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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS OF n–TH ORDER

N. PARHI AND SESHADEV PADHI

Abstract. This paper deals with property A and B of a class of canonical linear homogeneous delay differential equations of n-th order.

1.

In a recent paper [1], Dzurina has studied property (A) of n-th order linear delay-differential equations of the form

$$L_n y(t) + p(t)\, y(g(t)) = 0\,, \tag{1.1}$$

where $n \geq 2$, $p \in C([\sigma,\infty),[0,\infty))$, $g \in C([\sigma,\infty),R)$ is nondecreasing, $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$,

$$L_n y(t) = \left(\frac{1}{r_{n-1}(t)}\left(\dots\left(\frac{1}{r_1(t)} y'(t)\right)' \dots\right)'\right)'$$

and $r_i \in C([\sigma,\infty),R)$ such that $r_i(t) > 0$, $1 \leq i \leq n-1$. He has obtained sufficient conditions under which (1.1) has property (A). These conditions include non-existence of eventually positive solutions of first order linear delay-differential inequalities of the form

$$y'(t) + q_i(t)\, y(g(t)) \leq 0\,,$$

$1 \leq i \leq n-1$, where $q_i(t)$ is given in [1]. In another paper [2], he has studied property (B) of

$$L_n\, y(t) - p(t)\, y(g(t)) = 0 \tag{1.2}$$

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under the assumption that

$$y'(t) + q_\ell(t)\, y(w(t)) \le 0\,,$$

$1 \le \ell \le n-2$, has no eventually positive solutions, where $q_\ell(t)$ is given in [2], $g(t) < w(t) < t$,

$$L_n y(t) = \frac{1}{r_n(t)}\left(\frac{1}{r_{n-1}(t)}\left(\dots\left(\frac{1}{r_1(t)}\left(\frac{y(t)}{r_0(t)}\right)'\right)'\dots\right)\right)'\,,$$

p, g, r_i, $1 \le i \le n-1$, are same as in (1.1) and r_n, $r_0 \in C([\sigma,\infty], R)$ such that $r_n(t) > 0$ and $r_0(t) > 0$. However, $q_\ell(t)$ are different from $q_i(t)$ stated above.

The present work is motivated by our work on delay-differential equations of third order (see [9] and [10]) and the observation that the method developed to study property (A) could be applied to study property (B) and vice-versa. The latter problem was brought to our notice by Prof. Dzurina. In Section 2 we study property (A) of

$$L_n y(t) + p(t)\, y(g(t)) = 0\,, \tag{1.3}$$

where $n \ge 2$, $L_0 y(t) = y(t)/r_0(t)$, $L_i y(t) = (L_{i-1}y(t))'/r_i(t)$, $1 \le i \le n$, p, $r_i \in C([\sigma,\infty], R)$ such that $p(t) \ge 0$, $r_i(t) > 0$, $0 \le i \le n$ and $g \in C([\sigma,\infty], R)$ is increasing, $g(t) < t$ and $g(t) \to \infty$ as $t \to \infty$. We have considered two methods, one with $g(t)$ and the other with higher delay $w(t)$, and have compared them. Although our method with $g(t)$ has some similarity with the work in [1], they differ for higher i. Section 3 deals with the study of property (B) of

$$L_n y(t) - p(t)\, y(g(t)) = 0\,, \tag{1.4}$$

where p and g are same as in (1.3). We have compared our results with the work in [2] for better understanding. The technique employed here is different from that in [2].

We assume in the sequel that

$$\int_\sigma^\infty r_i(t)\, dt = \infty\,, \qquad 1 \le i \le n-1\,. \tag{1.5}$$

The operator L_n is said to be in canonical form if (1.5) holds. It is well-known that any differential operator of the form L_n can always be represented in a canonical form in an essentially unique way (see [11]). A nontrivial solution of (1.3) (or (1.4)) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1.3) (or (1.4)) is said to be oscillatory if all its solutions are oscillatory.

The asymptotic behaviour of solutions of (1.3) is described in the following lemma which is a generalization of a lemma due to Kiguradze [5, Lemma 3].

Lemma 1.1. *If $y(t)$ is a nonoscillatory solution of* (1.3) *on* (T_y,∞), $T_y \ge \sigma$, *then there is an integer* $\ell \in \{0,1,\dots,n-1\}$ *with* $n+\ell$ *odd and* $t_0 > T_y$ *such that*

$$\begin{aligned} y(t)L_i y(t) &> 0\,, \qquad 0 \le i \le \ell \\ (-1)^{i-\ell} y(t) L_i y(t) &> 0\,, \qquad \ell \le i \le n \end{aligned} \tag{1.6}$$

for all $t \ge t_0$.

If N denotes the set of all nonoscillatory solutions of (1.3) and N_ℓ denotes the set of all nonoscillatory solutions of (1.3) satisfying (1.6), then

$$\begin{aligned} N &= N_0 \cup N_2 \cup \dots \cup N_{n-1} \quad \text{for } n \text{ odd},\\ N &= N_1 \cup N_3 \cup \dots \cup N_{n-1} \quad \text{for } n \text{ even}. \end{aligned}$$

Following Kiguradze, Eq. (1.3) is said to have property (A) if for n odd $N = N_0$ and for n even $N = \emptyset$, that is, (1.3) is oscillatory.

The following lemma which is a generalization of a lemma due to Kiguradze [5, Lemma 3] describes the asymptotic behaviour of solutions of (1.4).

Lemma 1.2. *If $y(t)$ is a nonoscillatory solution of* (1.4) *on* $[T_y,\infty)$, $T_y \ge \sigma$, *then there is an integer* $\ell \in \{0,1,\dots,n\}$ *with* $\ell \equiv n (mod\, 2)$ *and* $t_0 > T_y$ *such that* (1.6) *holds for all* $t \ge t_0$.

If $\bar N$ denotes the set of all nonoscillatory solutions of (1.4) and $\bar N_\ell$ denotes the set of all nonoscillatory solutions of (1.4) satisfying (1.6), then

$$\begin{aligned} \bar N &= \bar N_1 \cup \bar N_3 \cup \dots \cup \bar N_n \quad \text{for } n \text{ odd},\\ \bar N &= \bar N_0 \cup \bar N_2 \cup \dots \cup \bar N_n \quad \text{for } n \text{ even}. \end{aligned}$$

Equation (1.4) is said to have property (B) if for n odd $\bar N = \bar N_n$ and for n even $\bar N = \bar N_0 \cup \bar N_n$.

Following [6], we define

$$\begin{aligned} &I_0 = 1 \\ &I_k\,(t,s;r_{i_k},\dots,r_{i_1}) = \int_s^t r_{i_k}(x)\, I_{k-1}(x,s;r_{i_{k-1}},\dots,r_{i_1})\,dx\,, \end{aligned} \tag{1.7}$$

where $i_k \in \{1,\dots,n-1\}$, $1 \le k \le n-1$, and $t,s \in [\sigma,\infty)$. It is easy to see that

$$\begin{aligned} &\text{(i)} \quad I_k\,(t,s;r_{i_k},\dots,r_{i_1}) = (-1)^k I_k\,(s,t;r_{i_1},\dots,r_{i_k}) \\ &\text{(ii)} \quad I_k\,(t,s;r_{i_k},\dots,r_{i_1}) = \int_s^t r_{i_1}(x)\, I_{k-1}\,(t,x;r_{i_k},\dots,r_{i_2})\;dx\,. \end{aligned} \tag{1.8}$$

The following lemma is a generalization of Taylor's formula with remainder. The proof is straightforward.

