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NATURAL VECTOR FIELDS AND 2-VECTOR FIELDS ON THE TANGENT BUNDLE OF A PSEUDO-RIEMANNIAN MANIFOLD

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ABSTRACT. Let M be a differentiable manifold with a pseudo-Riemannian metric g and a linear symmetric connection K. We classify all natural (in the sense of [4]) 0-order vector fields and 2-vector fields on TM generated by g and K. We get that all natural vector fields are of the form

$$E(u) = \alpha(h(u))\, u^H + \beta(h(u))\, u^V ,$$

where u^V is the vertical lift of $u \in T_xM$, u^H is the horizontal lift of u with respect to K, $h(u) = 1/2g(u,u)$ and α, β are smooth real functions defined on $\mathbb{R}$. All natural 2-vector fields are of the form

$$\Lambda(u) = \gamma_1(h(u))\, \Lambda(g,K) + \gamma_2(h(u))\, u^H \wedge u^V ,$$

where γ_1, γ_2 are smooth real functions defined on $\mathbb{R}$ and $\Lambda(g,K)$ is the canonical 2-vector field induced by g and K. Conditions for (E,Λ) to define a Jacobi or a Poisson structure on TM are disscused.

INTRODUCTION

In this paper M is a differentiable manifold with a of pseudo-Riemannian metric g. Let (x^i) be a typical local chart on M, then (∂_i) and (d^i) denote the canonical local bases of modules of vector fields and forms on M. In general relativistic theories $\dim M = 4$ and g is a Lorentz metric, but it is not relevant for our purposes; our considerations are correct for non-orientable manifolds if $\dim M \geq 2$ and for orientable manifolds if $\dim M \geq 4$.

The isomorphism $T^*M \to TM$ given by the metric tensor will be as usual denoted by $^\sharp$ and its inverse by $^\flat$.

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We consider the tangent bundle $p_M : TM \to M$ of M. The natural fibred coordinates on TM are denoted by $(x^i, \dot{x}^i)$ and the canonical local bases of modules of vector fields and forms are denoted by $(\partial_i, \dot{\partial}_i)$ and $(d^i, \dot{d}^i)$.

The canonical natural function (kinetic energy) on TM will be denoted by $h(u) = \frac{1}{2} g(u, u)$, $u \in TM$.

We assume a linear symmetric connection (gravitational field) K on M.

We use the term "natural operator" in the sense of [4, 6, 11]. Namely, a natural operator is defined to be a system of local operators $A_M : C^\infty(FM) \to C^\infty(GM)$, such that $A_N(f_F^* s) = f_G^* A_M(s)$ for any section $(s : M \to FM) \in C^\infty(FM)$ and any (local) diffeomorphism $f : M \to N$, where F, G are two natural bundles, [9]. A natural operator is said to be of order r if, for all sections $s, q \in C^\infty(FM)$ and every point $x \in M$, the condition $j^r_x s = j^r_x q$ implies $A_M s(x) = A_M q(x)$. Then we have the induced natural transformation $\mathcal{A}_M : J^r FM \to GM$ such that $A_M(s) = \mathcal{A}_M(j^r s)$, for all $s \in C^\infty(FM)$. It is well known, that the correspondence between natural operators of order r and the induced natural transformations is bijective. In this paper by natural operators we mean the corresponding natural transformations. Briefly speaking, a natural operator is a fibred manifold mapping which is invariant with respect to local diffeomorphisms of the underlying manifold.

In [1, 2] we have classified all natural 2-form fields on TM generated by g and K and we have found conditions for such 2-form fields to be symplectic. The aim of this paper is to classify all natural vector fields and 2-vector fields on TM generated by g and K. Finally, in Sections 5 and 6, we recall, [3], conditions under which natural vector fields and 2-vector fields define a Poisson or a Jacobi structure on TM. Evidently the conditions for the natural nondegenerate Poisson structure have to be equivalent with conditions for the natural symplectic structure we have found in [1, 2].

All manifolds and mappings are assumed to be smooth.

1. Schouten-Nijenhuis bracket

Let $\mathcal{V}^k(M)$ $\quad (\mathcal{V}^0(M) = C^\infty(M))$ denote the space of k-vector fields on a differentiable manifold M and $\mathcal{V}(M) = (\oplus_{k=0}^n \mathcal{V}^k(M), \wedge)$ be the contravariant Grassmann algebra of M. Let us recall that the *Schouten-Nijenhuis bracket* is a (natural) bilinear map

$$(P, Q) \in \mathcal{V}^p(M) \times \mathcal{V}^q(M) \to [P, Q] \in \mathcal{V}^{p+q-1}(M)$$

satisfying the following properties:

(1) $[P, Q] = (-1)^{pq}[Q, P]$;

(2) $(-1)^{(p-1)(s-1)}[P, [Q, S]] + (-1)^{(q-1)(p-1)}[Q, [S, P]]$
$\quad + (-1)^{(s-1)(q-1)}[S, [P, Q]] = 0$;

(3) $[P, Q \wedge S] = [P, Q] \wedge S + (-1)^{pq-q} Q \wedge [P, S]$;

(4) $[X, Q] = \mathcal{L}_X Q$, where $\mathcal{L}_X$ is the Lie derivative;

for any $P \in \mathcal{V}^p(M),\ Q \in \mathcal{V}^q(M),\ S \in \mathcal{V}^s(M),\ X \in \mathcal{V}^1(M)$.

