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REDUCTION OF THE MODIFIED POINCARÉ DIFFERENTIAL EQUATION TO BIRKHOFF MATRIX FORM

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Abstract. In this paper the reduction of the modified Poincaré linear differential equation with one n-tuple regular singularity to the Birkhoff canonical matrix form is given.

1. Introduction

In the previous paper [1] it is shown that the general Poincaré linear differential equation is reduced to the following Cauchy matrix form $(xI - D)\,\frac{dY}{dx} = QY$, where I is the $n \times n$ unit matrix, Q is an $n \times n$ constant matrix of the reduction, $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ and d_i $(1 \le i \le n)$ are distinct regular singularities. If we formally substitute $d_i = 0$ $(1 \le i \le n)$ into the last equality, then we easily find that the modified Poincaré linear differential equation with one n-tuple regular singularity is reduced to the following Birkhoff matrix form $xI\,\frac{dY}{dx} = QY$, which means that the Birkhoff canonical matrix system is a special case of the Cauchy matrix system. However, as we see below, the matrix Q is not constant in this case. It is just this type of reduction that will be an object of investigation in the present paper.

2. Preliminaries

Consider the modified Poincaré differential equation

$$x^n y^{(n)} = \sum_{i=1}^{n} \Big(\sum_{j=0}^{ri} a_{ij} x^j \Big) x^{n-i} y^{(n-i)} \,, \tag{1}$$

where r is a positive integer. For the equation (1) the characteristic constants ρ_i $(1 \le i \le n)$ of the regular singularity at the coordinate origin $x = 0$ are given

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as roots of the equation

$$I(\rho) \equiv [\rho]_n - \sum_{i=1}^{n} a_{i0}[\rho]_{n-i} = 0\,,$$

where

$$[\rho]_k = \rho(\rho-1)\cdots(\rho-k+1), \quad [\rho]_0 = 1\,,$$

such that $\rho_i \neq \rho_j \pmod 1$ $(i \neq j;\ 1 \le i,j \le n)$. The last equation is derived when we look for a solution of (1) behaving as x^ρ near $x=0$. The constants ρ_i $(1 \le i \le n)$ enter the formulation of our main result. For the sake of completeness we note that the principal characteristic constants λ_i $(1 \le i \le n)$ of the irregular singularity of rank 1 at infinity $x = \infty$ are given as roots of the equation

$$J(\lambda) \equiv \lambda^n - \sum_{i=1}^{n} a_{i,ri}\lambda^{n-i} = 0\,.$$

It is derived when looking for a solution of (1) behaving as $\exp\,(\lambda x^r/r)$ times an arbitrary power of x at infinity.

3. Main Result

We shall need the following

Lemma. *Let $\xi_0^1, \xi_0^2, \cdots, \xi_0^n$ be constants satisfying the equality*

$$[\rho]_n + \xi_0^1[\rho]_{n-1} + \cdots + \xi_0^n = \prod_{\nu=1}^{n}(\rho - \rho_\nu)\,. \tag{2}$$

Denote

$$f(\rho) = \begin{vmatrix} \mu_0^1 + \rho - (n-2) & -1 & 0 & \cdots & 0 \\ \mu_0^2 & \rho - (n-3) & -1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & \rho \end{vmatrix},$$

where μ_0^j $(1 \le j \le n-1)$ are some constants. Let the unknown variable ζ satisfy the following equalities

$$\xi_0^1 + [\zeta - (n-1)] = \mu_0^1\,,$$

$$\xi_0^j + [\zeta - (n-j)]\mu_0^{j-1} = \mu_0^j \quad (2 \le j \le n-1)\,, \tag{3}$$

$$\zeta\mu_0^{n-1} + \xi_0^n = 0\,.$$

If ζ takes one of the values ρ_ν, say $\zeta = \rho_n$, then

$$f(\rho) = [\rho]_{n-1} + \mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1} = \prod_{\nu=1}^{n-1}(\rho - \rho_\nu)\,. \tag{4}$$