Lemma 1.3. *If $y(t)$ is a solution of* (1.3) *or* (1.4) *on* $[T_y,\infty)$, *then*

$$\begin{aligned} L_i y(t) &= \sum_{j=1}^{k} (-1)^{j-i} L_j y(s)\, I_{j-i}\,(s,t;r_j,\dots,r_{i+1}) \\ &\quad + (-1)^{k-i+1} \int_t^s I_{k-i}\,(x,t;r_k,\dots,r_{i+1})\, r_{k+1}(x)\, L_{k+1}\, y(x)\, dx \end{aligned} \tag{1.9}$$

for $0 \le i \le k \le n-1$ *and* $t,s \in [T_y,\infty)$.

2.

In this section sufficient conditions are obtained so that Eq. (1.3) has property (A).

Theorem 2.1. *If the delay-differential inequality*

$$z'(t) + F_\ell(t,T)\, z(g(t)) \le 0\,, \tag{2.1}$$

$\ell \in \{1,\dots,n-1\}$, *does not admit eventually positive solutions for every large* $T>0$, *then Eq.* (1.3) *has property* (A), *where*

$$F_{n-1}(t,T) = r_n(t)\, p(t)\, r_0(g(t))\, I_{n-1}\,(g(t),T;r_1,\dots,r_{n-1})$$

and

$$\begin{aligned} F_\ell(t,T) &= r_{\ell+l}(t)\, I_\ell\,(g(t),T;r_1,\dots,r_\ell) \int_t^\infty r_{\ell+2}(s_{n-\ell-1}) \\ &\quad \times \int_{s_{n-\ell-1}}^\infty r_{\ell+3}(s_{n-\ell-2}) \cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, r_0(g(s_1))\, ds_1 \dots ds_{n-\ell-1} \end{aligned}$$

for $\ell \in \{1,2,\dots,n-2\}$.

Proof. If possible, suppose that Eq. (1.3) does not have property (A). Hence Eq. (1.3) admits a nonoscillatory solution $y(t)$ such that $y \in N_\ell$, where $\ell \in \{1,\dots,n-1\}$. We may assume, without any loss of generality, that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_1 > t_0$. Hence from Lemma 1.1 it follows that $n+\ell$ is odd and

$$L_i\, y(t) > 0\,,\ 0 \le i \le \ell \quad \text{and} \quad (-1)^{i-\ell} L_i\, y(t) > 0\,,\ \ell \le i \le n\,, \tag{2.2}$$

for $t \ge t_1$. Putting $i=0$, $k=\ell-1$, $t \ge s$ and $s=t_1$ in (1.9) we obtain

$$\begin{aligned} L_0 y(t) &= \sum_{j=0}^{\ell-1} (-1)^j L_j y(t_1)\, I_j(t_1,t;r_j,\dots,r_1) \\ &\quad + (-1)^\ell \int_t^{t_1} I_{\ell-1}(x,t;r_{\ell-1},\dots,r_1)\, r_\ell(x)\, L_\ell\, y(x)\, dx\,. \end{aligned}$$

The use of (1.7), (1.8) and (2.2) yields

$$
\begin{aligned}
L_0\, y(t) &= \sum_{j=0}^{\ell-1} L_j\, y(t_1)\, I_j(t, t_1; r_1, \dots, r_j) \\
&\quad + \int_{t_1}^{t} I_{\ell-1}(t, x; r_1, \dots, r_{\ell-1})\, r_\ell(x)\, L_\ell\, y(x)\, dx \\
&\ge \int_{t_1}^{t} I_{\ell-1}(t, x; r_1, \dots, r_{\ell-1})\, r_\ell(x)\, L_\ell\, y(x)\, dx \\
&\ge L_\ell\, y(t) \int_{t_1}^{t} I_{\ell-1}(t, x; r_1, \dots, r_{\ell-1})\, r_\ell(x)\, dx \\
&= L_\ell\, y(t)\, I_\ell(t, t_1; r_1, \dots, r_\ell)\,.
\end{aligned}
$$

For $t \ge t_2 \ge t_1$, we have $g(t) > t_1$. Thus, for $t \ge t_2$,

$$
L_0\, y(g(t)) \ge L_\ell\, y(g(t))\, I_\ell(g(t), t_1; r_1, \dots, r_\ell)\,, \tag{2.3}
$$

where $\ell \in \{1, 2, \dots, n-1\}$.

Let $\ell = n-1$. From (1.3) and (2.3) we obtain, for $t \ge t_2$,

$$
-L_n\, y(t) = p(t)\, y(g(t))\,, \tag{2.4}
$$

that is,

$$
\begin{aligned}
-(L_{n-1}\, y(t))' &= r_n(t)\, p(t)\, r_0(g(t))\, L_0\, y(g(t)) \\
&\ge r_n(t)\, p(t)\, r_0(g(t))\, L_{n-1}\, y(g(t))\, I_{n-1}(g(t), t_1; r_1, \dots, r_{n-1}) \\
&= F_{n-1}(t, t_1)\, L_{n-1}\, y(g(t))\,.
\end{aligned}
$$

Thus $z(t) = L_{n-1}\, y(t)$ is a positive solution of

$$
z'(t) + F_{n-1}(t, t_1)\, z(g(t)) \le 0
$$

for $t \ge t_2$, a contradiction to the given hypothesis. Next let $\ell \in \{1, \dots, n-2\}$. Repeated integration of (2.4) yields, due to (2.2), that

$$
\begin{aligned}
-(L_\ell\, y(t))' &\ge r_{\ell+1}(t) \int_t^\infty r_{\ell+2}(s_{n-\ell-1}) \int_{s_{n-\ell-1}}^\infty r_{\ell+3}(s_{n-\ell-2}) \\
&\qquad \cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, y(g(s_1))\, ds_1 \dots ds_{n-\ell-2}\, ds_{n-\ell-1} \\
&= r_{\ell+1}(t) \int_t^\infty r_{\ell+2}(s_{n-\ell-1}) \int_{s_{n-\ell-1}}^\infty r_{\ell+3}(s_{n-\ell-2}) \\
&\qquad \cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, r_0(g(s_1))\, L_0\, y(g(s_1))\, ds_1 \dots ds_{n-\ell-2}\, ds_{n-\ell-1}
\end{aligned}
$$

for $t \geq t_2$. Since $L_0 y(t)$ is increasing and $g(t)$ is nondecreasing, we get, using (2.3),

$$\begin{aligned} -(L_\ell\, y(t))' &\geq L_0\, y(g(t))\, r_{\ell+1}(t) \int_t^\infty r_{\ell+2}(s_{n-\ell-1}) \int_{s_{n-l-1}}^\infty r_{\ell+3}(s_{n-\ell-2}) \\ &\quad \cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, r_0(g(s_1))\, ds_1 \ldots ds_{n-\ell-2}\, ds_{n-\ell-1} \\ &\geq L_\ell\, y(g(t))\, F_\ell(t, t_1) \end{aligned}$$

for $t \geq t_2$. Thus $z(t) = L_\ell\, y(t)$ is a positive solution of

$$z'(t) + F_\ell(t, t_1)\, z(g(t)) \leq 0$$

for $t \geq t_2$, a contradiction. Hence the theorem is proved. $\square$

We need the following lemma (see [8, pp. 16, 19]) for our use in the sequel.

Lemma 2.2. *If*

$$\liminf_{t\to\infty} \int_{g(t)}^t p(s)\, ds > 1/e$$

or

$$\limsup_{t\to\infty} \int_{g(t)}^t p(s)\, ds > 1\,,$$

then $y'(t) + p(t)\, y(g(t)) \leq 0$ does not admit eventually positive solutions.