For any function $f \in \mathcal{V}^0(M) = C^\infty(M)$ the equality (3) implies

$$[P, fQ] = [P, f] \wedge Q + f[P, Q]\,,$$

where $[P, f]$ is defined by

$$[P, f](\rho_2, \dots, \rho_p) = P(df, \rho_2, \dots, \rho_p)$$

for any 1-form fields ρ_i on M.

The Schouten-Nijenhuis bracket can be characterized by

$$\begin{aligned} i([P, Q])\beta = {} & (-1)^{q(p+1)} i(P) d[i(Q)\beta] \\ & + (-1)^p i(Q) d[i(P)\beta] - i(P \wedge Q) d\beta \end{aligned} \tag{1.1}$$

for any $(p + q - 1)$-form field β.

A 2-vector field Λ defines on M a *Poisson structure*, [7, 12], if

$$[\Lambda, \Lambda] = 0\,. \tag{1.2}$$

A pair (E, Λ), where E is a vector field and Λ is a 2-vector field, defines on M a *Jacobi structure*, [8], if

$$[E, \Lambda] = 0, \quad [\Lambda, \Lambda] = 2E \wedge \Lambda\,. \tag{1.3}$$

2. Natural vector fields on TM

We can define two canonical natural vector fields on TM. The first one is the Liouville vector field $\ell(u)$, which can be considered as the *vertical lift* u^V of $u \in T_xM$. In coordinates

$$u^V = \ell(u) = \dot{x}^i \dot{\partial}_i\,. \tag{2.1}$$

A linear connection K on TM is a linear TTM-valued 1-form

$$K : TM \to T^*M \otimes TTM$$

with coordinate expression

$$K = d^i \otimes (\partial_i + K_i{}^j{}_k \dot{x}^k \dot{\partial}_j)\,, \quad K_i{}^j{}_k \in C^\infty(M)\,.$$

The space of linear connections is a natural bundle of 2nd order and we shall denote it by CM, by $C_\tau M$ we shall denote the subbundle of torsion-free linear connections on M. If we consider K as the mapping $K : TM \times_M TM \to TTM$, called the horizontal lift, we can define the *horizontal lift* of u as the vector field $u^H = K(u, u)$, i.e. in coordinates

$$u^H = \dot{x}^i \partial_i + K_r{}^i{}_s \dot{x}^r \dot{x}^s \dot{\partial}_i\,. \tag{2.2}$$

The vertical and the horizontal lifts of u are natural 0-order (with respect to the connection K) vector fields on TM, i.e. they are natural operators from $TM \times_M CM$ into TTM projectable (with respect to the projections $\mathrm{pr}_1 : TM \times_M CM \to TM$ and $p_{TM} : TTM \to TM$) over the identity of TM. As a consequence of the results of [10] we get that all 0-order natural vector fields on TM given by K are linear combination (with real coefficients) of the vertical and the horizontal lifts. In what follows we shall consider natural 0-order vector fields induced by a pseudo-Riemannian metric g and a torsion-free linear connection, i.e. we shall classify all 0-order natural operators from $TM\times_M reg\odot^2 T^*M\times_M C_\tau M$ into TTM projectable over the identity of TM.

Theorem 2.1. *Let M be a differentiable manifold with a pseudo-Riemannian metric g and a linear symmetric connection K. Then all natural 0-order vector fields on TM are of the form*

$$E(u)=\alpha(h(u))\,u^H+\beta(h(u))\,u^V\,, \tag{2.3}$$

where α,β are smooth real functions defined on $\mathbb{R}$.

Proof. According to the general theory of natural operators, [4, 6], all natural 0-order operators from $TM\times_M reg\odot^2 T^*M\times_M C_\tau M$ into TTM are given by invariant mappings from the standard fibre Q of the functor $T\times reg\odot^2 T^*\times C_\tau$ into the standard fibre S of the functor TT. To classify these invariant mappings we shall use the infinitesimal method of [6].

The standard fibre $Q=\mathbb{R}^n\times reg(\mathbb{R}^{*n}\odot\mathbb{R}^{*n})\times\mathbb{R}^n\otimes\mathbb{R}^{*n}\odot\mathbb{R}^{*n}$ with coordinates $(\dot x^i, g_{ij}, K_i{}^j{}_k)$, $g_{ij}=g_{ji}$, $\det(g_{ij})\neq 0$, $K_j{}^i{}_k=K_k{}^i{}_j$, and the action of the 2nd order differential group $G^2_n=invJ^2_0(\mathbb{R}^n,\mathbb{R}^n)_0$ given in coordinates by

$$\bar{\dot x}^i=a^i_p\dot x^p\,,\qquad \bar g_{ij}=\tilde a^p_i\tilde a^q_j g_{pq}\,,\qquad \bar K_i{}^k{}_j=a^k_r K_p{}^r{}_q\tilde a^p_i\tilde a^q_j-a^k_r\tilde a^r_{ij}\,, \tag{2.4}$$

where (a^i_j, a^i_{jk}) are the canonical coordinates on G^2_n and tilde denotes the inverse element.

