{\circ}{\mathcal{D}}\right) + \Omega_F\left(\tau\right)\,, \tag{58}$$

where $\overset{\circ}{\mathcal{D}}$ is an arbitrary, but otherwise fixed linear connection on E, τ is an arbitrary tensor field from $\mathcal{T}_2^1(E)$ and Φ_F, Ω_F are operators defined on the set of linear connections on E and on $\mathcal{T}_2^1(E)$ respectively, by the formulas:

$$\Phi_F\left(\mathcal{D}\right)_A = \tfrac{1}{2}\left(\mathcal{D}_A + F \circ \mathcal{D}_A \circ F\right),\ \Omega_F\left(\tau\right)_A = \tfrac{1}{2}\left(\tau_A + F \circ \tau_A \circ F\right)\,, \quad (\forall) A \in \mathcal{T}^1\left(E\right)\,. \tag{59}$$

Remark. See [4] concerning the geometrical structure of the set of connections and the simple method which follows it in order to determine the set of connections which are compatible with an almost product structure and thus with the normalization N.

If $\mathcal{D}$ is an N-connection on E, thus it satisfies the condition $\mathcal{D}F = 0$, then, using local coordinates and local N- adapted bases, we have the following formula:

(60)

$$\mathcal{D}_{\delta_i}\delta_j = \Gamma^k_{ij}\delta_k\,,\quad \mathcal{D}_{\partial_a}\delta_j = \Gamma^k_{aj}\delta_k\,,\quad \mathcal{D}_{\delta_i}\partial_b = \Gamma^c_{ib}\partial_c\,,\quad \mathcal{D}_{\partial_a}\partial_b = \Gamma^c_{ab}\partial_c\,.$$

Remark. An N-connection on a fibred manifold is called a linear d- connection in [12].

Using adapted coordinates and N-adapted bases the local functions $\{\Gamma^k_{ij},\ \Gamma^k_{aj},\ \Gamma^c_{ib},\ \Gamma^c_{ab}\}$ change according the formulas:

$$\Gamma^{i'}_{j'k'}(x^{i'}, y^{a'})\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^{k'}}{\partial x^k} = \Gamma^i_{jk}(x^i, y^a)\frac{\partial x^{i'}}{\partial x^i} - \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k} \tag{61}$$

$$\Gamma^{a'}_{j'c'}(x^{i'}, y^{d'})\frac{\partial x^{j'}}{\partial x^j}\frac{\partial y^{c'}}{\partial y^c} = \Gamma^a_{jc}(x^i, y^d)\frac{\partial y^{a'}}{\partial y^a} - \frac{\delta \partial y^{a'}}{\delta x^j \partial y^c} \tag{62}$$

$$\Gamma^{i'}_{b'k'}(x^{i'}, y^{a'})\frac{\partial y^{b'}}{\partial y^b}\frac{\partial x^{k'}}{\partial x^k} = \Gamma^i_{bk}(x^i, y^a)\frac{\partial x^{i'}}{\partial x^i} \tag{63}$$

$$\Gamma^{a'}_{b'c'}(x^{i'}, y^{d'})\frac{\partial y^{b'}}{\partial y^b}\frac{\partial y^{c'}}{\partial y^c} = \Gamma^a_{bc}(x^i, y^d)\frac{\partial y^{a'}}{\partial y^a} - \frac{\partial^2 y^{a'}}{\partial y^b \partial y^c}, \tag{64}$$

(see [12], using different notations).

In [12] are also given the structure equations of the linear N-connection D, using the local N-adapted bases $\{dx^i, \delta y^a\}$. So, consider:

$$\omega^i_k = \mu^i_k + \theta^i_k\,, \quad \mu^i_k = \Gamma^i_{jk} dx^j\,, \ \ \theta^i_k = \Gamma^i_{bk}\delta y^b \tag{65}$$

$$\bar{\omega}^a_c = \bar{\mu}^a_c + \bar{\theta}^a_c\,, \quad \bar{\mu}^a_c = \Gamma^a_{jc} dx^j\,, \ \theta^a_c = \Gamma^a_{bc}\delta y^b\,. \tag{66}$$

The curvature forms involved by the structure equations of D, are: $\{R^i_k, \bar{R}^a_c, P^i_k, \bar{P}^a_c, S^i_k, \bar{S}^a_c\}$, where $R^i_k = \dfrac{1}{2}R^i_{kjl}dx^j\wedge dx^l$, $\bar{R}^a_c = \frac{1}{2}R^a_{cjl}dx^j\wedge dx^l$, $P^i_k = R^i_{kjb}dx^j\wedge\delta y^b$, $\bar{P}^a_c = R^a_{cjb}dx^j\wedge\delta y^b$, $S^i_k = \frac{1}{2}R^i_{kb\delta}\delta y^b\wedge\delta y^\delta$ and $\bar{S}^a_c = \frac{1}{2}R^a_{cb\delta}\delta y^b\wedge\delta y^\delta$. Notice that all these local forms have a change rule as tensors on the intersection of two charts.