Corollary 2.3. *If, for $\ell \in \{1, 2, \ldots, n-1\}$ such that $n + \ell$ odd*

$$\liminf_{t\to\infty} \int_{g(t)}^t F_\ell(s, T)\, ds > 1/e$$

or

$$\limsup_{t\to\infty} \int_{g(t)}^t F_\ell(s, T)\, ds > 1\,,$$

for every large $T > 0$, then Eq. (1.3) has property (A), where $F_\ell(t, T)$ is same as in (2.1).

This follows from Theorem 2.1 and Lemma 2.2.

Remark. It is easy to verify that $F_\ell(t, T)$, $\ell \in \{1, \ldots, n-1\}$, and $q_\ell(t)$, $\ell \in \{1, \ldots, n-1\}$, (see [1]) differ for higher ℓ.

Example 1. Consider the canonical delay-differential equation

$$2\sqrt{2}\, t^{\sqrt{2}+1} \left(\frac{1}{2t^{\sqrt{2}-1}} \left(\frac{1}{4\sqrt{2}\, t^{\sqrt{2}-1}} \left(4t^{\sqrt{2}} y \right)' \right)' \right)' + 2^{4+\sqrt{2}} y\left(\frac{t}{2}\right) = 0\,, \quad t \geq 1\,.$$

For $T \geq 1$,

$$\int_{t/2}^{t} F_2(s,T)\,ds = 4\log 2 + \frac{2^{3+\sqrt{2}}T^{\sqrt{2}}}{\sqrt{2}}\frac{1}{t^{\sqrt{2}}} - \frac{2^{3+2\sqrt{2}}T^{\sqrt{2}}}{\sqrt{2}}\frac{1}{t^{\sqrt{2}}} - \frac{2^{2+2\sqrt{2}}T^{2\sqrt{2}}}{2\sqrt{2}}\frac{1}{t^{2\sqrt{2}}} + \frac{2^{2+4\sqrt{2}}T^{2\sqrt{2}}}{2\sqrt{2}}\frac{1}{t^{2\sqrt{2}}}$$

implies that

$$\liminf_{t\to\infty} \int_{t/2}^{t} F_2(s,T)\,ds = 4\log 2 > \frac{1}{e}$$

for every $T \geq 1$. Hence the equation has property (A) due to Corollary 2.3.

In the following we present another method of obtaining sufficient conditions so that Eq. (1.3) has property (A). This problem was brought to our notice by Prof. Dzurina.

Theorem 2.4. *If the delay-differential inequality*

$$z'(t) + Q_\ell(t,T)\,z(w(t)) \leq 0\,, \tag{2.5}$$

$\ell \in \{1,\dots,n-1\}$, *does not admit eventually positive solutions for every large* $T > 0$, *then Eq.* (1.3) *has property* (A), *where*

$$Q_{n-1}(t,T) = p(t)\,r_0(g(t))\,r_n(t)\,I_{n-1}(g(t),T;r_1,\dots,r_{n-1})$$

and

$$Q_\ell(t,T) = r_{\ell+1}(t)\int_t^{\tau(t)} I_{n-\ell-2}(x,t;r_{n-1},\dots,r_{\ell+2})\,r_n(x)\,p(x) \times r_0(g(x))\,I_\ell(g(x),T;r_1,\dots,r_\ell)\,dx\,,$$

$\ell \in \{1,\dots,n-2\}$, τ *and* w *are real valued continuous functions on* $[\sigma,\infty)$ *such that* $\tau(t) > t$ *and* $w(t) = g(\tau(t)) < t$.

Proof. Since $g(t)$ is nondecreasing, then $g(t) < w(t) < t$. Proceeding as in the proof of Theorem 2.1 we obtain, for $t \geq t_2$,

$$-(L_{n-1}\,y(t))' \geq Q_{n-1}(t,t_1)\,L_{n-1}\,y(g(t))\,.$$

Since $L_{n-1}\,y(t)$ is monotonic decreasing, then

$$-(L_{n-1}\,y(t))' \geq Q_{n-1}(t,t_1)\,L_{n-1}\,y(w(t))$$

for $t \geq t_2$. Thus $z(t) = L_{n-1}\,y(t)$ is a positive solution of

$$z'(t) + Q_{n-1}(t,t_1)\,z(w(t)) \leq 0$$

for $t \geq t_2$, a contradiction. Let $\ell \in \{1, \ldots, n-2\}$. Putting $i = \ell + 1$, $k = n - 1$ and $s \geq t \geq t_1$ in Lemma 1.3 and using (2.2) we obtain

$$\begin{aligned} L_{\ell+1}\, y(t) &= \sum_{j=\ell+1}^{n-1} (-1)^{j-\ell-1} L_j\, y(s)\, I_{j-\ell-1}(s,t;r_j,\ldots,r_{\ell+2}) \\ &\quad + (-1)^{n-\ell-1} \int_t^s I_{n-\ell-2}(x,t;r_{n-1},\ldots,r_{\ell+2})\, r_n(x)\, L_n\, y(x)\, dx \\ &\leq \int_t^s I_{n-\ell-2}(x,t;r_{n-1},\ldots,r_{\ell+2})\, r_n(x)\, L_n\, y(x)\, dx \\ &= \int_t^s I_{n-\ell-2}(x,t;r_{n-1},\ldots,r_{\ell+2})\, r_n(x)\, p(x)\, r_0(g(x))\, L_0\, y(g(x))\, dx\,. \end{aligned}$$

Letting $s \to \infty$, we get, using (2.3),

$$\begin{aligned} -L_{\ell+1}\, y(t) &\geq \int_t^{\infty} I_{n-\ell-2}(x,t;r_{n-1},\ldots,r_{\ell+2})\, r_n(x)\, p(x)\, r_0(g(x))\, L_0 y(g(x))\, dx \\ &\geq \int_t^{\tau(t)} I_{n-\ell-2}(x,t;r_{n-1},\ldots,r_{\ell+2}) r_n(x)\, p(x)\, r_0(g(x)) \\ &\quad \times I_\ell(g(x),t_1;r_1,\ldots,r_\ell) L_\ell\, y(g(x))\, dx \end{aligned}$$

for $t \geq t_2$. Since g is nondecreasing, $w(t) = g(\tau(t))$ and $L_\ell\, y$ is monotonic decreasing, then

$$\begin{aligned} -(L_\ell\, y(t))' &\geq r_{\ell+1}(t) L_\ell\, y(w(t)) \int_t^{\tau(t)} I_{n-\ell-2}(x,t;r_{n-1},\ldots,r_{\ell+2})\, r_n(x) \\ &\quad \times\ p(x)\, r_0(g(x)) I_\ell(g(x),t_1;r_1,\ldots,r_\ell)\, dx \\ &\geq Q_\ell(t,t_1)\, L_\ell\, y(w(t)) \end{aligned}$$

for $t \geq t_2$. Thus $z(t) = L_\ell\, y(t)$ is a positive solution of

$$z'(t) + Q_\ell(t,t_1)\, z(w(t)) \leq 0$$

for $t \geq t_2$, a contradiction which completes the proof of the theorem. □

Corollary 2.5. *If, for* $\ell \in \{1, \ldots n-1\}$,

$$\liminf_{t\to\infty} \int_{w(t)}^t Q_\ell(s,T)\, ds > \frac{1}{e}$$

or

$$\limsup_{t\to\infty} \int_{w(t)}^t Q_\ell(s,T)\, ds > 1\,,$$

for every large $T > 0$, *then Eq.* (1.3) *has property* (A), *where* $Q_\ell(t,T)$ *is same as in* (2.5).