The fundamental vector fields on Q relative to this action are

$$\zeta^q_p(Q)=\dot x^q\frac{\partial}{\partial\dot x^p}-2g_{ip}\frac{\partial}{\partial g_{iq}}+(\delta^i_p K^q_{j\,k}-\delta^q_j K_p{}^i{}_k-\delta^q_k K_j{}^i{}_p)\frac{\partial}{\partial K_j{}^i{}_k}\,, \tag{2.5}$$

$$\zeta^{qr}_p(Q)=\frac12\left(\frac{\partial}{\partial K_q{}^p{}_r}+\frac{\partial}{\partial K_r{}^p{}_q}\right)=\frac{\partial}{\partial K_q{}^p{}_r}\,. \tag{2.6}$$

The standard fibre $S=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ with coordinates $(\dot x^i,\xi^i,\Xi^i)$ and the action of the group G^2_n given in coordinates by

$$\bar{\dot x}^i=a^i_p\dot x^p\,,\qquad \bar\xi^i=a^i_p\xi^p\,,\qquad \bar\Xi^i=a^i_p\Xi^p+a^i_{pq}\dot x^p\xi^q\,. \tag{2.7}$$

The fundamental vector fields on S relative to this action are

$$\zeta^q_p(S)=\dot x^q\frac{\partial}{\partial\dot x^p}+\xi^q\frac{\partial}{\partial\xi^p}+\Xi^q\frac{\partial}{\partial\Xi^p}\,, \tag{2.8}$$

$$\zeta^{qr}_p(S)=\frac12(\xi^q\dot x^r+\dot x^q\xi^r)\frac{\partial}{\partial\Xi^p}\,. \tag{2.9}$$

A mapping $F:Q\to S$ is G^2_n-invariant if and only if the corresponding fundamental vector fields are F-related. Let $F=(\dot x^i(\dot x^r,g_{pq},K_p{}^r{}_q),\xi^i(\dot x^r,g_{pq},K_p{}^r{}_q),\Xi^i(\dot x^r,g_{pq},K_p{}^r{}_q))$ be a coordinate expression of F. The assumption that the operator is projectable over the identity of TM implies $\dot x^i=\dot x^i$. Further

$$\dot x^q\frac{\partial\xi^i}{\partial\dot x^p}-2g_{ip}\frac{\partial\xi^i}{\partial g_{iq}}+(\delta^i_p K_j{}^q{}_k-\delta^q_j K_p{}^i{}_k-\delta^q_k K_j{}^i{}_p)\frac{\partial\xi^i}{\partial K_j{}^i{}_k}=\delta^i_p\xi^q\,, \tag{2.10}$$

$$\dot x^q\frac{\partial\Xi^i}{\partial\dot x^p}-2g_{ip}\frac{\partial\Xi^i}{\partial g_{iq}}+(\delta^i_p K_j{}^q{}_k-\delta^q_j K_p{}^i{}_k-\delta^q_k K_j{}^i{}_p)\frac{\partial\Xi^i}{\partial K_j{}^i{}_k}=\delta^i_p\Xi^q\,, \tag{2.11}$$

and

$$\frac{\partial \xi^i}{\partial K_q{}^p{}_r} = 0 \,, \tag{2.12}$$

$$\frac{\partial \Xi^i}{\partial K_q{}^p{}_r} = \frac{1}{2}\delta^i_p(\xi^q \dot{x}^r + \dot{x}^q \xi^r) \,. \tag{2.13}$$

Putting (2.12) into (2.10) we get

$$\dot{x}^q \frac{\partial \xi^i}{\partial \dot{x}^p} - 2 g_{ip} \frac{\partial \xi^i}{\partial g_{iq}} = \delta^i_p \xi^q \,, \tag{2.14}$$

which implies, that ξ^i is a G^1_n-invariant mapping from $\mathbb{R}^n \times reg(\mathbb{R}^{*n} \odot \mathbb{R}^{*n})$ into $\mathbb{R}^n$ which is the G^1_n-invariant mapping corresponding to 0-order natural operator from $TM \times reg \odot^2 T^*M$ to TM. From the classical theory of differential invariants it is known that all such mappings are of the form

$$\xi^i = \alpha(h(u))\, \dot{x}^i \,, \tag{2.15}$$

where α is a smooth real function defined on $\mathbb{R}$.