Proposition 6.17. [12] *Using N-adapted bases, the structure equations of the linear N-connection D have the form:*

$$R^i_k + P^i_k + S^i_k = d\left(\mu^i_k + \theta^i_k\right) + \left(\mu^i_j + \theta^i_j\right)\wedge\left(\mu^j_k + \theta^j_k\right)$$

$$\bar{R}^a_c + \bar{P}^a_c + \bar{S}^a_c = d\left(\bar{\mu}^a_c + \bar{\theta}^a_c\right) + \left(\bar{\mu}^a_b + \bar{\theta}^a_b\right)\wedge\left(\bar{\mu}^b_c + \bar{\theta}^b_c\right)\,.$$

We deal now with the problem to find uniquely a linear N-connection from some suitable conditions.

If we impose the supplementary condition $T\circ(H\times V) = 0$, where T is the torsion field of $\mathcal{D}$, then we obtain:

$$\Gamma^k_{aj} = 0\,, \quad \Gamma^c_{ib} = \partial_b N^c_i \tag{67}$$

and the coefficients Γ^k_{ij} and Γ^c_{ab} remain to be determined.. In order to determine the coefficients Γ^k_{ij} we can give a linear connection ∇ on the base M and then take its horizontal lift ∇^h, thus $\Gamma^k_{ij} = \Gamma^k_{ij}(x)$. Or we can take a metric g on M, its Levi Civita linear connection ∇ and the horizontal lift ∇^h. In order to determine the coefficients Γ^c_{ab} we can take a linear connection on E along the vertical leaves, or the corresponding Levi-Civita R-connection of a metric γ^v on the fibers of the vertical bundle.

Proposition 6.18. *Consider a fibred manifold E with a normalization N, a linear connection ∇ on the base M and a linear connection D^v on E along the leaves of the vertical subbundle. Then the diagonal lift of the pair (∇, D^v) is the unique linear N-connection $\mathcal{D}$ on E which satisfies the conditions:*

$$\begin{gathered}\mathcal{D}_{X^h}Y^h = (\nabla_X Y)^h\,,\ \mathcal{D}_{A^v}B^v = D^v_{A^v}B^v\,,\ H\circ\mathcal{D}_{A^v}\circ V = 0\,,\\ V\circ\mathcal{D}_{X^h}\circ H = 0\,,\ T\circ(H\times V) = 0\,,\end{gathered} \tag{68}$$

$(\forall)X, Y\in\mathcal{T}^1(M)$, $A^v, B^v\in V\mathcal{T}^1(E)$.

In fact, from here it follows that $\mathcal{D}_{A^v}X^h = 0$, $\mathcal{D}_{X^h}B^v = [X^h, B^v]$ and all the conditions (55) are satisfied.

According to [12] we say that the fibred manifold E, with the canonical projection π, has a *vertical induced bundle*, if there is a vector bundle F_0 which has the same base M as E, such that the vertical vector bundle VE of E is isomorphic with the induced vector bundle $\pi^* F_0$. A remarkable example of a such fibred manifold is $\mathcal{O}sc^k(M)$, the osculator bundle of order k of a manifold M (see [10]). Notice that a tensor of type $(1,2)$ on F_0 defines uniquely a linear connection along the vertical leaves.

Proposition 6.19. *Consider a fibred manifold E with a vertical induced bundle, where $VE = \pi^* F_0$ and ∇ a linear connection on the base M. Then there is a unique linear N-connection on E which satisfies the conditions:*

$$\begin{array}{c} \mathcal{D}_{X^h} Y^h = (\nabla_X Y)^h\,,\ \mathcal{D}_{A^v} B^v = 0\,,\ H \circ \mathcal{D}_{A^v} \circ V = 0\,, \\ V \circ \mathcal{D}_{X^h} \circ H = 0\,,\ T \circ (H \times V) = 0\,, \end{array} \tag{69}$$

$(\forall) X, Y \in \mathcal{T}^1(M)$, $A^v, B^v \in V\mathcal{T}^1(E)$.