Remark. We may notice that $F_{n-1}(t,T) = Q_{n-1}(t,T)$. However, $F_\ell(t,T) \not\equiv Q_\ell(t,T)$ for $\ell < n-1$.

Example 2. Consider

$$\left(\log\frac{t}{2t_1}\right)\left(t\left(t\left(t\left(\left(\log\frac{t}{T}\right)^2 y(t)\right)'\right)'\right)'\right)' + \log\frac{t}{2t_1}\, y\left(\frac{t}{2}\right) = 0\,,$$

$$t > T > 1\,, \tag{2.6}$$

where

$$r_0(t) = \frac{1}{\left(\log\frac{t}{T}\right)^2}\,, r_1(t) = r_2(t) = r_3(t) = \frac{1}{t}\,, r_4(t) = \frac{1}{\log\frac{t}{2T}}\,,$$
$$p(t) = \log\frac{t}{2T} \quad \text{and} \quad g(t) = \frac{t}{2}\,.$$

Hence

$$\int_{s_2}^{\infty} r_4(s_1)\, p(s_1)\, r_0(g(s_1))\, ds_1 > s_2 \frac{1}{\log\frac{s_2}{2T}}\,,$$
$$\int_{t}^{\infty} r_3(s_2) \int_{s_2}^{\infty} r_4(s_1)\, p(s_1)\, r_0(g(s_1))\, ds_1\, ds_2 > \int_t^{\infty} \frac{1}{s}\cdot s \frac{1}{\log\frac{s}{2T}}\, ds$$
$$> t\left\{\lim_{\alpha\to\infty} \log\alpha - \log\left(\log\frac{t}{2T}\right)\right\}$$

and

$$I_1(g(t), T; r_1) = \int_T^{t/2} \frac{1}{s}\, ds = \log\frac{t}{2T}\,.$$

Hence

$$F_1(t,T) = r_2(t)\, I_1(g(t),T;r_1) \int_t^{\infty} r_3(s_2) \int_{s_2}^{\infty} r_4(s_1)\, p(s_1)\, r_0(g(s_1))\, ds_1\, ds_2 = \infty\,,$$

for every t. Thus

$$\liminf_{t\to\infty} \int_{g(t)}^{t} F_1(s,T)\, ds > \frac{1}{e}\,.$$

Further

$$I_3(g(t),T;r_1,r_2,r_3) = \int_T^{t/2} r_1(s_1) \int_T^{s_1} r_2(s_2) \int_T^{s_2} r_3(s_3)\, ds_3\, ds_2\, ds_1$$
$$= \frac{1}{6}\left(\log\frac{t}{2T}\right)^3$$

and

$$F_3(t,T) = \frac{1}{6}\log\frac{t}{2T}\,.$$

Hence

$$\liminf_{t\to\infty}\int_{g(t)}^{t} F_3(s,T)\,ds > \frac{1}{e}\,.$$

Thus, from Corollary 2.3, it follows that Eq. (2.6) has property (A). However, Corollary 2.5 cannot be applied to Eq. (2.6) because, for $\tau(t) = t+1 > t$, we obtain $w(t) = g(\tau(t)) = \frac{t+1}{2}$,

$$\begin{aligned} I_1(g(t),T;r_1) &= \log\frac{t}{2T}\,, \quad I_1(x,t;r_3) = \log\frac{x}{t}\,,\\ Q_1(t,T) &= \frac{1}{t}\int_t^{t+1}\log\frac{x}{t}\cdot\frac{1}{\log\frac{x}{2T}}\cdot\log\frac{x}{2T}\cdot\frac{1}{\left(\log\frac{x}{2T}\right)^2}\cdot\log\frac{x}{2T}\,dx\\ &< \frac{1}{t\log\frac{t}{2T}}\cdot\log\left(1+\frac{1}{t}\right) \end{aligned}$$

and

$$\begin{aligned} \liminf_{t\to\infty}\int_{w(t)}^{t} Q_1(s,T)\,ds &< \lim_{t\to\infty}\log\left(1+\frac{2}{t+1}\right)\cdot\frac{1}{\log(\frac{t+1}{4T})}\cdot\log\left(\frac{2}{1+\frac{1}{t}}\right)\\ &= 0 < \frac{1}{e} < 1\,. \end{aligned}$$

Remark. As the conditions in Corollaries 2.3 and 2.5 are not comparable, it would be interesting to find an example where Corollary 2.5 holds but Corollary 2.3 fails to hold.

In the following we state a result which is a particular case of Theorem 1 due to Fink and Kusano [3].

Theorem 2.6. *Let ℓ be an integer such that $0 \le \ell < n$ and $n+\ell$ odd. A necessary and sufficient condition for Eq. (1.3) to have a maximal solution $y(t)$ satisfying (1.6) is that*

$$\int_\sigma^\infty K_{n-\ell-1}(t,\sigma)\,p(t)|J_\ell(g(t),\sigma)|\,dt < \infty\,, \tag{2.7}$$

where

$$J_i(t,s) = r_0(t)\,I_i(t,s;r_1,\ldots,r_i) \tag{2.8}$$

and

$$K_i(t,s) = r_n(t)\,I_i(t,s;r_{n-1},\ldots,r_{n-i}), \quad 0 \le i \le n-1\,. \tag{2.9}$$

Remark. We may observe that, for $\ell = n-1$,

$$K_{n-\ell-1}(t,\sigma)\, p(t)\, J_\ell(g(t),\sigma) = F_\ell(t,\sigma) = Q_\ell(t,\sigma)$$

and, for $\ell \in \{0,1,\dots,n-2\}$,

$$K_{n-\ell-1}(t,\sigma)\, p(t)\, J_\ell(g(t),\sigma) \neq F_\ell(t,\sigma) \quad \text{and} \quad \neq Q_\ell(t,\sigma)\,.$$

In Example 2, $n = 4$ and hence from Corollary 2.3 it follows that all solutions of (2.6) are oscillatory. It is confirmed by Theorem 2.6 because (2.7) fails to hold for $\ell = 1$.

An attempt has been made in the following to compare property (A) of certain n-th order canonical ordinary differential equations with that of delay differential equations.

Theorem 2.7. *Let $g \in C^1([\sigma,\infty), R)$ such that $g'(t) > 0$. If the differential equation*

$$L_n x + \frac{p\left(g^{-1}(t)\right) r_n\left(g^{-1}(t)\right)}{r_n(t) g'\left(g^{-1}(t)\right)} x = 0 \tag{2.10}$$

has property (A)*, then Eq.* (1.3) *has property* (A)*.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.3). In order to complete the proof of the theorem it is enough to show, in view of Lemma 1.1, that $\ell = 0$. If possible, suppose that $\ell \neq 0$. Without any loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_0 > \sigma$. Hence $L_0 y(t) > 0$ and $L_n\, y(t) < 0$ for $t \geq t_1 > t_0$ by Lemma 1.1. Integrating (1.3) from t to ∞, $t > t_1$, we obtain

$$L_{n-1}\, y(t) > \int_t^\infty r_n(s_1)\, p(s_1)\, y(g(s_1))\, ds_1\,.$$

Further integration from t to ∞ yields

$$-L_{n-2}\, y(t) > \int_t^\infty r_{n-1}(s_2) \left(\int_{s_2}^\infty r_n(s_1)\, p(s_1)\, y(g(s_1))\, ds_1 \right) ds_2\,.$$

Repeating the process we get

$$L_\ell\, y(t) > \int_t^\infty r_{\ell+1}(s_{n-\ell}) \cdots \int_{s_3}^\infty r_{n-1}(s_2) \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, y(g(s_1))\, ds_1\, ds_2 \dots ds_{n-\ell}\,.$$