The right hand side of (2.13) we can rewrite in the form

$$\frac{1}{2}\delta^i_p(\xi^q \dot{x}^r + \dot{x}^q \xi^r) = \frac{1}{2}\delta^i_p(\delta^q_m \delta^r_l + \delta^q_l \delta^r_m)\xi^m \dot{x}^l = \frac{\partial K_m{}^i{}_l}{\partial K_q{}^p{}_r}\xi^m \dot{x}^l \,,$$

which implies that

$$\Xi^i = K_p{}^i{}_q \xi^p \dot{x}^q + Z^i \,, \tag{2.16}$$

where Z^i is a function independent on $K_j{}^i{}_k$. Putting (2.16) into (2.11) we get that Z^i has to satisfy (2.14), i.e.

$$Z^i = \beta(h(u))\, \dot{x}^i \,, \tag{2.17}$$

where β is a smooth real function defined on $\mathbb{R}$.

Now, putting (2.15) and (2.17) into (2.16), we have

$$\xi^i = \alpha(h(u))\, \dot{x}^i \,, \qquad \Xi^i = \alpha(h(u))\, K_p{}^i{}_q \dot{x}^p \dot{x}^q + \beta(h(u))\, \dot{x}^i \,, \tag{2.18}$$

and it is easy to see that the natural operator corresponding to (2.18) is the vector field (2.3). □

Lemma 2.2. *We have:*

(1) $[u^V, u^H] = u^H$; (2) $[u^V, u^V \wedge u^H] = u^V \wedge u^H$;

(3) $[u^H, u^V \wedge u^H] = 0$; (4) $[u^V \wedge u^H, u^V \wedge u^H] = 0$.

Proof. It is easy to prove the first equality by direct coordinate calculation by using (2.1) and (2.2). The others equalities follows from the properties of the Schouten-Nijenhuis bracket. □

Lemma 2.3. *Let γ be a smooth real function defined on $\mathbb{R}$. Then we have*

$$[u^V, \gamma(h(u))] = 2h(u)\dot{\gamma}(h(u)) \,, \tag{2.19}$$

$$[u^H, \gamma(h(u))] = \frac{1}{2}\dot{\gamma}(h(u))(\nabla g)(u,u)(u) \,, \tag{2.20}$$

where $\dot{\gamma} = \frac{d\gamma}{dt}$ *and* ∇ *is the covariant differentiation with respect to* K*.*

Proof. For any vector field E we have $[E, \gamma(h)] = \langle E, d\gamma(h)\rangle = \dot{\gamma}(h)\langle E, dh\rangle$. Lemma 2.3 now follows from (2.1), (2.2) and $dh = \frac{1}{2} g_{pq,i}\dot{x}^p\dot{x}^q d^i + g_{pi}\dot{x}^p \dot{d}^i$. $\square$

3. Canonical 2-vector field generated by g and K

Let us denote by ϑ the VTM-valued 1-form on M given by the vertical lift, i.e.

$$\vartheta : M \to T^*M \otimes VTM, \qquad \vartheta = d^i \otimes \dot{\partial}_i \,.$$

The metric g and the connection K induce naturally a 2-vector

$$\Lambda(g, K) = K \bar{\wedge} \vartheta : TM \to \wedge^2 T(TM)\,, \tag{3.1}$$

where $\bar{\wedge}$ denotes the wedge product followed by the contraction through the inverse metric $\tilde{g}$. In coordinates

$$\Lambda(g, K) = g^{ij}(\partial_i + K_i{}^m{}_k \dot{x}^k \dot{\partial}_m) \wedge \dot{\partial}_j \,. \tag{3.2}$$

Remark 3.1. The canonical 2-vector field $\Lambda(g, K)$ can be characterized by

$$\Lambda(g, K)(\rho^V, \sigma^V) = 0\,, \quad \Lambda(g, K)(\rho^V, \sigma^H) = \tilde{g}(\rho, \sigma)\,,$$
$$\Lambda(g, K)(\rho^H, \sigma^V) = -\tilde{g}(\rho, \sigma)\,, \quad \Lambda(g, K)(\rho^H, \sigma^H) = 0\,,$$

where ρ, σ are 1-form fields on M, ρ^V, σ^V are their vertical lifts (pullbacks) and ρ^H, σ^H are horizontal lifts with respect to K.

The metric g can be considered to be a T^*M-valued 1-form on M which will be denoted by $\bar{g}$ to distinguish it from the metric. Then, [4], we define the *covariant exterior differential* of $\bar{g}$ as a T^*M-valued 2-form field $d_K\bar{g}$ defined for any vector fields X_1, X_2, X_3 by

$$d_K\bar{g}(X_1, X_2)(X_3) = \big(\nabla_{X_1}\bar{g}(X_2) - \nabla_{X_2}\bar{g}(X_1) - \bar{g}([X_1, X_2])\big)(X_3)\,. \tag{3.3}$$