It follows from here $\mathcal{D}_{A^v} X^h = 0$, $\mathcal{D}_{X^h} B^v = [X^h, B^v]$ and also $\mathcal{D}F = 0$.

Remark. In the case when E is a vector bundle the N-connection given by the above proposition gives the Berwald connection when it is restricted to VE.

Taking a metric on M and a metric on the fibers of the vertical bundle, one get:

Proposition 6.20. *Given a normalization N on a fibred manifold E, a metric g on M and a metric γ^v on the fibers of the vertical bundle VE, then there is a unique N-connection $\mathcal{D}$ on E which satisfies the conditions:*

$$\begin{array}{c} \mathcal{D}_{X^h} Y^h = \left(\nabla^g_X Y\right)^h\,,\ \mathcal{D}_{A^v} B^v = \mathcal{D}^{\gamma_v}_{A^v} B^v\,,\ H \circ \mathcal{D}_{A^v} \circ V = 0\,, \\ V \circ \mathcal{D}_{X^h} \circ H = 0\,,\ T \circ (H \times V) = 0\,, \end{array} \tag{70}$$

$(\forall) X, Y \in \mathcal{T}^1(M)$, $A^v, B^v \in V\mathcal{T}^1(E)$, *where ∇^g is the Levi-Civita connection on M and ∇^{γ^v} is the Levi-Civita R-connection on the fibers of VE respectively.*

In fact from here it follows $\mathcal{D}_{A^v} X^h = 0$, $\mathcal{D}_{X^h} B^v = [X^h, B^v]$, then $\mathcal{D}F = 0$. We have also $\mathcal{D}g^h = 0$, $H \circ T \circ (H \times H) = 0$, $\mathcal{D}\gamma^v = 0$, $V \circ T \circ (V \times V) = 0$.

Remarks. 1. Using the notations in Proposition 6.20, then $G = g^h + \gamma^v$ (where g^h is the horizontal lift of g) is a metric on E, which generalizes the Sasaki metric and has the local form:

$$G = g_{ij}(x) d^i \otimes d^j + \gamma_{ab}(x,y) \delta^a \otimes \delta^b\,. \tag{71}$$

We also have:

(72)

$$\begin{array}{c} \mathcal{D}_{X^h} G\left(Y^h, Z^h\right) = \left(\nabla^g_X g\right)(Y, Z) \circ \pi\,,\ \mathcal{D}_{X^h} G\left(Y^h, B^v\right) = 0\,, \\ \mathcal{D}_{X^h} G\left(B^v, C^v\right) = L_{X^h} \gamma^v\left(B^v, C^v\right)\,,\ \ \mathcal{D}_{A^v} G\left(Y^h, Z^h\right) = 0\,, \\ \mathcal{D}_{A^v} G\left(Y^h, C^v\right) = 0\,,\ \mathcal{D}_{A^v} G\left(B^v, C^v\right) = \left(\nabla^{\gamma^v}_{A^v} \gamma^v\right)\left(B^v, C^v\right) = 0\,. \end{array}$$

2. The Levi-Civita connection of an R-connection is obtained using the same formula as for the Levi-Civita connection on the tangent bundle, but using on the vertical bundle the new bracket (which is the restriction on the vertical bundle of the Lie bracket defined on the tangent bundle) instead of the Lie bracket.

Proposition 6.21. *Using the notation from Proposition 6.20, the metric given by* (71) *satisfies the condition $\mathcal{D}G = 0$ iff X^h is a Killing vector field for the metric*

γ^v, *for every* $X \in \mathcal{T}^1(M)$. *The connection* $\overline{\mathcal{D}}_A = \frac{1}{2}\left(\mathcal{D}_A + G^{-1} \circ \mathcal{D}_A \circ G\right)$ *satisfies the condition* $\mathcal{D}G = 0$.

The proof of the second statement can be obtained using [4].