Integrating the above inequality from t_1 to t, one may obtain

$$\begin{aligned} L_{\ell-1}\, y(t) > \int_{t_1}^t r_\ell(s_{n-\ell+1}) \int_{s_{n-\ell+1}}^\infty & r_{\ell+1}(s_{n-\ell}) \\ \cdots \int_{s_2}^\infty & r_n(s_1)\, p(s_1)\, y(g(s_1))\, ds_1 \dots ds_{n-\ell+1}\,. \end{aligned}$$

Repeated integration yields

$$\begin{aligned} L_0\, y(t) &> K + \int_{t_1}^{t} r_1(s_n) \cdots \int_{t_1}^{s_{n-\ell+2}} r_\ell(s_{n-\ell+1}) \int_{s_{n-\ell+1}}^{\infty} r_{\ell+1}(s_{n-\ell}) \\ &\quad \cdots \int_{s_2}^{\infty} r_n(s_1)\, p(s_1)\, y(g(s_1))\, ds_1 \ldots ds_n \\ &= K + \int_{t_1}^{t} r_1(s_n) \cdots \int_{t_1}^{s_{n-\ell+2}} r_\ell(s_{n-\ell+1}) \int_{s_{n-\ell+1}}^{\infty} r_{\ell+1}(s_{n-\ell}) \\ &\quad \cdots \int_{g(s_2)}^{\infty} \frac{r_n(g^{-1}(s_1))\, p(g^{-1}(s_1))\, y(s_1)}{g'(g^{-1}(s_1))}\, ds_1 \ldots ds_n \\ &> K + \int_{t_1}^{t} r_1(s_n) \cdots \int_{t_1}^{s_{n-\ell+2}} r_\ell(s_{n-\ell+1}) \int_{s_{n-\ell+1}}^{\infty} r_{\ell+1}(s_{n-\ell}) \\ &\quad \cdots \int_{s_2}^{\infty} \frac{r_n(g^{-1}(s_1))\, p(g^{-1}(s_1))\, y(s_1)}{g'(g^{-1}(s_1))}\, ds_1 \ldots ds_n\,, \end{aligned}$$

where $K = L_0\, y(t_1) > 0$ and we have used the facts that g' exists, g is increasing and $g(t) < t$. Thus

$$\begin{aligned} L_0\, y(t) &> K + \int_{t_1}^{t} r_1(s_n) \cdots \int_{t_1}^{s_{n-\ell+2}} r_\ell(s_{n-\ell+1}) \int_{s_{n-\ell+1}}^{\infty} r_{\ell+1}(s_{n-\ell}) \\ &\quad \cdots \int_{s_2}^{\infty} \frac{r_n(g^{-1}(s_1))\, p(g^{-1}(s_1)) r_0(s_1)\, L_0\, y(s_1)}{g'(g^{-1}(s_1))}\, ds_1 \ldots ds_n\,. \end{aligned}$$

From Lemma 5 due to Kusano and Naito [6] it follows that the integral equation

$$\begin{aligned} v(t) &> K + \int_{t_1}^{t} r_1(s_n) \cdots \int_{t_1}^{s_{n-\ell+2}} r_\ell(s_{n-\ell+1}) \int_{s_{n-\ell+1}}^{\infty} r_{\ell+1}(s_{n-\ell}) \\ &\quad \cdots \int_{s_2}^{\infty} \frac{r_n(g^{-1}(s_1))\, p(g^{-1}(s_1))\, r_0(s_1)\, v(s_1)}{g'(g^{-1}(s_1))}\, ds_1 \ldots ds_n \end{aligned}$$

admits a solution $v(t)$, $t \ge t_1$, satisfying

$$K \le v(t) \le L_0\, y(t)\,, \quad t \ge t_1\,.$$

Hence $v(t) > 0$ for $t \ge t_1$. Setting $x(t) = r_0(t)\, v(t)$, we obtain $x(t) > 0$ for $t \ge t_1$ and

$$\begin{aligned} L_0\, x(t) &= K + \int_{t_1}^{t} r_1(s_n) \cdots \int_{t_1}^{s_{n-\ell+2}} r_\ell(s_{n-\ell+1}) \int_{s_{n-\ell+1}}^{\infty} r_{\ell+1}(s_{n-\ell}) \\ &\quad \cdots \int_{s_2}^{\infty} \frac{r_n(g^{-1}(s_1))\, p(g^{-1}(s_1))\, x(s_1)}{g'(g^{-1}(s_1))}\, ds_1 \ldots ds_n\,. \end{aligned}$$

Repeated differentiation yields that $x(t)$ is a solution of (2.10) satisfying (1.6) with $\ell \ne 0$. This contradicts the fact that (2.10) has property (A) and hence the theorem is proved. $\square$

3.

This section deals with property (B) of Eq. (1.4). We have the following theorem.

Theorem 3.1. *If the delay-differential inequality*

$$z'(t) + F_\ell(t)\, z(g(t)) \le 0\,, \tag{3.1}$$

$\ell \in \{1, \dots, n-2\}$ *such that* $n+\ell$ *is even, does not admit eventually positive solutions, then Eq.* (1.4) *has property* (B), *where*

$$F_{n-2}(t) = r_n(t)\, p(t)\, r_0(g(t))[R_{n-1}(g(t)) - R_{n-1}(g(g(t)))] \times I_{n-2}(g(g(t)), g(g(g(t))); r_1, \dots, r_{n-2})$$

and, for $1 \le \ell \le n-4$,

$$\begin{aligned} F_\ell(t) = {} & r_{\ell+2}(t)[R_{\ell+1}(g(t)) - R_{\ell+1}(g(g(t)))]\, I_\ell(g(g(t)), g(g(g(t))); r_1, \dots, r_\ell) \\ & \times \int_t^\infty r_{\ell+3}(s_{n-\ell-2}) \int_{s_{n-\ell-2}}^\infty r_{\ell+4}(s_{n-\ell-3}) \\ & \cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, r_0(g(s_1))\, ds_1 \dots ds_{n-\ell-3}\, ds_{n-\ell-2} \end{aligned}$$

and $R_i(t) = \int_\sigma^t r_i(s)\, ds$, $1 \le i \le n-1$.

Proof. Suppose that Eq. (1.4) does not have property (B). Hence there exists a solution $y(t)$ of (1.4) such that $y \in \bar{N}_\ell$, where $\ell \in \{1, \dots, n-2\}$. Without any loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \ge t_1 > t_0$. From Lemma 1.2 it follows that $n+\ell$ is even and

$$L_i\, y(t) > 0\,,\ 0 \le i \le \ell \quad \text{and} \quad (-1)^{i-\ell} L_i\, y(t) > 0\,,\ \ell \le i \le n\,, \tag{3.2}$$

for $t \ge t_1$. We may choose $t_2 > t_1$ such that $g(t) > t_1$ for $t \ge t_2$. Then putting $i = 0$, $k = \ell - 1$ and $t > s = g(t)$ for $t \ge t_2$ in (1.9) and using (1.8) (i) and (3.2) we obtain

$$\begin{aligned} L_0\, y(t) &= \sum_{j=0}^{\ell-1} (-1)^j L_j\, y(g(t))\, I_j(g(t), t; r_j, \dots, r_1) \\ &\quad + (-1)^\ell \int_t^{g(t)} I_{\ell-1}(x, t; r_{\ell-1}, \dots, r_1)\, r_\ell(x)\, L_\ell\, y(x)\, dx \\ &\ge (-1)^\ell \int_t^{g(t)} I_{\ell-1}(x, t; r_{\ell-1}, \dots, r_1)\, r_\ell(x)\, L_\ell\, y(x)\, dx \\ &\ge (-1)^{2\ell} \int_{g(t)}^t I_{\ell-1}(t, x; r_1, \dots, r_{\ell-1})\, r_\ell(x)\, L_\ell\, y(x)\, dx \\ &\ge L_\ell\, y(t) \int_{g(t)}^t I_{\ell-1}(t, x; r_1, \dots, r_{\ell-1})\, r_\ell(x)\, dx \\ &\ge (-1)^\ell L_\ell\, y(t) \int_t^{g(t)} I_{\ell-1}(x, t; r_{\ell-1}, \dots, r_1)\, r_\ell(x)\, dx \\ &= (-1)^\ell L_\ell\, y(t)\, I_\ell(g(t), t; r_\ell, \dots, r_1) = L_\ell\, y(t)\, I_\ell(t, g(t), r_1, \dots, r_\ell)\,. \end{aligned}$$

