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SOME STABLE OPERATOR IDEALS

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Abstract.

Introduction

For Banach spaces X and Y, let $L(X,Y)$ denote the space of bounded linear operators from X into Y. Let $L = UL(X,Y)$, where the union runs over all Banach spaces X and Y. Let Π be a subclass of L. The set $\Pi(X,Y) = \Pi \cap L(X,Y)$ is called a component of Π. Following Pietch [5], a subclass $\Pi \subseteq L$ is called an operator ideal if:

(i) Each component $\Pi(X,Y)$ is a vector space that contains the finite rank elements of $L(X,Y)$.
(ii) For all Banach spaces E, X, Y, and F, we have $L(Y,F) \circ \Pi(X,Y) \circ L(E,X) \subseteq \Pi(E,F)$.

If X and Y are Banach spaces then $X \overset{\vee}{\otimes} Y$ denotes the completion of the injective tensor product of X with Y. For $T_i \in L(E_i, F_i)$ we let $T_1 \overset{\vee}{\otimes} T_2$ denote the tensor product map of T_1 and T_2. An operator ideal Π is called stable if whenever $T_i \in \Pi(E_i, F_i)$, $i = 1, 2$, then $T_1 \overset{\vee}{\otimes} T_2 \in \Pi\left(E_1 \overset{\vee}{\otimes} E_2, F_1 \overset{\vee}{\otimes} F_2\right)$. We refer to [4] and [5] for more on tensor product of Banach spaces and tensor product of maps.

In [1] and [2] Holub proved that the ideals of p-summing operators, 1-nuclear operators and 1-integral operators are stable. It is the object of this paper to discuss the stability of other operator ideals. Indeed, we prove that r-nuclear operators and r-integral operators are stable. Further, we prove that if an operator ideal Π is stable, then the injective hull of Π is stable. This gives another proof for the stability of p-summing operators. Some other results are presented.

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1. Preliminaries and Notations

Let X and Y be Banach spaces and $T \in L(X,Y)$. Then:

(i) T is called p-summing operator if there exists $\lambda > 0$ such that:

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{\frac{1}{p}} \leq \lambda \sup_{\|x^*\|\leq 1} \left(\sum_{i=1}^{n} |\langle x_i, x^*\rangle|^p\right)^{\frac{1}{p}}$$

for all finite sequences $\{x_1, x_2, x_3, \ldots, x_n\} \subseteq X$.

Let $\Pi_p(X,Y)$ be the space of p-summing operators from X into Y, and Π_p be the operator ideal of p-summing operators.

(ii) T is called r-nuclear, $r \geq 1$, if T has a representation of the form

$$Tx = \sum_{i=1}^{\infty} \langle x_n^*, y^*\rangle y_n, \quad x_n^* \in X, \quad y_n \in Y, \left(\sum_{i=1}^{\infty} \|x_n^*\|^r\right)^{\frac{1}{r}} < \infty$$

and

$$\sup_{\|y^*\|\leq 1} \left(\sum_{i=1}^{n} |\langle y_n, y^*\rangle|^{r^*}\right)^{\frac{1}{r^*}} < \infty, \quad \left(\frac{1}{r} + \frac{1}{r^*} = 1\right).$$

It is well known, [5], that T is r-nuclear if and only if T has a factorization, $T = BT_0A$, where $B \in L(\ell^r, Y)$, $A \in L(X, \ell^\infty)$ and $T_0 \in L(\ell^\infty, \ell^r)$ is of the form $T_0(a_i) = (a_i\sigma_i)$, $(\sigma_i) \in \ell^\infty$.

Let $N_r(X,Y)$ denote the space of r-nuclear operators from X into Y, and N_r the operator ideal of r-nuclear operators.

(iii) T is called r-integral operator if T admits a factorization $J_Y T = BI_rA$, where $B \in L(L^r(\Omega,\mu), Y)$, $A \in L(X, C(\Omega))$, J_Y is the natural embedding of Y into Y^{**} and I_r is the inclusion map of $C(\Omega)$ into $L^r(\Omega,\mu)$ where Ω is some compact Hausdorff space, and μ a probability measure on Ω.

Let $L_r(X,Y)$ denote the space of r-integral operators from X into Y, and L_r the operator ideal of r-integral operators. We refer to Pietsch [5] for a full discussion of these ideals of operators.