7. N-projectable geometrical objects on E

Consider now the endomorphism H defined on the tensor algebra of E by:
$Hf = f$; $HA = HA$; $H\alpha = \alpha \circ H$;
$(HT)(\alpha_1, \ldots, \alpha_u, A_1, \ldots, A_v) = T(H(\alpha_1), \ldots, H(\alpha_u), H(A_1), \ldots, H(A_v))$,
$(\forall) f \in \mathcal{F}(E)$, $A, A_i \in \mathcal{T}^1(E)$, $\alpha, \alpha_j \in \mathcal{T}_1(E)$, $T \in \mathcal{T}_v^u(E)$.
It is easy to see that the values of the endomorphism H are in the horizontal algebra. Locally, we have:

$$H(\delta_i) = \delta_i\,, \quad H\left(d^j\right) = d^j\,, \quad H(\partial_a) = 0\,, \quad H\left(\delta^b\right) = 0\,.$$

A tensor field $T \in \mathcal{T}_v^u(E)$ is *N-projectable* if the N-horizontal tensor field HT is projectable.

Since for every vector field A on E we have locally, in N-adapted bases $A = A^i\delta_i + A^a\partial_a$ and $HA = A^i(\partial_i - N_i^a\partial_a)$, we obtain from Proposition 3.1:

Proposition 7.1. *A vector field A on E is N-projectable iff it satisfies one of the following equivalent conditions:*

1. *A is projectable;*
2. *using local adapted bases, the local functions A^i are projectable;*
3. *there is a vector field $X \in \mathcal{T}^1(M)$ such that $HA = X^h$ (i.e. the associated horizontal vector field is the horizontal lift of X).*

A differential 1-form $\alpha \in \mathcal{T}_1(E)$ has locally, in N-adapted bases, the form $\alpha = \alpha_i d^i + \alpha_a\delta^a$ and have $H\alpha = \alpha_i d^i$. From Proposition 3.2 it results:

Proposition 7.2. *A differential 1-form $\alpha \in \mathcal{T}_1(E)$ is N-projectable iff it satisfies one of the following equivalent conditions:*

1. *using local N-adapted bases, the local functions α_i are projectable;*
2. *there is a differential 1-form $\omega \in \mathcal{T}_1(M)$ such that $H\alpha = \omega^h$.*

Consider now a tensor field $T \in \mathcal{T}_v^u(E)$, which has in local N-adapted bases the form $T = T_{j_1\ldots j_v}^{i_1\ldots i_u}\delta_{i_1} \otimes \cdots \otimes \delta_{i_u} \otimes d^{j_1} \otimes \cdots \otimes d^{j_v} + \cdots + T_{b_1\ldots b_v}^{a_1\ldots a_u}\partial_{a_1} \otimes \cdots \otimes \partial_{a_u} \otimes \delta^{b_1} \otimes \cdots \otimes \delta^{b_v}$. We obtain $HT = T_{j_1\ldots j_v}^{i_1\ldots i_u}\delta_{i_1} \otimes \cdots \otimes \delta_{i_u} \otimes d^{j_1} \otimes \cdots \otimes d^{j_v}$. Replacing $\delta_{i_\alpha} = \partial_{i_\alpha} - N_{i_\alpha}^a\partial_a$, $\alpha = \overline{1,u}$ one obtains: $HT = T_{j_1\ldots j_v}^{i_1\ldots i_u}\partial_{i_1} \otimes \cdots \otimes \partial_{i_u} \otimes d^{j_1} \otimes \cdots \otimes d^{j_v} - T_{j_1\ldots j_v}^{i_1\ldots i_u}N_{i_1}^{a_1}\partial_{a_1} \otimes \partial_{i_2} \cdots \otimes \partial_{i_u} \otimes d^{j_1} \otimes \cdots \otimes d^{j_v} + \cdots + (-1)^u T_{j_1\ldots j_v}^{i_1\ldots i_u}N_{i_1}^{a_1} \cdots N_{i_u}^{a_u}\partial_{a_1} \otimes \cdots \otimes \partial_{a_u} \otimes d^{j_1} \otimes \cdots \otimes d^{j_v}$. Hence, considering HT as a tensor on E and using natural local bases (∂_i, ∂_a), (d^j, d^b), it has as coordinates $(HT)_{j_1\ldots j_v}^{i_1\ldots i_u} = T_{j_1\ldots j_v}^{i_1\ldots i_u}$, $(HT)_{b_1 j_2\ldots j_v}^{i_1\ldots i_u} = 0$,..., $(HT)_{b_1\ldots b_v}^{i_1\ldots i_u} = 0$. It follows that HT is projectable iff, using local N-adapted bases (δ_i, ∂_a), (d^j, δ^b), the coordinates $T_{j_1\ldots j_v}^{i_1\ldots i_u}$ of T are projectable. Thus from Proposition 3.3 we have:

Proposition 7.3. *A tensor field $T \in \mathcal{T}_q^p(E)$ is N-projectable iff it satisfies one of the following equivalent conditions:*

1. *in natural local N-adapted bases, the functions $T_{j_1 \dots j_v}^{i_1 \dots i_u}$ are projectable;*
2. *for all systems $\{\alpha_1, \dots, \alpha_u\}$ and $\{A_1, \dots A_v\}$ of horizontal projectable differentiable 1-forms and horizontal projectable vector fields respectively, the function $T(\alpha_1, \dots, \alpha_u, A_1, \dots A_v)$ is projectable, too;*
3. *for every vector field $A^v \in V\mathcal{T}^1(E)$ one has $(H \circ \mathcal{L}_{A^v} \circ H)T = 0$;*
4. *there is a tensor field $t \in \mathcal{T}_q^p(M)$ such that $HT = t^h$.*

In the proof one uses the definition of the horizontal lift, the fact that the Lie derivation with respect with a vertical field A^v satisfies the condition $\mathcal{L}_{A^v} X^h = [A^v, X^h]$ is a vertical field and $\mathcal{L}_{A^v} \omega^h = 0$.

Remarks. 1. A contravariant tensor field $T \in \mathcal{T}_0^u(E)$ is N-projectable iff it is projectable.

2. A covariant tensor field $T \in \mathcal{T}_v^0(E)$ is N-projectable iff $(\mathcal{L}_{A^v} \circ H)T = 0$, $(\forall) A^v \in V\mathcal{T}^1(E)$.

A derivation D in the tensor algebra $\mathcal{T}(E)$ is called *N-projectable* if for every projectable horizontal vector field D, the vector field DA is projectable, too.

Taking into account of the local form (39), a) of D, in a natural N-adapted base, we obtain:

Proposition 7.4. *A derivation D in tensor algebra $\mathcal{T}(E)$ is N-projectable iff it satisfies one of the following equivalent conditions:*

1. *in local N-adapted bases, the functions D^i and D_j^i are projectable;*
2. *for every N-projectable horizontal 1-form α and N-projectable horizontal vector field A, the function $\alpha(DA)$ is projectable;*
3. *there is a derivation $\tilde{D}$ in tensor algebra $\mathcal{T}(M)$ such that $H(DX^h) = (\tilde{D}X)^h$, $(\forall) X \in \mathcal{T}^1(M)$.*

Remarks. 1. The derivation $D = \mathcal{L}_A$, $A \in \mathcal{T}^1(E)$, is N-projectable iff A is projectable.

2. The derivation $i(S)$ on E, induced by $S \in \mathcal{T}_1^1(E)$, is N-projectable iff S is N-projectable.

If we consider a linear connection on E and we associate the derivation $D = \mathcal{D}_A$ with every vector field $A \in \mathcal{T}^1(E)$, then we have:

A linear connection $\mathcal{D}$ on E is *N-projectable* if for every projectable horizontal vector field A the derivation $\mathcal{D}_A$ is N-projectable.

Proposition 7.5. *A linear connection $\mathcal{D}$ on E is N-projectable iff it satisfies one of the following equivalent conditions:*

1. *using natural N-adapted local bases, the local coefficients Γ_{ij}^k are projectable;*
2. *for every N-projectable horizontal vector fields A and B, the vector field $\mathcal{D}_A B$ is N-projectable;*
3. *there is a linear connection ∇ on M such that $H(\mathcal{D}_{X^h} Y^h) = (\nabla_X Y)^h$, $(\forall) X, Y \in \mathcal{T}^1(M)$.*

Remark. If the linear connection $\mathcal{D}$ is N-projectable, then its torsion $\mathcal{T}$ is N-projectable and we have $(H\mathcal{T})(X^h, Y^h) = t(X,Y)^h$.