For $t \geq t_3 \geq t_2$, $g(t) \geq t_2$ and hence

$$L_0\, y(g(t)) \geq L_\ell\, y(g(t))\, I_\ell(g(t), g(g(t)); r_1, \dots, r_\ell)\,.$$

Since $L_0\, y(t)$ is monotonic increasing, we get, for $t \geq t_3$,

$$L_0\, y(t) \geq L_\ell\, y(g(t))\, I_\ell(g(t), g(g(t)); r_1, \dots, r_\ell)\,.$$

Further, for $t \geq t_4 \geq t_3$, $g(t) \geq t_3$ and we obtain

$$L_0\, y(g(t)) \geq L_\ell\, y(g(g(t)))\, I_\ell(g(g(t)), g(g(g(t))); r_1, \dots, r_\ell)\,. \tag{3.3}$$

Since $g(t)$ is increasing, then $g^{-1}(t)$ exists and increasing. Further, $g(t) < t$ implies that $t < g^{-1}(t)$. Integrating $(L_\ell\, y(t))' = r_{\ell+1}(t)\, L_{\ell+1}\, y(t)$ we obtain, for $t \geq t_4$,

$$L_\ell\, y(g^{-1}(t)) - L_\ell\, y(t) = \int_t^{g^{-1}(t)} r_{\ell+1}(s)\, L_{\ell+1} y(s)\, ds\,,$$

that is, for $t \geq t_4$,

$$\begin{aligned} -L_\ell\, y(t) &\leq L_{\ell+1}\, y(g^{-1}(t)) \int_t^{g^{-1}(t)} r_{\ell+1}(s)\, ds \\ &= L_{\ell+1}\, y(g^{-1}(t))[R_{\ell+1}(g^{-1}(t)) - R_{\ell+1}(t)]\,. \end{aligned}$$

For $t \geq t_5 > t_4$, we have $g(t) > t_4$ and hence

$$L_\ell\, y(g(t)) \geq -L_{\ell+1}\, y(t)[R_{\ell+1}(t) - R_{\ell+1}(g(t))]\,.$$

Thus, for $t \geq t_6$,

$$L_\ell\, y(g(g(t))) \geq -L_{\ell+1}\, y(g(t))[R_{\ell+1}(g(t)) - R_{\ell+1}(g(g(t)))]\,.$$

Hence, using (3.3), we get

$$\begin{aligned} L_0\, y(g(t)) \geq &- L_{\ell+1}\, y(g(t))[R_{\ell+1}(g(t)) - R_{\ell+1}(g(g(t)))] \\ &\times\ I_\ell(g(g(t)), g(g(g(t))); r_1, \dots, r_\ell) \end{aligned} \tag{3.4}$$

for $t \geq t_6$. From (1.4) we obtain, due to (3.4) with $\ell = n-2$,

$$\begin{aligned} (L_{n-1}\, y(t))' &= r_n(t)\, p(t)\, y(g(t)) \\ &= r_n(t)\, p(t)\, r_0(g(t))\, L_0\, y(g(t)) \\ &\geq -r_n(t)\, p(t)\, r_0(g(t))\, L_{n-1} y(g(t))[R_{n-1}(g(t)) - R_{n-1}(g(g(t)))] \\ &\quad \times\ I_{n-2}(g(g(t)), g(g(g(t))); r_1, \dots, r_{n-2})\,, \end{aligned}$$

that is,

$$-(L_{n-1}\, y(t))' - F_{n-2}(t)\, L_{n-1}\, y(g(t)) \le 0\,.$$

Hence $z(t) = -L_{n-1}\, y(t)$ is a positive solution of

$$z'(t) + F_{n-2}(t)\, z(g(t)) \le 0$$

for $t \ge t_6$, a contradiction. Next suppose that $\ell \in \{1, \dots, n-4\}$. Integrating (1.4) and using (3.2) we get

$$-L_{n-1}\, y(t) = \int_t^\infty r_n(s_1)\, p(s_1)\, y(g(s_1))\, ds_1\,.$$

Repeated integration and use of (3.2) yield, for $t \ge t_6$,

$$\begin{aligned}
(L_{\ell+1}\, y(t))' \ge{}& r_{\ell+2}(t) \int_t^\infty r_{\ell+3}(s_{n-\ell-2}) \int_{s_{n-\ell-2}}^\infty r_{\ell+4}(s_{n-\ell-3}) \\
&\cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, y(g(s_1))\, ds_1 \dots ds_{n-\ell-3}\, ds_{n-\ell-2} \\
={}& r_{\ell+2}(t) \int_t^\infty r_{\ell+3}(s_{n-\ell-2}) \int_{s_{n-\ell-2}}^\infty r_{\ell+4}(s_{n-\ell-3}) \\
&\cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, r_0(g(s_1))\, L_0\, y(g(s_1))\, ds_1 \dots ds_{n-\ell-2} \\
\ge{}& r_{\ell+2}(t)\, L_0\, y(g(t)) \int_t^\infty r_{\ell+3}(s_{n-\ell-2}) \int_{s_{n-\ell-2}}^\infty r_{\ell+4}(s_{n-\ell-3}) \\
&\cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, r_0(g(s_1))\, ds_1 \dots ds_{n-\ell-3}\, ds_{n-\ell-2}\,.
\end{aligned}$$

Hence using (3.4) we obtain, for $t \ge t_6$,

$$\begin{aligned}
(L_{\ell+1}\, y(t))' \ge{}& -r_{\ell+2}(t)\, L_{\ell+1}\, y(g(t))[R_{\ell+1}(g(t)) - R_{\ell+1}(g(g(t)))] \\
&\times\ I_\ell\,(g(g(t)), g(g(g(t))); r_1, \dots, r_\ell) \\
&\times \int_t^\infty r_{\ell+3}(s_{n-\ell-2}) \cdots \int_{s_2}^\infty r_n(s_1)\, p(s_1)\, r_0(g(s_1))\, ds_1 \dots ds_{n-\ell-2}\,,
\end{aligned}$$

that is, for $t \ge t_6$,

$$-(L_{\ell+1}\, y(t))' - F_\ell\,(t)\, L_{\ell+1}\, y(g(t)) \le 0\,.$$

Thus $z(t) = -L_{\ell+1}\, y(t)$ is a positive solution of

$$z'(t) + F_\ell(t)\, z(g(t)) \le 0$$

for $t \ge t_6$, a contradiction which completes the proof of the theorem. $\square$

Corollary 3.2. *If, for $\ell \in \{1, \dots, n-2\}$ such that $n+\ell$ is even,*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} F_\ell(s)\, ds > \frac{1}{e}$$

or

$$\limsup_{t\to\infty} \int_{g(t)}^{t} F_\ell(s)\, ds > 1\,,$$

then Eq. (1.4) *has property* (B)*, where F_ℓ is same as in* (3.1)*.*

This follows from Lemma 2.2 and Theorem 3.1.