Proof. Since

$$\begin{vmatrix} \mu_0^1 + y_1 & -1 & 0 & \cdots & 0 \\ \mu_0^2 & y_2 & -1 & \cdots & 0 \\ \vdots & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & y_{n-1} \end{vmatrix}$$
$$= y_1 y_2 \cdots y_{n-1} + \mu_0^1 y_2 y_3 \cdots y_{n-1} + \mu_0^2 y_3 y_4 \cdots y_{n-1} + \cdots + \mu_0^{n-1}\,,$$

then

$$f(\rho) = [\rho]_{n-1} + \mu_0^1[\rho]_{n-2} + \cdots + \mu_0^{n-1}$$

and

$$\begin{aligned}(\rho - \rho_n) f(\rho) &= [\rho]_n - (\rho_n - n + 1)[\rho]_{n-1} + \mu_0^1[\rho]_{n-1} - (\rho_n - n + 2)\mu_0^1[\rho]_{n-2} \\ &\quad + \cdots + \mu_0^{n-1}[\rho]_1 - \rho_n \mu_0^{n-1} \\ &= [\rho]_n + \xi_0^1[\rho]_{n-1} + \cdots + \xi_0^n = \prod_{\nu=1}^{n}(\rho - \rho_\nu) = (\rho - \rho_n)\prod_{\nu=1}^{n-1}(\rho - \rho_\nu)\,.\end{aligned}$$

Now we will prove the following result.

Theorem. *The equation* (1) *is reduced to the Birkhoff matrix form*

$$xI\,\frac{dU}{dx} = Q(x)U\,, \tag{5}$$

where

$$Q(x) = \begin{bmatrix} q_{11}(x) & x^r & 0 & \cdots & 0 & 0 \\ q_{21}(x) & q_{22}(x) & x^r & \cdots & 0 & 0 \\ \vdots & & & & & \\ q_{n-1,1}(x) & q_{n-1,2}(x) & q_{n-1,3}(x) & \cdots & q_{n-1,n-1}(x) & x^r \\ q_{n1}(x) & q_{n2}(x) & q_{n3}(x) & \cdots & q_{n,n-1}(x) & q_{nn}(x) \end{bmatrix} \tag{6}$$

and

$$q_{ij}(x) = \sum_{\nu=0}^{r-1} q_{ij}^\nu x^\nu \ (j \le i \ne n)\,, \qquad q_{nj}(x) = \sum_{\nu=0}^{r} q_{nj}^\nu x^\nu \ (j \le n) \tag{7}$$

(in particular, $q_{ii}^0 = \rho_i - (i-1)r$, $1 \le i \le n$, *and* $q_{nj}^r = a_{n+1-j,r(n+1-j)}$, $1 \le j \le n$*)* *by a linear transformation with polynomials in* x^{-1} *as its coefficients.*

Proof. Let us denote

$$y_p(x) = x^{-(r-1)p} y^{(p)} . \tag{8}$$

After a differentiation this yields

$$x y_p' = x^r y_{p+1} + s_{p+1} y_p \quad (0 \le p \le n-1), \tag{9}$$

where

$$s_i = -(i-1)(r-1) \quad (1 \le i \le n) .$$

If we multiply the equation (1) by x^{-rn}, then it takes on the form

$$y_n = \sum_{i=1}^{n} F_i(x) y_{n-i} , \tag{10}$$

where

$$F_i(x) = \sum_{j=0}^{ri} a_{ij} x^{j-ri} \quad (1 \le i \le n) . \tag{11}$$

Let us denote $\mathcal{D} \equiv x \frac{d}{dx}$. Now according to (9) and (10) we obtain

$$\begin{aligned}
\mathcal{D}Y \equiv \mathcal{D} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix}
&= \begin{bmatrix} s_1 & x^r & 0 & \cdots & 0 & 0 \\ 0 & s_2 & x^r & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & s_{n-1} & x^r \\ x^r F_n(x) & x^r F_{n-1}(x) & x^r F_{n-2}(x) & \cdots & x^r F_2(x) & s_n + x^r F_1(x) \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{n-2} \\ y_{n-1} \end{bmatrix} \\
&\equiv L(x) Y ,
\end{aligned}$$

i.e.