Then

$$d_K^2\bar{g} = R \wedge \bar{g}\,,$$

where $R = R^i{}_{jkr}\dot{x}^r\, \partial_i \otimes d^j \wedge d^k$ is the curvature tensor field of K. The T^*M-valued 3-form $R \wedge \bar{g}$ is defined by

$$(R \wedge \bar{g})(X_1, X_2, X_3)(X_4) = \frac{1}{2!}\sum_{\sigma} |\sigma| g(X_{\sigma(3)}, R(X_{\sigma(1)}, X_{\sigma(2)})(X_4))\,,$$

where σ is a permutation and $|\sigma|$ is its sign. In coordinates

$$d_K^2\bar{g} = R^p{}_{ijr} g_{pk}\, d^r \otimes d^i \wedge d^j \wedge d^k \,. \tag{3.4}$$

Lemma 3.2. *Let* X_1, X_2, X_3 *be three vector fields on* M*. Then*

$$d_K\bar{g}(X_1, X_2)(X_3) = 0 \tag{A}$$

if and only if

$$(\nabla_{X_1} g)(X_2, X_3) = (\nabla_{X_2} g)(X_1, X_3)\,,$$

i.e. if and only if ∇g *is a section* $\nabla g : M \to \odot^3 T^*M$*.*

Proof. We have

$$\nabla_{X_1} X_2 - \nabla_{X_2} X_1 = [X_1, X_2]$$

and

$$\nabla_{X_1} g(X_2, X_3) - g(X_2, \nabla_{X_1} X_3) = \nabla_{X_1} \bar{g}(X_2)(X_3) .$$

Then

$$(\nabla_{X_1} g)(X_2, X_3) - (\nabla_{X_2} g)(X_1, X_3) = d_K \bar{g}(X_1, X_2)(X_3)$$

implies Lemma 3.2. □

Corollary 3.3. *The condition* (A) *is satisfied if and only if*

$$\begin{aligned} X_1 g(X_2, X_3) &- X_2 g(X_1, X_3) \\ &= g(X_2, \nabla_{X_1} X_3) - g(X_1, \nabla_{X_2} X_3) + g([X_1, X_2], X_3) \end{aligned}$$

for any three vector fields X_1, X_2, X_3 on M.

Theorem 3.4. *$\Lambda(g, K)$ defines a Poisson structure if and only if the condition* (A) *is satisfied.*

Proof. We have from (1.1)

$$\begin{aligned} [\Lambda(g, K), \Lambda(g, K)] &= g^{ip} g^{rj} g^{mk} (g_{pm,r} + g_{ms} K_p{}^s{}_r) \partial_i \wedge \dot{\partial}_j \wedge \dot{\partial}_k \\ &\quad + (R^{ijk}{}_r + R^{jki}{}_r + R^{kij}{}_r) \dot{x}^r \dot{\partial}_i \wedge \dot{\partial}_j \wedge \dot{\partial}_k . \end{aligned} \tag{3.5}$$

On the other hand

$$d_K \bar{g} = (g_{ij,k} + g_{js} K_i{}^s{}_k) d^i \otimes d^j \wedge d^k .$$

The condition $d_K \bar{g} = 0$ imply $d_K^2 \bar{g} = 0$ which is from (3.4) equivalent to

$$R_{ijkr} + R_{jkir} + R_{kijr} = 0 , \tag{3.6}$$

i.e. by increasing indices

$$R^{ijk}{}_r + R^{jki}{}_r + R^{kij}{}_r = 0 ,$$

which implies Theorem 3.4. □

Lemma 3.5. *Let γ be a smooth real function defined on $\mathbb{R}$. Then we have:*
(1) $[\Lambda(g, K), \gamma(h(u))] = \dot{\gamma}(h(u)) \big(- u^H + \frac{1}{2} (((\nabla g)(u, u))^\sharp)^V \big)$,
(2) $[\Lambda(g, K), u^V] = -\Lambda(g, K)$.

Proof. It is easy to prove it in coordinates. □

Lemma 3.6.

$$(((\nabla g)(u, u))^\sharp)^V \wedge u^H = 0$$

if and only if

$$\nabla g = 0 .$$

Proof. In coordinates we have

$$(((\nabla g)(u, u))^\sharp)^V \wedge u^H = (\nabla_m g_{rs}) \dot{x}^r \dot{x}^s g^{mi} \dot{\partial}_i \wedge (\dot{x}^j \partial_j + K_p{}^j{}_q \dot{x}^p \dot{x}^q \dot{\partial}_j) .$$

which vanishes if and only if $\nabla_m g_{rs} = 0$. □

Lemma 3.7. *The following relations are equivalent:*

(B) $g(X_1, X_3)(\nabla_{X_2} g)(X_4, X_5) = g(X_2, X_3)(\nabla_{X_1} g)(X_4, X_5)$ *for any vector fields* X_i, $i = 1, \dots, 5$;

(1) $(((\nabla g)(u,u))^\sharp)^V \wedge \Lambda(g, K) = 0$;

(2) $(((\nabla g)(u,u))^\sharp)^V \wedge u^V = 0$.