Concerning the curvature of an N-projectable linear connection, we have:

Proposition 7.6. *If $\mathcal{D}$ is an N-projectable linear connection, then the curvature tensor field $\mathcal{R}$ of $\mathcal{D}$ is an N-projectable tensor field iff $\mathcal{D}_A(V\mathcal{D}_B C) - \mathcal{D}_B(V\mathcal{D}_A C) - \mathcal{D}_{[A,B]}C$ is an N-projectable vector field, for every horizontal N-projectable vector fields A, B, C on E.*

Proof. Since $\mathcal{R}(A,B)C = \mathcal{D}_A\mathcal{D}_B C - \mathcal{D}_B\mathcal{D}_A C - \mathcal{D}_{[A,B]}C$, using Propositions 7.5 2) and 7.3 the conclusion follows. □

If A and B are horizontal N-projectable vector fields, then $H[A,B]$ is an horizontal N-projectable vector field and $V[A,B] = -N_H(A,B)$ $(= -\Omega(A,B))$. It follows the following result:

Proposition 7.7. *If $\mathcal{D}$ is an N-projectable linear connection, then the curvature tensor field $\mathcal{R}$ of $\mathcal{D}$ is an N-projectable tensor field iff $\mathcal{D}_A(V\mathcal{D}_B C) - \mathcal{D}_B(V\mathcal{D}_A C) + \mathcal{D}_{\Omega(A,B)}C$ is an N-projectable vector field, for every horizontal N-projectable vector fields A, B, C on E.*

Remark. If $\mathcal{D}$ is an N-projectable linear N-connection, then the curvature tensor field $\mathcal{R}$ of $\mathcal{D}$ is an N-projectable tensor field iff $\mathcal{D}_{[A,B]}C$ (or, equivalently, $\mathcal{D}_{\Omega(A,B)}C$) is an N-projectable vector field, for every horizontal N-projectable vector fields A, B, C on E. Particularly, if $\Omega = 0$ (i.e. the horizontal distribution HE is integrable), then the curvature tensor field $\mathcal{R}$ of an N-projectable linear N-connection $\mathcal{D}$ is an N-projectable tensor field.

References

[1] Ako, M., *Fibred spaces with almost complex structure*, Kodai Math. Sem. Rep., **29** (1972), 482–505.

[2] Bejancu, A., *Finsler Geometry and Applications*, Math. and its appl., Chichester, 1988.

[3] Cruceanu, V., *Sur la structure presque produit assiciée à une connection sur un espace fibré*, An. St. Univ. "Al.I.Cuza" Iassy, **XV** (1969), 159–167.

[4] Cruceanu, V.,*Connections compatibles avec certaines structures sur un fibré vectoriel banachique*, Czechoslovak Math. J., **24** (1974), 126–142.

[5] Cruceanu, V., *A New definition for certaines lifts on a Vector Bundle*, An. St. Univ. "Al.I.Cuza" Iassy, **42** (1996), 59–72.

[6] Gray, A., *Pseudo-Riemannian almost products manifolds and submersions*, J.Math.and Mech. **16** (1967), 715–738.

[7] Grifone, J., *Structures presque-tangente et connexions*, I, II, Ann. Inst. Fourier, **22** (1972), 287–334, 291–338.

[8] Ishihara, S. and Konishi, M., *Differential Geometry of Fibred Spaces*, Publ. Study Group of Geometry, **8** (1973), 1–200.

[9] Mangiarotti, L. and Modugno, M., *Connections and Differential Calculus on Fibred Manifolds*, Istituto de Mat. Appl. "G. Sansone", Italy, 1989, 147 pp.

[10] Miron, R. *The geometry of Higher-Order Lagrange Spaces. Aplications to Mechanics and Phisics.* Kluwer Acad. Publ. FTPH (1996).

[11] Miron, R. and Anastasiei, M., *The Geometry of Lagrange Spaces. Theory and Applications.* Kluwer Acad. Publ. **59** (1994).

[12] Popescu, M. and Popescu, P., *d-Linear Connections on Fibred Manifolds*, An. Univ. Craiova, Ser. mat., **XXIII** (1996), 52–59.

[13] Popescu, P., *On the geometry of R-tangent spaces*, Rev. Roum. Math. Pures Appl., **37**(1992), 8, 727–733.

[14] Yano, K. and Ishihara, S., *Differential Geometry of Fibred Spaces*, Kodai Math. Sem. Rep., **19** (1967), 257–288.

Vasile CRUCEANU
"Al.I.Cuza" University of Iassy
Department of Geometry
11, Copou st., 6600, ROMANIA
E-mail: cruv@uaic.ro

Marcela POPESCU and Paul POPESCU
University of Craiova
Department of Mathematics
P.O. Box 4-66, Craiova 1100, ROMANIA
E-mail: marcelapopescu@email.com paulpopescu@email.com