$$\mathcal{D}Y = L(x) Y . \tag{12}$$

By the linear transformation of Turrittin [2]

$$U = C(x) Y , \tag{13}$$

where

$$C(x)=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0\\ c_{21}(x) & 1 & 0 & \cdots & 0 & 0\\ c_{31}(x) & c_{32}(x) & 1 & \cdots & 0 & 0\\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ c_{n1}(x) & c_{n2}(x) & c_{n3}(x) & \cdots & c_{n,n-1}(x) & 1 \end{bmatrix},$$

we obtain the equality

$$\mathcal{D}U=[\mathcal{D}C(x)+C(x)L(x)]C^{-1}(x)U=Q(x)U\,, \tag{14}$$

and hence it follows that

$$\mathcal{D}C(x)+C(x)L(x)=Q(x)C(x)\,. \tag{15}$$

From (15) $c_{ij}(x)$ $(i>j)$ can be uniquely determined so that all entries of the matrix $Q(x)$ are polynomials of the required form (7).

If we denote

$$C(x)L(x)=\Big[p_{ij}(x)\Big]_{n\times n} \quad \text{and} \quad Q(x)C(x)=\Big[t_{ij}(x)\Big]_{n\times n},$$

then we have

$$\begin{aligned} p_{ij}(x)&=x^r c_{i,j-1}(x)+s_j c_{ij} \quad (1\le i\le n-1)\,,\\ p_{nj}(x)&=x^r c_{n,j-1}(x)+s_j c_{nj}(x)+x^r F_{n-j+1}(x) \end{aligned} \tag{16}$$

and

$$t_{ij}(x)=q_{ij}(x)+\sum_{\nu=j+1}^{i} q_{i\nu}(x)c_{\nu j}(x)+x^r c_{i+1,j}(x)\,, \tag{17}$$

where

$$c_{ii}(x)=1\,,\quad c_{ij}(x)=0\quad (j>i),\quad c_{i0}(x)=0\,.$$

Then (15) can be written as

$$\mathcal{D}c_{ij}(x)+p_{ij}(x)=t_{ij}(x)\quad (j<i+2). \tag{18}$$

These relations are identically satisfied for the entries above the diagonal because $p_{i,i+1}(x)=x^r$ and $t_{i,i+1}=x^r$.

For the diagonal entries (18) becomes

$$p_{ii}(x)=t_{ii}(x)\,,$$

which implies that

$$x^r c_{i,i-1}(x) + s_i = q_{ii}(x) + x^r c_{i+1,i}(x) \quad (1 \le i \le n-1)\,, \tag{19}$$

$$x^r c_{n,n-1}(x) + s_n + x^r F_1(x) = q_{nn}(x)\,, \tag{20}$$

and for the j-th subdiagonal entries we obtain

$$\begin{aligned} &\mathcal{D} c_{i,i-j}(x) + x^r c_{i,i-j-1}(x) + s_{i-j} c_{i,i-j}(x) \\ &= q_{i,i-j}(x) + \sum_{\nu=0}^{j-1} q_{i,i-\nu}(x) c_{i-\nu,i-j}(x) + x^r c_{i+1,i-j}(x) \quad (2 \le i \le n-1)\,, \end{aligned} \tag{21}$$

$$\begin{aligned} &\mathcal{D} d c_{n,n-j}(x) + x^r c_{n,n-j-1}(x) + s_{n-j} c_{n,n-j}(x) + x^r F_{j+1}(x) \\ &= q_{n,n-j}(x) + \sum_{\nu=0}^{j-1} q_{n,n-\nu}(x) c_{n-\nu,n-j}(x) \quad (1 \le j \le n-1)\,. \end{aligned} \tag{22}$$

From the equalities (19) – (22) the subdiagonal entries of $C(x)$ can be determined as polynomials of the same degree with respect to x^{-1}. Now we will substitute

$$\begin{gathered} c_{i,i-j}(x) = x^{-r} c^1_{i,i-j}(x) + x^{-2r} c^2_{i,i-j}(x) + \cdots + x^{-jr} c^j_{i,i-j}(x) \\ (2 \le i \le n; \quad 1 \le j \le i-1)\,, \end{gathered} \tag{23}$$

where $c^m_{i,i-j}(x)$ $(1 \le m \le j)$ are polynomials of x of degree at most $r-1$. The coefficients $F_i(x)$ $(1 \le i \le n)$ can be represented in the form

$$F_i(x) = a_{i,ri} + x^{-r} F^1_i(x) + x^{-2r} F^2_i(x) + \cdots + x^{-ir} F^i_i(x) \quad (1 \le i \le n)\,, \tag{24}$$

where $F^m_i(x)$ $(1 \le m \le i)$ are polynomials of degree $r-1$, i.e.