An operator $T \in L(X,Y)$ is said to belong to Π^{s}, the surjective hull of the operator ideal Π, if $TQ_X \in \Pi(X^{\mathrm{sur}}, Y)$, where $X^{\mathrm{sur}} = \ell^1(B_1(X)), B_1(X)$ is the unit ball of X, and Q_X is the canonical surjection of $\ell^1(B_1(X))$ onto X, [5].

The operator T is said to belong to Π^{i}, *the injective hull of* Π, if $J_Y T \in \Pi(X, Y^{\mathrm{inj}})$, where $Y^{\mathrm{inj}} = \ell^\infty(B_1(Y^*))$, and J_Y is the canonical surjection of Y onto $\ell^\infty(B_1(Y^*))$. It is known,[5], that Π^{s} and Π^{i} are operator ideals.

2. The Stability of Some Operator Ideals

In this section we establish the stability of r-nuclear and r-integral operators, $r \geq 1$.

Theorem 2.1. *The ideal N_r is stable.*

Proof. Let $T_i \in N_r(E_i, F_i)$, $i = 1, 2$. Then T_i has a factorization

$$E_i \xrightarrow{A_i} \ell^\infty \xrightarrow{S_i} \ell^r \xrightarrow{B_i} F_i$$

where S_i is a diagonal operator, $S_i(\eta_n) = (\sigma_i(n)\eta_n)$, where $\sigma_i \in \ell^r$, $i = 1, 2$. Hence $T_1 \overset{\vee}{\otimes} T_2$ has a factorization:

$$T_1 \overset{\vee}{\otimes} T_2 = (B_1 \overset{\vee}{\otimes} B_2) \circ (S_1 \overset{\vee}{\otimes} S_2) \circ (A_1 \overset{\vee}{\otimes} A_2)\,.$$

Consider

$$J_1 : \ell^\infty \overset{\vee}{\otimes} \ell^\infty \to \ell^\infty(N \times N)$$
$$J_2 : \ell^r(N \times N) \to \ell^r \overset{\vee}{\otimes} \ell^r\,,$$

where J_1 and J_2 are the inclusion maps. Let $\ell^r \underset{\alpha_r}{\otimes} \ell^r$ denote the r-nuclear tensor product of ℓ^r with itself, [3]. It is well known that $\ell^r \underset{\alpha_r}{\otimes} \ell^r \cong \ell^r(N \times N)$, [3]. Let $S : \ell^\infty(N \times N) \to \ell^r(N \times N)$

$$S(a(n,m)) = (\sigma_1(n)\sigma_2(m)\, a(n,m))\,,$$

with $\sigma_1 \cdot \sigma_2 \in \ell^\infty(N \times N)$.

Consequently $S_1 \overset{\vee}{\otimes} S_2$ has the factorization

$$\ell^\infty \overset{\vee}{\otimes} \ell^\infty \xrightarrow{J_1} \ell^\infty(N \times N) \xrightarrow{s} \ell^r(N \times N) \xrightarrow{J_2} \ell^r \overset{\vee}{\otimes} \ell^r$$

and so $T_1 \overset{\vee}{\otimes} T_2$ has the factorization

$$T_1 \overset{\vee}{\otimes} T_2 = J_1 \circ (A_1 \overset{\vee}{\otimes} A_2) \circ S \circ (B_1 \overset{\vee}{\otimes} B_2) \circ J_2\,.$$

Since S is a diagonal operator, $T_1 \overset{\vee}{\otimes} T_2$ is r-nuclear. This ends the proof. $\square$

Theorem 2.2. *The ideal L_r is stable.*

Proof. Let $T_i \in L_r(E_i, F_i)$, $i = 1, 2$. Then $J_{F_i}T_i$ has the factorization

$$E_i \xrightarrow{A_i} C(K_i) \xrightarrow{I_r} L^r(K_i, \mu_i) \xrightarrow{B_i} F_i^{**}\,.$$

Since $C(K_1) \overset{\vee}{\otimes} C(K_2) = C(K_1 \times K_2)$, and $I_r : C(K_i) \to L^r(K_i, \mu_i)$ is just the inclusion map, it follows that

$$I_r \overset{\vee}{\otimes} I_r : C(K_1) \overset{\vee}{\otimes} C(K_2) \to L^r(K_1, \mu_1) \overset{\vee}{\otimes} L^r(K_2, \mu_2)$$

is the inclusion map of $C(K_1 \times K_2)$ into $L^r(K_1,\mu_1) \overset{\vee}{\otimes} L^r(K_2,\mu_2)$ with range in

$$L^r(K_1 \times K_2, \mu_1 \otimes \mu_2) = L^r(K_1,\mu_1) \underset{\alpha_r}{\otimes} L^r(K_2,\mu_2)\,.$$

