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QUASICONFORMALITY AND EQUIVALENT NORMS

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Abstract. We study the behaviour of a quasiconformal mapping when we change the norms of the considered normed spaces by other equivalent norms. We propose a new metric definition with which we can study the interdependence between a quasiconformal homeomorphism and the new equivalent norms of the normed spaces.

Let E, F be two normed spaces, $D \subset E$, $D' \subset F$ open sets and $f : D \to D'$ a homeomorphism. The scalar derivatives of f at a point x are defined by

$$D^+ f(x) = \limsup_{x' \to x} \frac{\|f(x') - f(x)\|_F}{\|x' - x\|_E} \, , \; D^- f(x) = \liminf_{x' \to x} \frac{\|f(x') - f(x)\|_F}{\|x' - x\|_E} \, .$$

We recall also the linear dilatation of f at x as defined by

$$H(x,f) = \limsup_{r \to 0} \frac{L(x,f,r)}{l(x,f,r)}$$

where

$$L(x,f,r) = \sup \{\|f(x') - f(x)\|_F \, , \, x' \in D, \; \|x' - x\|_E = r\}$$
$$l(x,f,r) = \inf \{\|f(x') - f(x)\|_F \, , \, x' \in D \, , \|x' - x\|_E = r\} \, .$$

Definition 1. $f : D \to D'$ is K-quasiconformal in the metric sense, $K \geq 1$, $(K - MQC)$, if

$$H(x,f) \leq K, \, (\forall) \, x \in D \, .$$

Definition 2. $f : D \to D'$ is K-quasiconformal in the analytical sense, $K \geq 1$, $(K - AQC)$, if

(i) $D^- f(x) > 0$, $D^+ f(x) < \infty$, $(\forall)$ $x \in D$,
(ii) $D^+ f(x) \leq K \cdot D^- f(x)$, $(\forall)$ $x \in D$.

Definition 3. If in the previous definitions $K = 1$, we say that f is conformal in the metric sense (MC) respectively in the analytical sense (AC).

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Theorem 1. *Let $f : D \to D'$ be a $K - AQC$ homeomorphism. If we replace the norms of E and F by some equivalent norms, then, for a certain K', f becomes $K' - AQC$ homeomorphism.*

Proof. Let $\|\cdot\|_{1E}$, $\|\cdot\|_{1F}$ be two norms equivalent to the initial norms of E, respectively F and m_1, M_1, m_1', M_1' strictly positive numbers such that

$$m_1 \cdot \|x\|_E \le \|x\|_{1E} \le M_1 \cdot \|x\|_E \, , \, (\forall) \, x \in E$$

and

$$m_1' \cdot \|y\|_F \le \|y\|_{1F} \le M_1' \cdot \|y\|_F \, , \, (\forall) \, y \in F$$

respectively.

We will have

$$\begin{aligned} D^+_{11} f(x) &= \limsup_{x' \to x} \frac{\|f(x') - f(x)\|_{1F}}{\|x' - x\|_{1E}} \\ &\le \frac{M_1'}{m_1} \cdot \limsup_{x' \to x} \frac{\|f(x') - f(x)\|_F}{\|x' - x\|_E} = \frac{M_1'}{m_1} \cdot D^+ f(x), \\ D^-_{11} f(x) &= \liminf_{x' \to x} \frac{\|f(x') - f(x)\|_{1F}}{\|x' - x\|_{1E}} \\ &\ge \frac{m_1'}{M_1} \cdot \liminf_{x' \to x} \frac{\|f(x') - f(x)\|_F}{\|x' - x\|_E} = \frac{m_1'}{M_1} \cdot D^- f(x) \end{aligned}$$

and

$$\begin{aligned} D^+_{11} f(x) &\le \frac{M_1'}{m_1} \cdot D^+ f(x) \le \frac{M_1'}{m_1} \cdot K \cdot D^- f(x) \\ &\le \frac{M_1'}{m_1} \cdot \frac{M_1}{m_1'} \cdot K \cdot D^-_{11} f(x) \, . \end{aligned}$$

i.e. the required result with $K' = \frac{M_1'}{m_1} \cdot \frac{M_1}{m_1'} \cdot K$. □

For the metric definition we prove the invariance of the quasiconformality only for the renorming of the arrival space.