$$F^m_i(x) = F^m_i(0) + F^m_{i1} x + \cdots + F^m_{i,r-1} x^{r-1}\,. \tag{25}$$

In particular, $F^i_i(0) = a_{i0}$ $(1 \le i \le n)$. Now we will introduce the following notation

$$\begin{aligned} \mathcal{D}\Big[x^{-mr} c^m(x)\Big] &\equiv \mathcal{D}\Big(x^{-mr} \sum_{j=0}^{r-1} \mu_j x^j\Big) = x^{-mr} \Big[\sum_{j=0}^{r-1} \mu_j (-mr + j) x^j\Big] \\ &\equiv x^{-mr} \mathcal{D}_{mr} c^m(x)\,, \end{aligned}$$

$$\begin{aligned} q(x)c(x) &\equiv \Big(\sum_{j=0}^{r} t_j x^j\Big)\Big(\sum_{j=0}^{r-1} \mu_j x^j\Big) \\ &= x^r \big[(t_r\mu_0 + t_{r-1}\mu_1 + \cdots + t_1\mu_{r-1}) + (t_r\mu_1 + t_{r-1}\mu_2 + \cdots + t_2\mu_{r-1})x \\ &\quad + \cdots + (t_r\mu_{r-1})x^{r-1}\big] \\ &\quad + \big[(t_0\mu_0) + (t_0\mu_1 + t_1\mu_0)x + \cdots + (t_0\mu_{r-1} + t_1\mu_{r-2} + \cdots + t_{r-1}\mu_0)x^{r-1}\big] \\ &\equiv x^r \big[q(x)c(x)\big]^0 + \big[q(x)c(x)\big]^1\,. \end{aligned}$$

Substituting (23) and (24) into (21) and (22), we obtain

$$\begin{aligned}
&\mathcal{D}_{mr} c^m_{i,i-j} + s_{i-j} c^m_{i,i-j} + c^{m+1}_{i,i-j-1} \\
&= \sum_{\nu=0}^{r-1} \Big\{ \Big[q_{i,i-\nu} c^m_{i-\nu,i-j} \Big]^1 + \Big[q_{i,i-\nu} c^{m+1}_{i-\nu,i-j} \Big]^0 \Big\} + c^{m+1}_{i+1,i-j} \,,
\end{aligned} \tag{26}$$

$$q_{i,i-j} = c^1_{i,i-j-1} - \sum_{\nu=0}^{j-1} \Big[q_{i,i-\nu} c^1_{i-\nu,i-j} \Big]^0 - c^1_{i+1,i-j} \,, \tag{27}$$

$$\begin{aligned}
&\mathcal{D}_{mr} c^m_{n,n-j} + s_{n-j} c^m_{n,n-j} + c^{m+1}_{n,n-j-1} + F^{m+1}_{j+1} \\
&= \sum_{\nu=0}^{r-1} \Big\{ \Big[q_{n,n-\nu} c^m_{n-\nu,n-j} \Big]^1 + \Big[q_{n,n-\nu} c^{m+1}_{n-\nu,n-j} \Big]^0 \Big\} \,,
\end{aligned} \tag{28}$$

$$\begin{gathered}
q_{n,n-j} = c^1_{n,n-j-1} + F^1_{j+1} + a_{j+1,r(j+1)} x^r - \sum_{\nu=0}^{j-1} \Big[q_{n,n-\nu} c^1_{n-\nu,n-j} \Big]^0 \\
(1 \le m \le j) \,,
\end{gathered} \tag{29}$$

where

$$c^m_{i-\nu,i-j} = 0 \qquad (m > j - \nu) \,.$$

From (26) it just follows that

$$q_{n,n-j} = a_{j+1,r(j+1)} x^r + \text{ lower order terms } \quad (0 \le j \le n-1) \,. \tag{30}$$

Now we are able to determine all coefficients of the polynomials $c^m_{ij}(x)$ and $q_{ij}(x)$.