Since $\|\cdot\|_\vee \le \|\cdot\|_{\alpha_p}$, it follows that for any $\psi \in C(K_1 \times K_2)$

$$\left\| \left(I_r \overset{\vee}{\otimes} I_r \right)(\psi) \right\|_\vee \le \left\| \left(I_r \overset{\vee}{\otimes} I_r \right)(\psi) \right\|_{\alpha_p}\,.$$

But by Theorem 17.3.3 and Proposition 17.3.8 of [5] we get by

$$\left\| \left(I_r \overset{\vee}{\otimes} I_r \right)(\psi) \right\|_\vee \le \lambda \left(\int_{K_1 \times K_2} |\psi(x,y)|^r \, d(\mu_1 \otimes \mu_2) \right)^{\frac{1}{r}}\,,$$

for some $\lambda > 0$. Hence Proposition 17.3.8 of [5],

$$I_r \overset{\vee}{\otimes} I_r \in \Pi_r \left(C(K_1 \times K_2), L^r(K_1) \overset{\vee}{\otimes} L^r(K_2) \right)\,.$$

Consequently [5], $I_r \overset{\vee}{\otimes} I_r$ has the factorization:

$$I_r \overset{\vee}{\otimes} I_r = D \circ \widetilde{I_r} : C(K_1 \times K_2) \xrightarrow{\widetilde{I_r}} L^r(K_1 \times K_2, \mu_1 \otimes \mu_2) \xrightarrow{D} L^r(K_1) \overset{\vee}{\otimes} L^r(K_2)$$

where $\widetilde{I_r}$ is the inclusion map. Hence we have

$$\begin{aligned} J \circ K_{F_1} \overset{\vee}{\otimes} K_{F_2} \circ T_1 \overset{\vee}{\otimes} T_2 &= J \circ B_1 \overset{\vee}{\otimes} B_2 \circ I_r \overset{\vee}{\otimes} I_r \circ A_1 1 \overset{\vee}{\otimes} A_2 \\ &= J \circ B_1 \overset{\vee}{\otimes} B_2 \circ D \circ \widetilde{I_r} \circ A_1 \overset{\vee}{\otimes} A_2 \end{aligned}$$

where J is *the natural inclusion* of $F_1^{**} \overset{\vee}{\otimes} F_2^{**}$ into $\left(F_1 \overset{\vee}{\otimes} F_2 \right)^{**}$. But

$$J \circ \left(K_{F_1} \overset{\vee}{\otimes} K_{F_2} \right) = K_{F_1 \overset{\vee}{\otimes} F_2}\,.$$

Hence $K_{F_1 \overset{\vee}{\otimes} F_2} \circ T_1 \overset{\vee}{\otimes} T_2$ has a factorization $K_{F_1 \overset{\vee}{\otimes} F_2} \circ T_1 \overset{\vee}{\otimes} T_2 = B_1 \overset{\vee}{\otimes} B_2 \circ D \circ \widetilde{I_r} \circ A_1 \overset{\vee}{\otimes} A_2$. Hence $T_1 \overset{\vee}{\otimes} T_2$ is an r-integral. This ends the proof. □

3. Hull Stability of Operator Ideals

In this section we prove the stability of the hulls Π^{i} and Π^{s} for any stable operator ideal Π.

Theorem 3.1. Π^{i} *is stable for any stable operator ideal* Π.

Proof. Let Π be a stable operator ideal and $T_i \in \Pi^{\mathrm{i}}(E_i, F_i)$, $i = 1, 2$. Consider

$$J_{F_1 \overset{\vee}{\otimes} F_2} \circ T_1 \overset{\vee}{\otimes} T_2 : E_1 \overset{\vee}{\otimes} E_2 \to F_1 \overset{\vee}{\otimes} F_2 \to \ell^\infty \left(B_1 \left(F_1 \overset{\vee}{\otimes} F_2 \right)^* \right) .$$