Theorem 2. *Let $f : D \to D'$ be $K - MQC$ homeomorphism. If we replace the norm of F by an equivalent norm, then, for a certain K', f becomes $K' - MQC$ homeomorphism.*

Proof. Let $\|\cdot\|_{1F}$ be a norm equivalent to the initial norm of F and, as in the preceding proof, m_1', M_1' strictly positive numbers such that

$$m_1' \cdot \|y\|_F \le \|y\|_{1F} \le M_1' \cdot \|y\|_F \, , \, (\forall) \, y \in F \, .$$

Let $\eta > 0$. There exists $r_\eta > 0$ so that for any $0 < r < r_\eta$,

$$\frac{L(x, f, r)}{l(x, f, r)} < K + \eta \, .$$

Let $\varepsilon > 0$. Then there exist

$$x'_\varepsilon, \, x''_\varepsilon \in D$$

so that

$$\|x'_\varepsilon - x\| = \|x''_\varepsilon - x\| = r$$

and

$$\begin{aligned} L_{11}(x,f,r) - \varepsilon &= \sup\{\|f(x') - f(x)\|_{1F}\,;\, x' \in D,\ \|x' - x\| = r\} - \varepsilon \\ &< \|f(x'_\varepsilon) - f(x)\|_{1F} \end{aligned}$$

$$\begin{aligned} l_{11}(x,f,r) + \varepsilon &= \inf\{\left\|f(x'') - f(x)\right\|_{1F}\,;\, x'' \in D,\ \|x'' - x\| = r\} + \varepsilon \\ &> \|f(x''_\varepsilon) - f(x)\|_{1F} \end{aligned}$$

respectively.

In the previous inequalities r can be fixed so that $r < r_\eta$.

We have

$$\begin{aligned} L_{11}(x,f,r) - \varepsilon &< \|f(x'_\varepsilon) - f(x)\|_{1F} \le M'_1 \cdot \|f(x'_\varepsilon) - f(x)\|_F \\ &\le M'_1 \cdot \sup\{\|f(x') - f(x)\|_F\,;\, x' \in D,\ \|x' - x\| = r\} = M'_1 \cdot L(x,f,r) \end{aligned}$$

and

$$\begin{aligned} l_{11}(x,f,r) + \varepsilon &> \|f(x''_\varepsilon) - f(x)\|_{1F} \ge m'_1 \cdot \|f(x''_\varepsilon) - f(x)\|_F \\ &\ge m'_1 \cdot \inf\{\|f(x'') - f(x)\|_F\,;\, x'' \in D,\ \|x'' - x\| = r\} = m'_1 \cdot l(x,f,r) \end{aligned}$$

such that, finally, we can write

$$\frac{L_{11}(x,f,r) - \varepsilon}{l_{11}(x,f,r) + \varepsilon} < \frac{M'_1}{m'_1} \cdot \frac{L(x,f,r)}{l(x,f,r)}\,.$$

But $r < r_\eta$, and

$$\frac{L_{11}(x,f,r) - \varepsilon}{l_{11}(x,f,r) + \varepsilon} < \frac{M'_1}{m'_1} \cdot \frac{L(x,f,r)}{l(x,f,r)} < \frac{M'_1}{m'_1} \cdot (K + \eta)\,.$$

If $\varepsilon \to 0$ and $r \to 0$ we obtain

$$H_{11}(x,f) = \limsup_{r \to 0} \frac{L_{11}(x,f,r)}{l_{11}(x,f,r)} < \frac{M'_1}{m_1{\prime}} \cdot (K + \eta)$$

and for $\eta \to 0$,

$$H_{11}(x,f) = \limsup_{r \to 0} \frac{L_{11}(x,f,r)}{l_{11}(x,f,r)} \le \frac{M'_1}{m_1{\prime}} \cdot K$$

for any $x \in D$, i.e. the required result with $K' = \frac{M'_1}{m_1{\prime}} \cdot K$. □

Proposition 1. *If the spaces E, F are norm isomorphic by $f : E \to F$, then f is MC and AC.*

Proof. By hypothesis f is a one-to-one and a bicontinuous mapping and we have also

$$\|f(x)\|_F = \|x\|_E \ \ (\forall)\ x \in E\,.$$

Hence,

$$\|f(x') - f(x)\|_F = \|f(x' - x)\|_F = \|x' - x\|_E \ (\forall)\ x \in E$$

and

$$L(x,f,r) = l(x,f,r) = r, \ \ (\forall)\ r > 0\,,$$

such that finally we obtain $H(x,f) = 1$ for all $x \in E$. □

Similarly we obtain $D^+f(x) = D^-f(x) = 1$ for all $x \in E$.