First we will start by the determination of $c^j_{n,n-j}(x)$ $(1 \le j \le n-1)$ and $q_{nn}(x)$. From (20) and (28) we obtain

$$\begin{gathered}
c^1_{n,n-1} + s_n + F^1_1 = q_{nn} - a_{1r} x^r \,, \\
\mathcal{D}_{jr} c^j_{n,n-j} + s_{n-j} c^j_{n,n-j} + c^{j+1}_{n,n-j-1} + F^{j+1}_{j+1} = [q_{nn} c^j_{n,n-j}]^1 \\
(1 \le j \le n-1) \,.
\end{gathered} \tag{31}$$

If we substitute

$$c^j_{n,n-j} = \mu^j_0 + \mu^j_1 x + \cdots + \mu^j_{r-1} x^{r-1} \quad (1 \le j \le n-1) \,,$$

$$q_{nn} - a_{1r} x^r = \sum_{\nu=0}^{r-1} q^\nu_{nn} x^\nu \,,$$

then from (31) it follows that

$$\mu_0^1 = q_{nn}^0 - s_n - a_{10} = q_{nn}^0 + (n-1)r - (n-1) - a_{10}\,, \tag{32}$$

$$\mu_m^1 = q_{nn}^m - F_{1m}^1 \quad (1 \le m \le r-1)\,, \tag{33}$$

$$\begin{aligned} \mu_0^{j+1} &= (q_{nn}^0 + jr - s_{n-j})\mu_0^j - a_{j+1,0} \\ &= [q_{nn}^0 + (n-1)r - (n-j-1)]\mu_0^j - a_{j+1,0}\,, \end{aligned} \tag{34}$$

$$\begin{aligned} \mu_m^{j+1} = {} & [q_{nn}^0 + (n-1)r - m - (n-j-1)]\mu_m^j + q_{nn}^m \mu_0^j \\ & + \sum_{\nu=1}^{m-1} q_{nn}^\nu \mu_{m-\nu}^j - F_{j+1,m}^{j+1} \\ & (1 \le j \le n-1;\ 1 \le m \le r-1)\,, \end{aligned} \tag{35}$$

where $\mu_m^n = 0$ and $\sum_{\nu=1}^{0} = 0$. Applying the Lemma to (32) and (34), it follows that $q_{nn}^0 + (n-1)r$ is one of the roots ρ_i $(1 \le i \le n)$ of the equation

$$[\rho]_n - \sum_{i=1}^{n} a_{i0}[\rho]_{n-i} = 0\,.$$

By the substitution

$$q_{nn}^0 + (n-1)r = \rho_n \tag{36}$$

the coefficients μ_0^j $(1 \le j \le n-1)$ can be determined. Using (33) and (35), we will determine μ_m^j $(1 \le j \le n-1)$ and then q_{nn}^m successively for $m = 1, \cdots, r-1$. In fact, according to the equation (4) the determinant

$$\begin{vmatrix} \mu_0^1 + \rho_n - m - (n-2) & -1 & 0 & \cdots & 0 \\ \mu_0^2 & \rho_n - m - (n-3) & -1 & \cdots & 0 \\ \vdots & & & & \\ \mu_0^{n-2} & 0 & 0 & \cdots & -1 \\ \mu_0^{n-1} & 0 & 0 & \cdots & \rho_n - m \end{vmatrix}$$

$$= [\rho_n - m]_{n-1} + \mu_0^1[\rho_n - m]_{n-2} + \cdots + \mu_0^{n-1} = \prod_{\nu=1}^{n-1}(\rho_n - m - \rho_\nu)$$

is again different from zero, which means that the coefficients of $c^j_{n,n-j}(x)$ and $q_{nn}(x)$ can be uniquely determined.