Proof. The condition (B) in coordinates reads as

$$g_{ij} \nabla_k g_{rs} - g_{kj} \nabla_i g_{rs} = 0 .$$

(B) $\Leftrightarrow$ (1) In coordinates we have

$$(((\nabla g)(u,u))^\sharp)^V \wedge \Lambda(g, K) \\ = -g^{pk} g^{mj} (\nabla_m g_{rs}) \dot x^r \dot x^s (\delta^i_p \partial_i \wedge \dot\partial_j \wedge \dot\partial_k + K_p{}^i{}_q \dot x^q \dot\partial_i \wedge \dot\partial_j \wedge \dot\partial_k) .$$

Now

$$\begin{aligned}(g^{pk} g^{mj} - g^{pj} g^{mk})(\nabla_m g_{rs}) &= (g^{ak} g^{mj} g_{ab} g^{bp} - g^{aj} g^{mk} g_{ab} g^{bp})(\nabla_m g_{rs}) \\ &= g^{ak} g^{mj} g^{bp} (g_{ab} \nabla_m g_{rs} - g_{mb} \nabla_a g_{rs})\end{aligned}$$

which implies the first equivalence.

(B) $\Leftrightarrow$ (2) In coordinates we have

$$(((\nabla g)(u,u))^\sharp)^V \wedge u^V = g^{mi} \dot x^j (\nabla_m g_{rs}) \dot x^r \dot x^s \dot\partial_i \wedge \dot\partial_j .$$

The equivalence now follows from

$$(g^{mj} \dot x^k - g^{mk} \dot x^j)(\nabla_m g_{rs}) = g^{jm} g^{kp} \dot x^q (g_{pq} \nabla_m g_{rs} - g_{mq} \nabla_p g_{rs}) . \qquad \square$$

Lemma 3.8. *We have*

$$[\Lambda(g, K), u^H] = 0 \Longleftrightarrow \nabla g = 0 .$$

Proof. In coordinates

$$[\Lambda(g, K), u^H] = \dot x^r \nabla_r g^{ij} \partial_i \wedge \dot\partial_j \\ + \big(R^i{}_r{}^j{}_s + K_m{}^i{}_r \nabla_s g^{mj}\big) \dot x^r \dot x^s \dot\partial_i \wedge \dot\partial_j . \tag{3.7}$$

If K is the metric connection, i.e. $\nabla g = 0$ then $g(R(X_1, X_2)(X_3), X_3) = 0$ which in coordinates reads as

$$R_{rijs} \dot x^r \dot x^s = 0 .$$

Then, by using the cyclic permutations of the first three indices, we have

$$0 = R_{rijs} \dot x^r \dot x^s = (R_{irjs} - R_{jris}) \dot x^r \dot x^s$$

which, by increasing indices, implies $(R^i{}_r{}^j{}_s - R^j{}_r{}^i{}_s) \dot x^r \dot x^s = 0$. $\square$

Lemma 3.9. *Let* $X_i, i = 1, \dots, 5$, *be vector fields on* M. *Then the conditions* (A) *and* (B) *imply*

$$\nabla(g \otimes \tilde g) = 0 . \tag{3.8}$$

and

$$\sum g(X_1, X_4) g(R(X_2, X_3)(X_4), X_4) = 0 , \tag{3.9}$$

where $\sum$ is taken over the cyclic permutation of X_1, X_2, X_3.

Proof. For any vector fields $X_1, \dots, X_5$ we have

$$\begin{aligned}\nabla_{X_5}(g\otimes\tilde g)(X_1,X_2,X_3^\flat,X_4^\flat) &= \tilde g(X_3^\flat,X_4^\flat)\nabla_{X_5}g(X_1,X_2)+g(X_1,X_2)\nabla_{X_5}\tilde g(X_3^\flat,X_4^\flat)\\ &= g(X_3,X_4)\nabla_{X_5}g(X_1,X_2)-g(X_1,X_2)\nabla_{X_5}g(X_3,X_4)\end{aligned}$$

which follows from

$$\nabla_{X_5}\tilde g(X_3^\flat,X_4^\flat) = -\nabla_{X_5}g(X_3,X_4)\,.$$

(3.8) now follows from the fact that the conditions (A) and (B) imply that $g\otimes\nabla g$ is a symmetric tensor, i.e. $g\otimes\nabla g : M\to\odot^5T^*M$.

To prove (3.9) we consider the covariant derivative of the equation

$$g(X_1,X_4)(\nabla_{X_2}g)(X_4,X_4)-g(X_2,X_4)(\nabla_{X_1}g)(X_4,X_4)=0$$

with respect to X_3. Then taking the sum with respect to the cyclic permutation of X_1, X_2, X_3, (A), (B) and

$$(\nabla^2g)(X_4,X_4,X_1,X_2)-(\nabla^2g)(X_4,X_4,X_2,X_1)=-2g(R(X_1,X_2)(X_4),X_4)$$