Proposition 2. *If $f : E \to F$ is an isomorphism, then there exist $K \geq 1$ such that f is $K - MQC$ and $K - AQC$.*

Proof. From the hypothesis there exist $m > 0$, $M > 0$ such that

$$m \cdot \|x\|_E \leq \|f(x)\|_F \leq M \cdot \|x\|_E\,,\ (\forall)\ x \in E\,.$$

Then

$$\begin{aligned}
L(x,f,r) &= \sup\left\{\|f(x') - f(x)\|_F\,,\ x' \in E,\ \|x' - x\|_E = r\right\} \\
&= \sup\left\{\|f(x' - x)\|_F\,;\ x' \in E,\ \|x' - x\|_E = r\right\} \\
&\leq \sup\left\{M \cdot \|x' - x\|_E\,;\ x' \in E,\ \|x' - x\|_E = r\right\} = M \cdot r, \\
l(x,f,r) &= \inf\left\{\|f(x') - f(x)\|_F\,,\ x' \in E,\ \|x' - x\|_E = r\right\} \\
&= \inf\left\{\|f(x' - x)\|_F\,;\ x' \in E,\ \|x' - x\|_E = r\right\} \\
&\geq \inf\left\{m \cdot \|x' - x\|_E\,;\ x' \in E,\ \|x' - x\|_E = r\right\} = m \cdot r
\end{aligned}$$

such that we will have

$$H(x,f) \leq \frac{M}{m}\,,\ (\forall)\ x \in E\,.$$

Similarly, we obtain

$$D^-(x,f) \geq m,\ D^+(x,f) \leq M,\ (\forall)\ x \in E\,,$$

and obviously the conditions (i), (ii) in definition (2) are satisfied. □

We use the previous proposition to prove that, if we consider some adjacent conditions for a quasiconformal homeomorphism in the metric sense, we have the invariance if we change also the norm of the normed space E.

Theorem 3. *Let $f : D \to D'$ be $K - MQC$ homeomorphism so that f is Fréchet-differentiable and $f'(x)$ is a bijection for any $x \in D$. If we replace the norms of E and F by some equivalent norms, then, for a certain K', f becomes $K' - MQC$ homeomorphism.*

Proof. From [2], the product of a $K - MQC$ homeomorphism f and a $K' - MQC$ homeomorphism g, both with bijective Fréchet derivatives is a $K \cdot K' - MQC$ homeomorphism.

Let $\|\cdot\|_{1E}$, $\|\cdot\|_{1F}$ be two norms equivalent to the initial norms of E, respectively F and $m_1,\ M_1,\ m_1',\ M_1'$ strictly positive numbers such that

$$m_1 \cdot \|x\|_E \leq \|x\|_{1E} \leq M_1 \cdot \|x\|_E\,,\ (\forall)\ x \in E$$

and

$$m_1' \cdot \|y\|_F \le \|y\|_{1F} \le M_1' \cdot \|y\|_F \, , \, (\forall) \; y \in F$$

respectively.

The identity $i : (E, \|\cdot\|_{1E}) \to (E, \|\cdot\|_E)$ is an isomorphism and it results from the proposition 2 that i is $\frac{M_1}{m_1} - MQC$. Similarly the identity $j : (F, \|\cdot\|_F) \to (F, \|\cdot\|_{1F})$ is $\frac{M_1'}{m_1'} - MQC$.

Then $f_1 : D \subset (E, \|\cdot\|_{1E}) \to D' \subset (F, \|\cdot\|_{1F})$ defined by $f_1(x) = f(x)$ for any $x \in D$, can be written as $f_1 = j \circ f \circ i$ and it results that f_1 is $\frac{M_1}{m_1} \cdot \frac{M_1'}{m_1'} \cdot K - MQC$. $\square$

Example. We give now an example of a K-quasiconformal homeomorphism $f : E \to F$ in the metric sense, so that it becomes conformal if we replace the norm of F by an equivalent norm.

Let $E = F = \mathbb{R}^2$ be normed by the equivalent norms

$$\|(u, v)\| = \max(|u|, |v|), \; \|(u, v)\|_1 = |u| + |v| \, .$$

We have the inequalities

$$\frac{1}{2} \|(u, v)\|_1 \le \|(u, v)\| \le \|(u, v)\|_1$$

for any $(u, v) \in \mathbb{R}^2$.