Further the coefficients $c^j_{i,i-j}(x)$ $(1 \le j \le i-1)$ and $q_{ii}(x)$ will be successively determined. From (19) and (26), it follows that

$$
\begin{aligned}
&c^1_{i,i-1} + s_i = q_{ii} + c^1_{i+1,i}\,, \\
&\mathcal{D}_{jr} c^j_{i,i-j} + s_{i-j} c^j_{i,i-j} + c^{j+1}_{i,i-j-1} = [q_{ii} c^j_{i,i-j}]^1 + c^{j+1}_{i+1,i-j} \quad (1 \le j \le i-1)\,.
\end{aligned}
\tag{37}
$$

Further we apply the mathematical induction. Let

$$
c^{j+1}_{i+1,i-j}(x) = \xi^{j+1}_0 + \xi^{j+1}_1 x + \cdots + \xi^{j+1}_{r-1} x^{r-1} \qquad (0 \le j \le i-1)
$$

be polynomials and let the constants ξ^j_0 $(1 \le j \le i)$ satisfy the equation

$$
[\rho]_i + \xi^1_0 [\rho]_{i-1} + \cdots + \xi^i_0 = \prod_{\nu=1}^{i} (\rho - \rho_\nu)\,. \tag{38}
$$

This is so for $i = n-1$ by virtue of the Lemma.

By the changes

$$
c^j_{i,i-j} = \mu^j_0 + \mu^j_1 x + \cdots + \mu^j_{r-1} x^{r-1} \quad (1 \le j \le i-1)\,, \qquad q_{ii} = \sum_{\nu=0}^{r-1} q^\nu_{ii} x^\nu
$$

(of course, the coefficients μ^j_ν $(1 \le \nu \le r-1)$ of $c^j_{i,i-j}$ are in general different from those of $c^j_{n,n-j}$ but we suppress the dependence on i) we obtain

$$
\mu^1_0 = q^0_{ii} + (i-1)r - (i-1) + \xi^1_0, \tag{39}
$$

$$
\mu^1_m = q^m_{ii} + \xi^1_m \quad (1 \le m \le r-1), \tag{40}
$$

$$
\mu^{j+1}_0 = [q^0_{ii} + (i-1)r - (i-j-1)]\mu^j_0 + \xi^{j+1}_0, \tag{41}
$$

$$
\begin{gathered}
\mu^{j+1}_m = [q^0_{ii} + (i-1)r - m - (i-j-1)]\mu^j_m + q^m_{ii}\mu^j_0 + \sum_{\nu=1}^{m-1} q^\nu_{ii}\mu^j_{m-\nu} + \xi^{j+1}_m \\
(1 \le j \le i-1; \quad 1 \le m \le r-1)
\end{gathered}
\tag{42}
$$

where $\mu^i_m = 0$. Applying the Lemma to the equations (39), (41), it follows that

$$
q^0_{ii} + (i-1)r = \rho_i \tag{43}
$$

and

$$[\rho]_{i-1} + \mu_0^1[\rho]_{i-2} + \cdots + \mu_0^{i-1} = \prod_{\nu=1}^{i-1}(\rho - \rho_\nu). \tag{44}$$

Again, assuming that (6) and (7) hold, $c^j_{i,i-j}(x)$ $(1 \le j \le i-1)$ and $q_{ii}(x)$ can be determined. Thus by mathematical induction we proved that for each i $(1 \le i \le n)$ $c^j_{i,i-j}(x)$ $(1 \le j \le i-1)$ and $q_{ii}(x)$ can be determined. By using (36), (38), (43) and (44), we also obtain that $q_{ii}(x)$ has the form

$$q_{ii}(x) = \rho_i - (i-1)r + \sum_{\nu=1}^{r} q^\nu_{ii} x^\nu \qquad (1 \le i \le n),$$

where $q^r_{ii} = 0$ $(i \ne n)$ and $q^r_{nn} = a_{1r}$.

Finally, we will prove that the sets of polynomials

$$\{q_{n,n-j}(x),\, c^m_{n-j-m}(x); \quad 1 \le m \le n-j-1\}$$

and

$$\{q_{i,i-j}(x),\, c^m_{i,i-j-m}(x); \quad 1 \le m \le i-j-1\} \quad (n-1 \ge i \ge 1)$$

can be determined successively for the values n and i $(1 \le i \le n-1)$. It will be proved by induction on j. Let the sets $\{q_{n,n-\nu},\, c^m_{n-\nu-m}\}$ and $\{q_{i,i-\nu},\, c^m_{i-\nu-m}\}$ $(1 \le i \le n-1)$ be known for $0 \le \nu \le j-1$. Then from (28) we obtain

