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COINCIDENCE POINTS AND R–WEAKLY COMMUTING MAPS

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Abstract. In this paper we extend the concept of R-weak commutativity to the setting of single-valued and multivalued mappings. We also establish a coincidence theorem for pairs of R-weakly commuting single-valued and multivalued mappings satisfying a contractive type condition.

Recently, Cho, Fisher and Jeong [2] and Rashwan [9] extended independently the notion of compatibility [4] for multivalued mappings, which is a slight varient of the notions given by Kaneko and Sessa [6] and Beg and Azam [1]. They also showed that every weakly commuting pair of multivalued mappings is compatible but the converse is not true. In this paper we produce some examples which show that weak commutativity does not imply the existence of sequences of points satisfying the compatibility condition. For the single-valued case we refer the reader to Singh (Math. Rev. 89h:54030). We may mention that in such cases, the condition of compatibility is satisfied in a vacuous sort of way. We also extend the concept of R-weak commutativity introduced by Pant [8] to the setting of single-valued and multivalued mappings. Finally, we obtain a coincidence theorem for pairs of R-weakly commuting single-valued and multivalued mappings satisfying a contractive type condition. Related (but different) problems were also studied in [2, 9].

Throughout this paper, X stands for a metric space with the metric d whereas $CB(X)$ denotes the family of all nonempty closed bounded subsets of X. Let

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}\,,$$

where $A, B \in CB(X)$ and $d(x,A) = \inf\{d(x,y) : y \in A\}$. The function H is a metric on $CB(X)$ and is called the Hausdorff metric. It is well known that if X is a complete metric space, then so is the metric space $(CB(X), H)$. The mappings $f : X \to X$ and $T : X \to CB(X)$ are said to be (1) weakly commuting [2, 9]

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if for all $x \in X$, $fTx \in CB(X)$ and $H(fTx, Tfx) \le d(fx, Tx)$; (2) compatible [2, 9] if $\lim_{n\to\infty} d(fy_n, Tfx_n) = 0$ whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = z$ for some $z \in X$, where $y_n \in Tx_n$ for $n = 1, 2, \ldots$; (3) R-weakly commuting if for all $x \in X$, $fTx \in CB(X)$ and there exists some positive real number R such that

$$H(Tfx, fTx) \le Rd(fx, Tx)\,.$$

In fact, every weakly commuting pair of mappings (f, T) is compatible but the converse is not true in general. However, the following examples show that weak commutativity does not imply the existence of sequences of points satisfying the compatibility condition.

Example 1. Let $X = [1, \infty)$ be endowed with the usual metric d. Let $fx = 2x$, $Tx = [1, x]$ for all $x \in X$. Then

$$H(fTx, Tfx) = 1 \le x = d(fx, Tx)$$

for all $x \in X$. The mappings f and T are thus weakly commuting but there exist no sequences $\{x_n\}$, $\{y_n\}$ in X such that the condition of compatibility is satisfied.

Example 2. Let $X = [0, \frac{3}{5}]$ and d the usual metric. Define $f : X \to X$ and $T : X \to CB(X)$ by $fx = x^2$, $Tx = \left[0, \frac{x^4+1}{2}\right]$ for all $x \in X$. Then

$$\begin{aligned} H(fTx, Tfx) = \frac{(x^4-1)^2}{4} &= \frac{(x^2+1)^2}{2}\frac{(x^2-1)^2}{2} \\ &\le \frac{1}{2}\left(\frac{9}{25}+1\right)^2 d(fx, Tx) \\ &< d(fx, Tx) \end{aligned}$$

for all $x \in X$. This shows that f and T are weakly commuting. But there exist no sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying the compatibility condition.

Example 3. Let $X = [2, \infty)$ be endowed with usual metric d. Let $fx = x^2$, $Tx = [2, 2x-1]$ for all $x \in X$. Then

$$H(fTx, Tfx) = 2d(fx, Tx)\,.$$

Hence f and T are R-weakly commuting with $R = 2$ but they are not weakly commuting.

For $A, B \in CB(X)$, let $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$.

Lemma 4 [7]. *Let $A, B \in CB(X)$ and $k > 1$. Then for each $a \in A$, there exists an element $b \in B$ such that*

$$d(a, b) \le kH(A, B)\,.$$

Theorem 5. *Let X be a complete metric space, f, $g : X \to X$ continuous mappings and S, $T : X \to CB(X)$ H-continuous mappings such that $TX \subseteq fX$, $SX \subseteq gX$. If the pairs (f,S) and (g,T) are R-weakly commuting and*

$$(1)\quad H^p(Sx,Ty) \leq \frac{q[\alpha d^p(fx,Sx)d^p(gy,Ty)+\beta d^{2p}(fx,gy)+\gamma d^p(fx,Ty)d^p(gy,Sx)]}{\alpha\max\{\delta^p(fx,Sx),\delta^p(gy,Ty)\}+\beta d^p(fx,gy)}$$

for all $x, y \in X$ with

$$\alpha\max\{\delta(fx,Sx),\delta(gy,Ty)\}+\beta d(fx,gy)\neq 0\,,$$

$0\leq q<1$, $p\geq 1$ and α, β, $\gamma\geq 0$, (not all zero), then there exists a sequence $\{x_n\}$ in X such that