We consider the identical function $i : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|), i(z) = z$ and the function $i_1 : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|_1), i_1(z) = z$.

For i_1 we take $z_0 = (u_0, v_0) \in \mathbb{R}^2$ and $r > 0$. For $z = (u, v) \in \mathbb{R}^2$ with

$$\|z - z_0\| = \max(|u - u_0|, |v - v_0|) = r$$

we have

$$\|i_1(z) - i_1(z_0)\|_1 = |u - u_0| + |v - v_0| =$$

$$= \left\{ \begin{array}{l} r, \text{ if } 0 = |u - u_0| < |v - v_0| = r \text{ or } 0 = |v - v_0| < |u - u_0| = r \\ 2r, \text{ if } 0 < |u - u_0| = |v - v_0| = r \\ \alpha + r \text{ if } 0 < |u - u_0| < |v - v_0| = r \text{ or } 0 < |v - v_0| < |u - u_0| = r \end{array} \right\} .$$

where $0 < \alpha < r$. It results that

$$L(z_0, i_1, r) = \sup\{\|i_1(z) - i_1(z_0)\|_1 \; ; \; z \in \mathbb{R}^2, \; \|z - z_0\| = r\} = 2r$$
$$l(z_0, i_1, r) = \inf\{\|i_1(z) - i_1(z_0)\|_1 \; ; \; z \in \mathbb{R}^2, \; \|z - z_0\| = r\} = r$$

and,

$$H(z_0, i_1) = \limsup_{r \to 0} \frac{L(z_0, i_1, r)}{l(z_0, i_1, r)} = 2$$

for any $z_0 \in \mathbb{R}^2$. So, the homeomorphism $i_1 : (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|_1)$ is 2-quasiconformal and, if we replace the norm $\|\cdot\|_1$ by the equivalent norm $\|\cdot\|$ we obtain the conformal homeomorphism i.

More generally, we can prove that an isomorphism $f : E \to F$ becomes conformal in the metric sense if we replace the norm of F by the equivalent norm $y \to \|y\|_1 = \left\|f^{-1}(y)\right\|_E$, $(\forall)\ y \in F$.

Theorem 4. *An isomorphism $f : E \to F$ becomes conformal in the metric sense if we replace the norm of F by the equivalent norm*

$$y \to \|y\|_1 = \left\|f^{-1}(y)\right\|_E \ , (\forall)\ y \in F .$$

Proof. Let first remark that, if we take $y = f(x)$, the double inequality

$$m \cdot \|x\|_E \leq \|f(x)\|_F \leq M \cdot \|x\|_E \ , (\forall)\ x \in E .$$

becomes

$$m \cdot \|y\|_1 \leq \|y\|_F \leq M \cdot \|y\|_1 \ , (\forall)\ y \in F$$

whence the fact that the two norms are equivalent in F.

For $f : E \to (F, \|\cdot\|_1)$ we have

$$\begin{aligned} L_1(x, f, r) &= \sup\left\{\|f(x') - f(x)\|_1 , \ x' \in E, \ \|x' - x\|_E = r\right\} \\ &= \sup\left\{\|f(x' - x)\|_1 , \ x' \in E, \ \|x' - x\|_E = r\right\} \\ &= \sup\left\{\|x' - x\|_E , \ x' \in E, \ \|x' - x\|_E = r\right\} = r . \end{aligned}$$

Similarly, we obtain $l_1(x, f, r) = r$, and, finally, $H(x, f) = 1$. □

The same result is true for the definition with scalar derivatives.

Theorem 5. *In the hypothesis of the preceding theorem, f becomes also conformal in the analytical sense.*

Proof. We will have

$$\begin{aligned} D_1^+ f(x) &= \limsup_{x' \to x} \frac{\|f(x') - f(x)\|_1}{\|x' - x\|_E} \\ &= \limsup_{x' \to x} \frac{\|f(x' - x)\|_1}{\|x' - x\|_E} = \limsup_{x' \to x} \frac{\|x' - x\|_E}{\|x' - x\|_E} = 1 \end{aligned}$$

and similarly, $D_1^- f(x) = 1$ whence the fact that the conditions (i), (ii) in the Definition 2 are satisfied with $K = 1$, i.e. f is conformal in analytical sense. □

Remark 1. In the case of the analytical definition, from the first theorem, results the invariance of the quasiconformality when we change the norms of both spaces E and F by some equivalent norms . For the metric definition that is true if we suppose the Fréchet-differentiability of the mapping f and if $f'(x)$ is a bijection for any x. The last two theorems give us an example of a K-quasiconformal homeomorphism that becomes conformal if we replace the norm of F by a suitable equivalent norm.