Let $J_{F_i} : F_i \to \ell^\infty(B_1(F_i^*))$ be the canonical embedding, $i = 1, 2$. Since J_{F_i} is an injection, then the operator

$$J_{F_1} \overset{\vee}{\otimes} J_{F_2} : F_1 \overset{\vee}{\otimes} F_2 \to \ell^\infty(B_1(F_1^*)) \overset{\vee}{\otimes} \ell^\infty(B_1(F_2^*))$$

is an injection. But $\ell^\infty(B_1(F_1 \overset{\vee}{\otimes} F_2)^*)$ has the metric extension property, [5]. Consequently, $J_{F_1 \overset{\vee}{\otimes} F_2}$ has the factorization

$$F_1 \overset{\vee}{\otimes} F_2 \xrightarrow{J_{F_1} \overset{\vee}{\otimes} J_{F_2}} \ell^\infty(B_1(F_1)^*) \overset{\vee}{\otimes} \ell^\infty(B_1(F_2)^*) \xrightarrow{S} \ell^\infty(B_1(F_1 \overset{\vee}{\otimes} F_2)^*)$$

for some bounded linear operator S. Hence

$$\begin{aligned} J_{F_1 \overset{\vee}{\otimes} F_2} \circ T_1 \overset{\vee}{\otimes} T_2 &= S \circ \left(J_{F_1} \overset{\vee}{\otimes} J_{F_2} \right) \circ T_1 \overset{\vee}{\otimes} T_2 \\ &= S \circ \left[(J_{F_1} \circ T_1) \overset{\vee}{\otimes} (J_{F_2} \circ T_2) \right] . \end{aligned}$$

Since Π is assumed to be stable, then $(J_{F_1} \circ T_1) \overset{\vee}{\otimes} (J_{F_2} \circ T_2) \in \Pi$. This implies that $J_{F_1 \overset{\vee}{\otimes} F_2} \circ T_1 \overset{\vee}{\otimes} T_2 \in \Pi$. This ends the proof. □

As a corollary we give a different proof for the stability of the ideal Π_r:

Corollary 3.2 (Holub, [1]). Π_r *is a stable ideal.*

Proof. By Theorem 19.2.7 of [5], we have $\Pi_r = (L_r)^{\mathrm{i}}$. Theorem 2.2 implies that L_r is stable. Hence by Theorem 3.1, Π_r is stable. This ends the proof. □

Stability with respect to the projective tensor product is defined as that with respect to the injective case. In that respect we prove:

Theorem 3.3. *Let* Π *be a stable operator ideal with respect to the projective tensor product. Then* Π^{s} *is similarly stable.*

Proof. Let $T_i \in \Pi^{\mathrm{s}}(E_i, F_i)$, $i = 1, 2$, and $Q_{E_1 \overset{\wedge}{\otimes} E_2}$ be the canonical surjection of $\ell^1(B_1(E_1 \overset{\wedge}{\otimes} E_2))$ into $E_1 \overset{\wedge}{\otimes} E_2$. Q_{E_i} is defined similarly, $i = 1, 2$. Since Q_{E_i} : $\ell^1(B_1(E_i)) \to E_i$ is a surjection, [6], then

$$Q_{E_1} \overset{\wedge}{\otimes} Q_{E_2} : \ell^1(B_1(E_1)) \overset{\wedge}{\otimes} \ell^1(B_1(E_2)) \to E_1 \overset{\wedge}{\otimes} E_2$$

is a surjection. Since ℓ^1-spaces have the lifting property [5], it follows that $Q_{E_1 \hat{\otimes} E_2}$ has a factorization

$$\ell^1(B_1(E_1 \hat{\otimes} E_2)) \xrightarrow{S} \ell^1(B_1(E_1)) \hat{\otimes} \ell^1(B_1(E_2)) \xrightarrow{Q_{E_1} \hat{\otimes} Q_{E_2}} E_1 \hat{\otimes} E_2$$

for some bounded linear operator S. Hence

$$\begin{aligned} Q_{E_1 \hat{\otimes} E_2} \circ T_1 \hat{\otimes} T_2 &= (Q_{E_1} \hat{\otimes} Q_{E_2}) \circ (T_1 \hat{\otimes} T_2) \circ S \\ &= [(Q_{E_1} \circ T_1) \hat{\otimes} (Q_{E_2} \circ T_2)] \circ S\,. \end{aligned}$$

By the assumption on Π we get

$$(Q_{E_1} \circ T_1) \hat{\otimes} (Q_{E_2} \circ T_2) \in \Pi(E_1^{\mathrm{sur}} \hat{\otimes} E_2^{\mathrm{sur}}, F_1 \hat{\otimes} F_2)\,.$$

Consequently Π^{s} is stable. This ends the proof. □

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