(a) *for every n, $fx_{2n-1}\in Tx_{2n-2}$, $gx_{2n}\in Sx_{2n-1}$,*
(b) *there exists $z=\lim\limits_{n\to\infty} gx_{2n}=\lim\limits_{n\to\infty} fx_{2n-1}$,*
(c) *$fz\in Sz$, $gz\in Tz$.*

Proof. Choose a real number k such that $1<k<\left(\frac{1}{q}\right)^{1/p}$. Let x_0 be an arbitrary point of X. Since $TX\subseteq fX$, there exists $x_1\in X$ such that $fx_1\in Tx_0$. It follows from Lemma 4 that there exists $u_1\in Sx_1$ such that

$$d(u_1,fx_1)\leq kH(SX_1,Tx_0)\,,$$

where $k>1$. Moreover, since $SX\subseteq gX$, there exists a point x_2 in X such that $u_1=gx_2$ and

$$d(gx_2,fx_1)\leq kH(Sx_1,Tx_0)\,.$$

Proceeding in this way, we can obtain a sequence $\{x_n\}$ in X such that for each n

$$(2)\qquad d(gx_{2n},fx_{2n-1})\leq kH(Sx_{2n-1},Tx_{2n-2})$$

and

$$(3)\qquad d(fx_{2n+1},gx_{2n})\leq kH(Tx_{2n},Sx_{2n-1})$$

where

$$gx_{2n}\in Sx_{2n-1}\quad\text{and}\quad fx_{2n-1}\in Tx_{2n-2}\,.$$

Using (1), (2) and (3), we get

$$\begin{aligned}
d^p(gx_{2n},&fx_{2n+1})\leq\\
&\leq\frac{k^pq[\alpha d^p(fx_{2n-1},Sx_{2n-1})d^p(gx_{2n},Tx_{2n})+\beta d^{2p}(fx_{2n-1},gx_{2n})]}{\alpha\max\{\delta^p(fx_{2n-1},Sx_{2n-1}),\delta^p(gx_{2n},Tx_{2n})\}+\beta d^p(fx_{2n-1},gx_{2n})}\\
&\leq k^pqd^p(fx_{2n-1},gx_{2n})\\
&\leq(k^pq)(k^p)H^p(Sx_{2n-1},Tx_{2n-2})\\
&\leq(k^{2p}q^2)d^p(gx_{2n-2},fx_{2n-1})\,,
\end{aligned}$$

and hence

$$d(gx_{2n}, fx_{2n+1}) \le \left(kq^{\frac{1}{p}}\right)^{2n} d(fx_1, gx_0)\,. \tag{4}$$

Similarly, we have

$$d(gx_{2n}, fx_{2n-1}) \le \left(kq^{\frac{1}{p}}\right)^{2n-1} d(fx_1, gx_0) \tag{5}$$

for all n. It follows from (4) and (5) that $\{fx_{2n-1}\}$ is a Cauchy sequence in X, and so there exists $z \in X$ such that $\lim_{n\to\infty} fx_{2n-1} = z$. In view of (5), we also have $\lim_{n\to\infty} gx_{2n} = z$. We now show that z is a coincidence point of f and S, that is, $fz \in Sz$. The H-continuity of S ensures that

$$H(SfX_{2n-1}, Sz) \to 0\,.$$

Since $\lim_{n\to\infty} fx_{2n-1} = z = \lim_{n\to\infty} gx_{2n}$ and $gx_{2n} \in Sx_{2n-1}$, it follows that

$$d(fx_{2n-1}, Sx_{2n-1}) \to 0\,.$$

Further, since f and S are R-weakly commuting, we have

$$\begin{aligned} d(fgx_{2n}, Sz) &\le H(fSx_{2n-1}, Sz) \\ &\le H(fSx_{2n-1}, Sfx_{2n-1}) + H(Sfx_{2n-1}, Sz) \\ &\le Rd(fx_{2n-1}, Sx_{2n-1}) + H(Sfx_{2n-1}, Sz)\,. \end{aligned}$$

On letting $n \to \infty$ the above inequality yields $d(fz, Sz) = 0$, which implies $fz \in Sz$. Similarly, we can show that $gz \in Tz$. This establishes the theorem. □

If we put $\alpha = \gamma = 0$ and $p = 1$ in Theorem 5, we get at once the following corollary.

Corollary 6. *Let X be a complete metric space, f, $g : X \to X$ continuous mappings, and S, $T : X \to CB(X)$ H-continuous mappings such that $TX \subseteq fX$ and $SX \subseteq gX$. If the pairs (f, S) and (g, T) are R-weakly commuting and*

$$H(Sx, Ty) \le qd(fx, gy)$$

for all $x, y \in X$, where $0 \le q < 1$, then there exists a sequence $\{x_n\}$ in X such that

(a) *for every n, $fx_{2n-1} \in Tx_{2n-2}$, $gx_{2n} \in Sx_{2n-1}$,*
(b) *there exists $z = \lim_{n\to\infty} gx_{2n} = \lim_{n\to\infty} fx_{2n-1}$,*
(c) *$fz \in Sz$, $gz \in Tz$.*

Remark 7. Theorem 5 generalizes many fixed point and coincidence theorems, we mention only those due to Hadzic [3], Kaneko [5] and Nadler [7].

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