In a recent paper (see [2, pp. 152]), Dzurina has obtained the following result.

Theorem 3.3. *If, for $\ell \in \{1, \dots, n-2\}$ such that $n+\ell$ even,*

$$\liminf_{t\to\infty} \int_{w(t)}^{t} Q_\ell(s)\, ds > \frac{1}{e}$$

or

$$\limsup_{t\to\infty} \int_{w(t)}^{t} Q_\ell(s)\, ds > 1\,,$$

then Eq. (1.4) *has property* (B)*, where*

$$\begin{aligned} Q_\ell(t) = r_{\ell+1}(t) \int_t^{\tau(t)} & r_n(s)\, r_0(g(s))\, p(s)\, I_{n-\ell-2}(s, t; r_{n-1}, \dots, r_{\ell+2}) \\ \times\ & I_\ell(g(s), t_1; r_1, \dots, r_\ell)\, ds \end{aligned}$$

for sufficiently large t_1 with $g(t) > t_1$.

In the following we give some examples to which Corollary 3.2 can be employed but Theorem 3.3 cannot be applied.

Example 3. Consider

$$\log\frac{t}{2t_1}\left(t\left(t\left(\frac{y(t)}{\log\left(\frac{t}{t_1}\right)}\right)'\right)'\right)' - \frac{1}{\log\left(\frac{t}{2t_1}\right)}\, y\left(\frac{t}{2}\right) = 0\,, \quad t > t_1 > 1\,, \tag{3.5}$$

where $r_0(t) = \log\frac{t}{t_1}$, $r_1(t) = r_2(t) = \frac{1}{t}$, $r_3(t) = \frac{1}{\log\left(\frac{t}{2t_1}\right)}$,

$$p(t) = \frac{1}{\log\left(\frac{t}{2t_1}\right)} \quad \text{and} \quad g(t) = \frac{t}{2}\,.$$

Hence

$$R_2(g(t)) - R_2(g(g(t))) = \int_{t/4}^{t/2} \frac{1}{s}\, ds = \log 2$$

and

$$I_1(g(g(t)), g(g(g(t))); r_1) = \int_{t/8}^{t/4} \frac{1}{s}\, ds = \log 2\,.$$

Then

$$F_1(t) = \frac{1}{\log\left(\frac{t}{2t_1}\right)} \cdot \frac{1}{\log\left(\frac{t}{2t_1}\right)} \cdot \log\left(\frac{t}{2t_1}\right) \log 2 \cdot \log 2 = \frac{(\log 2)^2}{\log\left(\frac{t}{2t_1}\right)}$$

implies that

$$\lim_{t\to\infty} \int_{g(t)}^{t} F_1(s)\, ds = \lim_{t\to\infty} (\log 2)^2 \int_{t/2}^{t} \frac{1}{\log\left(\frac{s}{2t_1}\right)}\, ds > (\log 2)^2 \lim_{t\to\infty} \frac{1}{2\log\left(\frac{t}{2t_1}\right)}$$
$$= \frac{(\log 2)^2}{2} \cdot \lim_{t\to\infty} \frac{t}{\log\left(\frac{t}{2t_1}\right)} = \infty\,.$$

Thus, by Corollary 3.2, Eq. (3.5) has property (B). On the other hand, Theorem 3.3 cannot be applied to Eq. (3.5) because, setting $\tau(t) = t+1$, we obtain $\tau(t) > t$ and $w(t) = g(\tau(t)) = \frac{t+1}{2}$ and

$$Q_1(t) = \frac{1}{t} \int_t^{t+1} \frac{1}{\log\left(\frac{s}{2t_1}\right)} \cdot \log\left(\frac{s}{2t_1}\right) \frac{1}{\log\left(\frac{s}{2t_1}\right)} \cdot \log(\frac{s}{2t_1})\, ds = \frac{1}{t}\,.$$

Hence

$$\lim_{t\to\infty} \int_{w(t)}^{t} Q_1(s)\, ds = \lim_{t\to\infty} \int_{\frac{t+1}{2}}^{t} \frac{1}{s}\, ds = \log 2 = 0.3010 < \frac{1}{3} < \frac{1}{e} < 1\,.$$

Example 4. Consider

(3.6)
$$\log\left(\frac{t}{2t_1}\right) \left(t\left(t\left(t\left(\left(\log\left(\frac{t}{t_1}\right)\right)^2 y(t)\right)'\right)'\right)'\right)' - \log\left(\frac{t}{2t_1}\right) y\left(\frac{t}{2}\right) = 0\,,$$
$$t > t_1 > 1\,,$$

where

$$r_0(t) = \frac{1}{\left(\log\left(\frac{t}{t_1}\right)\right)^2}\,, r_1(t) = r_2(t) = r_3(t) = \frac{1}{t}\,, r_4(t) = \frac{1}{\log\left(\frac{t}{2t_1}\right)}\,,$$
$$p(t) = \log\left(\frac{t}{2t_1}\right) \quad \text{and} \quad g(t) = \frac{t}{2}\,.$$

Hence

$$R_3(g(t)) - R_3(g(g(t))) = \int_{t/4}^{t/2} \frac{1}{s}\, ds = \log 2$$

and

$$I_2(g(g(t)), g(g(g(t))); r_1, r_2) = \int_{t/8}^{t/4} \frac{1}{s_1} \left(\int_{t/8}^{s_1} \frac{1}{s_2}\, ds_2 \right) ds_1 = \frac{1}{2}(\log 2)^2 .$$

Then

$$F_2(t) = \frac{1}{\log(\frac{t}{2t_1})} \cdot \log\left(\frac{t}{2t_1}\right) \cdot \frac{1}{\left(\log\left(\frac{t}{2t_1}\right)\right)^2} \cdot \log 2 \cdot \frac{1}{2}(\log 2)^2 = \frac{(\log 2)^3}{2} \cdot \frac{1}{\left(\log\left(\frac{t}{2t_1}\right)\right)^2}$$

implies that

$$\begin{aligned}\lim_{t\to\infty} \int_{g(t)}^{t} F_2(s)\, ds &= \frac{(\log 2)^3}{2} \lim_{t\to\infty} \int_{t/2}^{t} \frac{1}{\left(\log\left(\frac{s}{2t_1}\right)\right)^2}\, ds \\ &\geq \frac{(\log 2)^3}{4} \cdot \lim_{t\to\infty} \frac{1}{\left(\log\left(\frac{t}{2t_1}\right)\right)^2} = \frac{(\log 2)^3}{8} \cdot \lim_{t\to\infty} t > \frac{1}{e} .\end{aligned}$$

Thus, from Corollary 3.2, it follows that (3.6) has property (B). However, Theorem 3.3 cannot be employed to (3.6). Indeed, setting $\tau(t) = t + 1 > t$, we obtain

$$\begin{aligned} w(t) &= g(\tau(t)) = \frac{t+1}{2} , \\ I_2(g(t), t_1; r_1, r_2) &= \int_{t_1}^{t/2} \frac{1}{s_1} \left(\int_{t_1}^{s_1} \frac{1}{s_2}\, ds_2 \right) ds_1 \leq \left(\int_{t_1}^{t/2} \frac{1}{s}\, ds \right)^2 \\ &= \left(\log\left(\frac{t}{2t_1}\right)\right)^2 \end{aligned}$$

and

$$Q_2(t) \leq \frac{1}{t} \int_{t}^{t+1} \frac{1}{\log\left(\frac{s}{2t_1}\right)} \cdot \frac{1}{\left(\log\left(\frac{s}{2t_1}\right)\right)^2} \cdot \log\left(\frac{s}{2t_1}\right) \cdot \left(\log\left(\frac{s}{2t_1}\right)\right)^2 ds = \frac{1}{t} .$$

Hence

$$\begin{aligned}\lim_{t\to\infty} \int_{w(t)}^{t} Q_2(s)\, ds &= \lim_{t\to\infty} \int_{\frac{t+1}{2}}^{t} \frac{1}{s}\, ds = \lim_{t\to\infty} \log\left(\frac{2}{1+\frac{1}{t}}\right) \\ &= \log 2 = 0.3010 < \frac{1}{3} < \frac{1}{e} < 1 .\end{aligned}$$

Remark. Existence of a solution of Eq. (3.5) or Eq. (3.6) is obvious. However, we could not find explicit solutions to these equations. In the following we give an example of an equation which has property (B). Here an explicit solution of the equation is given.