we get (3.9). □

Lemma 3.10. *The conditions* (A) *and* (B) *imply*

$$(\nabla g)(u,u)(u)u^V\wedge\Lambda(g,K)=0 \tag{3.10}$$

and

$$[\Lambda(g,K),u^H]\wedge u^V=0\,. \tag{3.11}$$

Proof. In coordinates (3.8) reads as

$$g^{pq}\nabla_m g_{rs} = -g_{rs}\nabla_m g^{pq}\,.$$

Then we have

$$\begin{aligned}&(\nabla g)(u,u)(u)u^V\wedge\Lambda(g,K)\\ &\qquad=(\nabla_m g_{rs})\dot x^r\dot x^s\dot x^m\dot x^j g^{qk}(\delta^i_q\partial_i\wedge\dot\partial_j\wedge\dot\partial_k+K_p{}^i{}_q\dot x^p\dot\partial_i\wedge\dot\partial_j\wedge\dot\partial_k)\end{aligned}$$

which vanishes if and only if

$$\begin{aligned}0=\dot x^r\dot x^s\dot x^m(\dot x^jg^{qk}-\dot x^kg^{qj})\nabla_mg_{rs}&=\dot x^r\dot x^s\dot x^m(\dot x^kg_{rs}\nabla_mg^{qj}-\dot x^jg_{rs}\nabla_mg^{qk})\\ &=2h(u)\dot x^m(\dot x^k\nabla_mg^{jq}-\dot x^j\nabla_mg^{kq})\,.\end{aligned}$$

But

$$\begin{aligned}\dot x^m(\dot x^k\nabla_mg^{qj}-\dot x^j\nabla_mg^{qk})&=\dot x^m\dot x^s(g^{jp}g^{qk}\nabla_mg_{ps}-g^{kp}g^{qj}\nabla_mg_{ps})\\ &=\dot x^m\dot x^sg^{jp}g^{qu}g^{tk}(g_{ut}\nabla_mg_{ps}-g_{pu}\nabla_mg_{ts})\\ &=\dot x^m\dot x^sg^{jp}g^{qu}g^{tk}(g_{ut}\nabla_pg_{rs}-g_{pu}\nabla_tg_{rs})\end{aligned} \tag{3.12}$$

which proves the first part of Lemma 3.10.

In coordinates

$$\begin{aligned}[\Lambda(g,K),u^H]\wedge u^V&=\dot x^r\dot x^k\nabla_rg^{mj}(\delta^i_m\partial_i\wedge\dot\partial_j\wedge\dot\partial_k+K_m{}^i{}_s\dot x^s\dot\partial_i\wedge\dot\partial_j\wedge\dot\partial_k)\\ &\quad+R^i{}_r{}^j{}_s\dot x^r\dot x^s\dot x^k\dot\partial_i\wedge\dot\partial_j\wedge\dot\partial_k\,.\end{aligned}$$

The first term on the right hand side vanishes because of (3.12). The second term vanishes because of (3.9). Really in coordinates (3.9) reads as

$$\dot{x}^r \dot{x}^s \dot{x}^t (g_{ir} R_{sjkt} + g_{jr} R_{skit} + g_{kr} R_{sijt}) = 0$$

which, by using (3.6) and increasing indices, implies

$$\dot{x}^r \dot{x}^s (\dot{x}^i R^j{}_r{}^k{}_s - \dot{x}^i R^k{}_r{}^j{}_s + \dot{x}^j R^k{}_r{}^i{}_s - \dot{x}^j R^i{}_r{}^k{}_s + \dot{x}^k R^i{}_r{}^j{}_s - \dot{x}^k R^j{}_r{}^i{}_s) = 0 \,. \quad \square$$

4. Classification of natural 2-vector fields generated by g and K

Any 2-vector field on TM has coordinate expression $\Lambda = \Lambda_1^{ij} \partial_i \wedge \partial_j + \Lambda_2^{ij} \partial_i \wedge \dot{\partial}_j + \Lambda_3^{ij} \dot{\partial}_i \wedge \dot{\partial}_j$, $\Lambda_1^{ij} = -\Lambda_1^{ji}, \Lambda_3^{ij} = -\Lambda_3^{ji}$. The canonical 2-vector field $\Lambda(g,K)$ is given by $\Lambda_1^{ij} = 0$, $\Lambda_2^{ij} = g^{ij}$ and $\Lambda_3^{ij} = (g^{mj} K_m{}^i{}_r - g^{mi} K_m{}^j{}_r)\dot{x}^r$. $\Lambda(g,K)$ is a natural 0-order operator from $TM \times_M reg \odot^2 T^*M \times_M C_\tau M$ into $\wedge^2 T(TM)$ projectable over the identity of TM. In this section we shall classify all operators of this type.

Let us recall the standard fibre Q of the functor $T \times reg \odot^2 T^* \times C_\tau$ with the action of the group G_n^2 on Q given by (2.4) and the fundamental vector fields (2.5) and (2.6).