Some open questions are:

- can we prove the invariance for the metric definition in the same conditions as for the analytic definition or find a counterexample?

- can we find for any K-quasiconformal homeomorphism f, some equivalent norms so that f becomes conformal? Or, how much can we decrease the value of K by changing the norms of E and F by some equivalent norms ?

In [4], the author considers, for E and F general metric spaces,

$$H_a(x,f) = \limsup_{r\to 0} \frac{L_a(x,f,r)}{l_a(x,f,r)}$$

where

$$L_a(x,f,r) = \sup\{\|f(x') - f(x)\|_F\,;\, x' \in D,\ \|x'-x\|_E \le r\}$$
$$l_a(x,f,r) = \inf\{\|f(x') - f(x)\|_F\,;\, x' \in D,\ \|x'-x\|_E \ge r\}\,.$$

Using this notations, we can consider the a-metric definition.

Definition 4. $f : D \to D'$ is K-quasiconformal in the a-metric sense, $K \ge 1$, if

$$H_a(x,f) \le K,\ (\forall)\, x \in D\,.$$

We propose to consider another version. For a constant $\alpha \in (0,1]$ we note

$$L(x,f,r,\alpha) = \sup\{\|f(x') - f(x)\|_F\,;\, x' \in D,\ \alpha r \le \|x'-x\|_E \le r\}$$
$$l(x,f,r,\alpha) = \inf\{\|f(x') - f(x)\|_F\,;\, x' \in D,\ \alpha r \le \|x'-x\|_E \le r\}$$

and

$$H(x,f,\alpha) = \limsup_{r\to 0} \frac{L(x,f,r,\alpha)}{l(x,f,r,\alpha)}\,.$$

Definition 5. $f : D \to D'$ is (K,α)-quasiconformal in the metric sense, $K \ge 1$, and $\alpha \in (0,1]$ if

$$H(x,f,\alpha) \le K,\ (\forall)\, x \in D\,.$$

In the last definition, if $\alpha = 1$ we obtain the metric definition.

Proposition 3. 1) *If $f : D \to D'$ is K-quasiconformal in the a-metric sense then f is K-quasiconformal in the metric sense.*

2) *If $f : D \to D'$ is (K,α)-quasiconformal in the metric sense then f is K-quasiconformal in the metric sense.*

Proof. These affirmations are consequences of the relations

$$\{x' \ /\ x' \in D,\ \|x'-x\|_E \ge r\} \supseteq \{x' \ /\ x' \in D,\ \|x'-x\|_E = r\}$$

$$\{x' \ /\ x' \in D,\ \|x'-x\|_E \le r\} \supseteq \{x' \ /\ x' \in D,\ \|x'-x\|_E = r\}$$

for the first affirmation, and

$$\{x' \ /\ x' \in D,\ \alpha r \le \|x'-x\|_E \le r\} \supseteq \{x' \ /\ x' \in D,\ \|x'-x\|_E = r\}$$

for the last affirmation. $\square$

Theorem 6. *Let $f : D \to D'$ be a (K,α)-quasiconformal homeomorphism in the metric sense. If we replace the norms of E and F by some equivalent norms, $\|\cdot\|_{1E}$, $\|\cdot\|_{1F}$ so that*

$$m_1 \cdot \|x\|_E \leq \|x\|_{1E} \leq M_1 \cdot \|x\|_E \,,\ (\forall)\ x \in E$$

and

$$m_1' \cdot \|y\|_F \leq \|y\|_{1F} \leq M_1' \cdot \|y\|_F \,,\ (\forall)\ y \in F$$

and if

$$\alpha \frac{M_1}{m_1} < 1$$

then, for a certain K' and α', f becomes a (K',α')-quasiconformal homeomorphism in the metric sense.