Example 5. Consider

$$y^{(iv)}(t) - \frac{1944}{t^4}\, y\left(\frac{t}{3}\right) = 0\,, \qquad t \geq 1\,. \tag{3.7}$$

Since $r_i(t) = 1$, $0 \leq i \leq 4$, and $g(t) = t/3$, then $g(g(t)) = t/9$ and $g(g(g(t))) = t/27$ and $R_3(g(t)) - R_3(g(g(t))) = 2t/9$. Further,

$$I_2(g(g(t)), g(g(g(t))); r_1, r_2) = \int_{t/27}^{t/9} \left(\int_s^{t/9} d\theta \right) ds = \frac{2t^2}{27^2}$$

and hence

$$F_2(t) = \frac{1944}{t^4} \times \frac{2t}{9} \times \frac{2t^2}{27^2} = \frac{32}{27} \cdot \frac{1}{t}\,.$$

Thus

$$\liminf_{t\to\infty} \int_{g(t)}^{t} F_2(s)\, ds = \frac{32}{27} \log 3 > \frac{1}{e}\,.$$

On the other hand, for $\tau(t) = t + 1 > t$, we have $w(t) = g(\tau(t)) = \frac{(t+1)}{3} < t$ and

$$\begin{aligned} Q_2(t) &= 1944 \int_t^{t+1} \frac{1}{s^4}\, I_2(g(s), t_1; r_1, r_2)\, ds \\ &= 1944 \int_t^{t+1} \frac{1}{s^4} \left(\frac{s^2}{18} - \frac{t_1 s}{3} + \frac{1}{2} t_1^2 \right) ds \end{aligned}$$

for sufficiently large t such that $g(t) > t_1 > 1$. Clearly,

$$\liminf_{t\to\infty} \int_{w(t)}^{t} Q_2(s)\, ds = 0 < \frac{1}{e}\,.$$

Hence Corollary 3.2 can be employed to Eq. (3.7) to conclude that it has property (B), where as Theorem 3.3 cannot be applied to Eq. (3.7). In particular, $y(t) = t^4$ is a nonoscillatory solution of (3.7) with $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$, $y'''(t) > 0$ and $y^{(iv)}(t) > 0$.

In the following we obtain a result which ensures the existence of a nonoscillatory solution of (1.4) whether n is even or odd.

Theorem 3.4. *Eq.* (1.4) *admits a nonoscillatory solution satisfying*

$$y(t)\, L_i\, y(t) > 0\,, \qquad 0 \leq i \leq n\,.$$

Proof. From a result due to Kusano et all. (Lemma 2, [7]) it follows that the equation

$$L_n x - p^*(t) x = 0\,, \tag{3.8}$$

where $p^*(t) = (r_0(t))^{-1} p(t)\, r_0(g(t))$, admits a nonoscillatory solution $x(t)$ satisfying

$$x(t)\, L_i\, x(t) > 0\,, \qquad 0 \le i \le n$$

for large t. We may assume, without any loss of generality, that $x(t) > 0$ for $t \ge t_0 > \sigma$. Hence $L_i\, x(t) > 0$ for $t \ge t_0$ and $x(g(t)) > 0$ for $t \ge t_1 > t_0$ and $0 \le i \le n$. Successive integration of (3.8) from t_1 to t yields

$$\begin{aligned} L_0\, x(t) &\ge K + \int_{t_1}^{t} r_1(s_1) \int_{t_1}^{s_1} r_2(s_2) \cdots \int_{t_1}^{s_{n-1}} r_n(s_n)\, p^*(s_n)\, x(s_n)\, ds_n \ldots ds_2\, ds_1 \\ &\ge K + \int_{t_1}^{t} r_1(s_1) \int_{t_1}^{s_1} r_2(s_2) \\ &\qquad \cdots \int_{t_1}^{s_{n-1}} r_n(s_n)\, p^*(s_n)\, r_0(s_n)\, L_0\, x(s_n)\, ds_n \ldots ds_2\, ds_1\,, \end{aligned}$$

where $K = L_0\, x(t_1) > 0$. Since $L_1\, x(t) > 0$ for $t \ge t_1$, then

$$\begin{aligned} L_0\, x(t) \ge K &+ \int_{t_1}^{t} r_1(s_1) \int_{t_1}^{s_1} r_2(s_2) \\ &\cdots \int_{t_1}^{s_{n-1}} r_n(s_n)\, p^*(s_n)\, r_0(s_n)\, L_0\, x(g(s_n))\, ds_n \ldots ds_2\, ds_1\,. \end{aligned}$$

From Lemma 5 due to Kusano and Naito [6] if follows that the integral equation

$$v(t) = K + \int_{t_1}^{t} r_1(s_1) \int_{t_1}^{s_1} r_2(s_2) \ldots \int_{t_1}^{s_{n-1}} r_n(s_n)\, p^*(s_n)\, r_0(s_n)\, v(g(s_n)) ds_n \ldots ds_2\, ds_1$$

admits a solution $v(t)$, $t \ge t_1$, satisfying

$$K \le v(t) \le L_0\, x(t)\,, \quad t \ge t_1\,.$$

Hence $v(t) > 0$ for $t \ge t_1$. Setting $y(t) = r_0(t)\, v(t)$ we obtain $y(t) > 0$ for $t \ge t_1$ and

$$L_0\, y(t) = K + \int_{t_1}^{t} r_1(s_1) \int_{t_1}^{s_1} r_2(s_2) \cdots \int_{t_1}^{s_{n-1}} r_n(s_n)\, p(s_n)\, y(g(s_n))\, ds_n \ldots ds_2\, ds_1\,.$$

Successive differentiation shows that $y(t)$ is a positive solution of (1.4) satisfying $L_i\, y(t) > 0$ for $t \ge t_1$, $0 \le i \le n$. Hence the theorem is proved. □

Theorem 3.5. *Suppose that $g \in C^1([\sigma, \infty), R)$ such that $g'(t) > 0$. If the differential equation*

$$L_n x - \frac{p(g^{-1}(t)) r_n(g^{-1}(t))}{r_n(t) g'(g^{-1}(t))}\, x = 0 \tag{3.9}$$

has property (B), *then Eq.* (1.4) *has property* (B).

Proof. Let $y(t)$ be a nonoscillatory solution of (1.4). It is sufficient to show, in view of Lemma 1.2, that $\ell = 0$ or n for n even and $\ell = n$ for n odd. If possible, let $\ell \in \{1, 2, \ldots, n-2\}$. Then proceeding as in the proof of Theorem 2.7 one may show that (3.9) admits a solution $x \in \bar{N}_\ell$, which contradicts the assumption that (3.9) has property (B). Thus the theorem is proved. □

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Department of Mathematics, Berhampur University
Berhampur - 760 007, INDIA