$$\begin{aligned} &\mathcal{D}_{mr} c^m_{n,n-j-m} + s_{n-j-m} c^m_{n,n-j-m} + c^{m+1}_{n,n-j-m-1} \\ &\quad = [q_{nn} c^m_{n,n-j-m}]^1 + [q_{n,n-j} c^m_{n-j,n-j-m}]^1 \\ &\quad + \sum_{\nu=1}^{j-1} [q_{n,n-\nu} c^m_{n-\nu,n-j-m}]^1 + \sum_{\nu=1}^{j-1} [q_{n,n-\nu} c^{m+1}_{n-\nu,n-j-m}]^0 - F^{m+1}_{j+m+1} \\ &\qquad (1 \le m \le n-j-1)\,. \end{aligned} \tag{45}$$

By the change

$$c^j_{n-j,n-j-m} = \xi^m_0 + \xi^m_1 x + \cdots + \xi^m_{r-1} x^{r-1} \qquad (1 \le m \le n-j-1)$$

and using (38) and (44), we obtain

$$[\rho]_{n-j-1} + \xi^1_0[\rho]_{n-j-2} + \cdots + \xi^{n-j-1}_0 = \prod_{i=1}^{n-j-1} (\rho - \rho_i)\,. \tag{46}$$

If we substitute

$$q_{n,n-j} = \sum_{\nu=0}^{r} q^\nu_{n,n-j} x^\nu\,,$$

$$c^m_{n,n-j-m} = \mu^m_0 + \mu^m_1 x + \cdots + \mu^m_{r-1} x^{r-1} \qquad (1 \le m \le n-j-1)\,,$$

then from (29) there follow the equalities

$$q^r_{n,n-j} = a_{j+1,r(j+1)}\,, \tag{47}$$

$$q^\nu_{n,n-j} = \mu^1_\nu + \text{ known terms} \qquad (0 \le \nu \le r-1)\,, \tag{48}$$

and from (45) we obtain

$$(-mr + \nu + s_{n-j-m})\mu^m_\nu + \mu^{m+1}_\nu = \sum_{k=0}^{\nu} q^k_{nn}\mu^m_{\nu-k} + \sum_{k=0}^{\nu} q^k_{n,n-j}\xi^m_{\nu-k} + \cdots \tag{49}$$

$$(1 \le m \le n-j-1; \quad 0 \le \nu \le r-1)\,,$$

The equation (49) can be represented in the following form

$$\begin{aligned} \mu^{m+1}_\nu &= [\rho_n - jr - \nu - (n-j-m-1)]\mu^m_\nu + q^\nu_{n,n-j}\xi^m_0 \\ &\quad + \sum_{k=1}^{\nu} q^k_{nn}\mu^m_{\nu-k} + \sum_{k=0}^{\nu-1} q^k_{n,n-j}\xi^m_{\nu-k} + \cdots \end{aligned} \tag{50}$$

and hence we obtain the determinant

$$\begin{vmatrix} \xi^1_0+\rho_n-jr-\nu-(n-j-2) & -1 & 0 & \cdots & 0 \\ \xi^2_0 & \rho_n-jr-\nu-(n-j-3) & -1 & \cdots & 0 \\ \vdots & & & & \\ \xi^{n-j-2}_0 & 0 & 0 & \cdots & -1 \\ \xi^{n-j-1}_0 & 0 & 0 & \cdots & \rho_n-jr-\nu \end{vmatrix}$$

$$\begin{aligned} &= [\rho_n - jr - \nu]_{n-j-1} + \xi^1_0[\rho_n - jr - \nu]_{n-j-2} + \cdots + \xi^{n-j-1}_0 \\ &= \prod_{i=1}^{n-j-1} (\rho_n - rj - \nu - \rho_i) \ne 0\,. \end{aligned}$$

Thus, the set $\{q_{n,n-j}, c^m_{n,n-j-m}\}$ is uniquely determined. In order to prove that the sets $\{q_{i,i-j}\ c^m_{i,i-j-m}\}$ $(1 \le i \le n-1)$ can be uniquely determined, we again apply mathematical induction. The arguments are similar to those above, so we omit the details. □

References

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