Proof. Let $\eta > 0$. From

$$H(x,f,\alpha) = \limsup_{r\to 0} \frac{L(x,f,r,\alpha)}{l(x,f,r,\alpha)} < K\,,$$

there exists $r_\eta > 0$ so that for any r_1, $0 < r_1 < r_\eta$,

$$\frac{L(x,f,r_1,\alpha)}{l(x,f,r_1,\alpha)} < K + \eta\,.$$

Let $\varepsilon > 0$ and r be so that $0 < r < r_\eta \cdot m_1$. For $\alpha_1 = \alpha \dfrac{M_1}{m_1}$, we note

$$L^{11}(x,f,r,\alpha_1) = \sup\left\{\|f(x') - f(x)\|_{1F}\,,\ x' \in D,\ \alpha_1 r \leq \|x' - x\|_{1E} \leq r\right\}$$

and

$$l^{11}(x,f,r,\alpha_1) = \inf\left\{\|f(x') - f(x)\|_{1F}\,,\ x' \in D,\ \alpha_1 r \leq \|x' - x\|_{1E} \leq r\right\}.$$

There exists x'_ε, $x''_\varepsilon \in D$ so that

$$\alpha_1 r \leq \|x'_\varepsilon - x\|_{1E} \leq r\ ,\ \alpha_1 r \leq \|x''_\varepsilon - x\|_{1E} \leq r$$

and

$$L^{11}(x,f,r,\alpha_1) - \varepsilon < \|f(x'_\varepsilon) - f(x)\|_{1F}\,,\ l^{11}(x,f,r,\alpha_1) + \varepsilon > \|f(x''_\varepsilon) - f(x)\|_{1F}\ .$$

But x'_ε verifies the inequalities

$$m_1 \|x'_\varepsilon - x\|_E \leq \|x'_\varepsilon - x\|_{1E} \leq r\ ;\ M_1 \|x'_\varepsilon - x\|_E \geq \|x'_\varepsilon - x\|_{1E} \geq \alpha_1 r$$

so,

$$\frac{\alpha_1 r}{M_1} \leq \|x'_\varepsilon - x\|_E \leq \frac{r}{m_1}\,.$$

For x''_ε we obtain also

$$\frac{\alpha_1 r}{M_1} \leq \|x''_\varepsilon - x\|_E \leq \frac{r}{m_1}\,.$$

The last two inequalities can be written

$$\alpha_1 \frac{m_1}{M_1} \frac{r}{m_1} \leq \|x'_\varepsilon - x\|_E \leq \frac{r}{m_1}\,,$$

$$\alpha_1 \frac{m_1}{M_1} \frac{r}{m_1} \leq \|x''_\varepsilon - x\|_E \leq \frac{r}{m_1}\,.$$

If we note $r_1 = \dfrac{r}{m_1}$, we have $0 < r_1 < r_\eta$ and if we replace α_1 we obtain

$$\alpha r_1 \leq \|x'_\varepsilon - x\|_E \leq r_1\,, \quad \alpha r_1 \leq \|x''_\varepsilon - x\|_E \leq r_1\,.$$

It results that,

$$\begin{aligned} L^{11}(x,f,r,\alpha_1) - \varepsilon &< \|f(x'_\varepsilon) - f(x)\|_{1F} \leq M'_1 \|f(x'_\varepsilon) - f(x)\|_F \\ &\leq M'_1 \sup\{\|f(x') - f(x)\|_F\,,\, x' \in D\,, \alpha r_1 \leq \|x' - x\|_E \leq r_1\} \\ &= M'_1 L(x,f,r_1,\alpha) \end{aligned}$$

and,

$$\begin{aligned} l^{11}(x,f,r,\alpha_1) + \varepsilon &> \|f(x''_\varepsilon) - f(x)\|_{1F} \geq m'_1 \|f(x''_\varepsilon) - f(x)\|_F \\ &\geq m'_1 \inf\{\|f(x') - f(x)\|_F\,,\, x' \in D,\, \alpha r_1 \leq \|x' - x\|_E \leq r_1\} \\ &= m'_1 l(x,f,r_1,\alpha)\,. \end{aligned}$$

Finally,

$$\frac{L^{11}(x,f,r,\alpha_1) - \varepsilon}{l^{11}(x,f,r,\alpha_1) + \varepsilon} < \frac{M'_1}{m'_1} \frac{L(x,f,r_1,\alpha)}{l(x,f,r_1,\alpha)} < \frac{M'_1}{m'_1}(K + \eta)\,.$$

If $\varepsilon \to 0$, $r \to 0$ and $\eta \to 0$ it results that

$$H^{11}(x,f,\alpha_1) = \limsup_{r \to 0} \frac{L^{11}(x,f,r,\alpha_1)}{l^{11}(x,f,r,\alpha_1)} \leq \frac{M'_1}{m'_1} K\,. \qquad \square$$

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