Let us denote by $S = \mathbb{R}^n \times \wedge^2 \mathbb{R}^{2n*}$ the standard fibre of $\wedge^2 T(T)$ with the canonical coordinates $(\dot{x}^i, \begin{pmatrix} \Lambda_1^{ij} & \Lambda_2^{ij} \\ -\Lambda_2^{ji} & \Lambda_3^{ij} \end{pmatrix})$, $\Lambda_1^{ij} = -\Lambda_1^{ji}$, $\Lambda_3^{ij} = -\Lambda_3^{ji}$. The action of G_n^2 on S is given by

$$\begin{aligned} \text{(4.1)} \qquad \bar{\dot{x}}^i &= a^i_p \dot{x}^p, \qquad \bar{\Lambda}_1^{ij} = a^i_p a^j_q \Lambda_1^{pq}, \qquad \bar{\Lambda}_2^{ij} = a^i_p a^j_q \Lambda_2^{pq} + a^i_p a^j_{rq} \dot{x}^r \Lambda_1^{pq}, \\ \bar{\Lambda}_3^{ij} &= a^i_p a^j_q \Lambda_3^{pq} + (a^i_{pr} a^j_q - a^j_{pr} a^i_q)\dot{x}^r \Lambda_2^{pq} + a^i_{pr} a^j_{qs} \dot{x}^r \dot{x}^s \Lambda_1^{pq} \,. \end{aligned}$$

The fundamental vector fields on S relative to this action are

$$\text{(4.2)} \qquad \xi_p^q(S) = \dot{x}^q \frac{\partial}{\partial \dot{x}^p} + 2\Lambda_1^{qi} \frac{\partial}{\partial \Lambda_1^{pi}} + \Lambda_2^{iq} \frac{\partial}{\partial \Lambda_2^{ip}} + \Lambda_2^{qi} \frac{\partial}{\partial \Lambda_2^{pi}} + 2\Lambda_3^{qi} \frac{\partial}{\partial \Lambda_3^{pi}},$$

$$\text{(4.3)} \qquad 2\xi_p^{qr}(S) = \Lambda_1^{ir} \dot{x}^q \frac{\partial}{\partial \Lambda_2^{ip}} + \Lambda_1^{iq} \dot{x}^r \frac{\partial}{\partial \Lambda_2^{ip}} + 2\big(\Lambda_2^{qj} \dot{x}^r + \Lambda_2^{rj} \dot{x}^q\big) \frac{\partial}{\partial \Lambda_3^{pj}} \,.$$

First we shall prove

Theorem 4.1. *All G_n^2-equivariant mappings $F : Q \to S$ (over the identity of $\mathbb{R}^n$) are given by*

$$\text{(4.4)} \qquad \begin{aligned} F_1^{ij} &= \gamma_1^{ij} \,, \\ F_2^{ij} &= \gamma_1^{ir} K_r{}^j{}_q \dot{x}^q + \gamma_2^{ij} \,, \\ F_3^{ij} &= \gamma_1^{pq} K_p{}^i{}_k K_q{}^j{}_m \dot{x}^k \dot{x}^m + (\gamma_2^{qi} K_q{}^j{}_k - \gamma_2^{qj} K_q{}^i{}_k)\dot{x}^k + \gamma_3^{ij} \,, \end{aligned}$$

where γ_α^{ij} *are functions on* Q *which are solutions of the following system of partial differential equations*

$$\frac{\partial\tilde{\zeta}^{ij}}{\partial K_p{}^r{}_q} = 0\,, \tag{4.5}$$

$$\dot{x}^q\frac{\partial\tilde{\zeta}^{ij}}{\partial\dot{x}^p} - 2g_{rp}\frac{\partial\tilde{\zeta}^{ij}}{\partial g_{rq}} = \tilde{\zeta}^{iq}\delta_p^j + \tilde{\zeta}^{qj}\delta_p^i\,, \tag{4.6}$$

Moreover, $\gamma_1^{ij} = -\gamma_1^{ji}, \gamma_3^{ij} = -\gamma_3^{ji}$.

Proof. A mapping $F : Q \to S$ is G_n^2-equivariant if and only if the corresponding fundamental vector fields (2.5), (2.6) and (4.2), (4.3) are F-related. If F has the coordinate expression

$$\dot{x}^i = \dot{x}^i, \quad \Lambda_\alpha^{ij} = F_\alpha^{ij}(\dot{x}^p, g_{pq}, K_q{}^p{}_r), \quad \alpha = 1, 2, 3,$$

then F_α^{ij} have to satisfy the following system of partial differential equations

$$\begin{aligned}\dot{x}^q\frac{\partial F_\alpha^{ij}}{\partial\dot{x}^p} - 2g_{rp}\frac{\partial F_\alpha^{ij}}{\partial g_{rq}} + (\delta_p^r K_s{}^q{}_t - \delta_s^q K_p{}^r{}_t - \delta_t^q K_s{}^r{}_p)\frac{\partial F_\alpha^{ij}}{\partial K_s{}^r{}_t}& \\ = F_\alpha^{iq}\delta_p^j + F_\alpha^{qj}\delta_p^i\,,&\end{aligned} \tag{4.7}$$

$$\frac{\partial F_1^{ij}}{\partial K_q{}^p{}_r} = 0\,, \tag{4.8}$$

$$\frac{\partial F_2^{ij}}{\partial K_q{}^p{}_r} = \frac{1}{2}\big(F_1^{ir}\dot{x}^q\delta_p^j + F_1^{iq}\dot{x}^r\delta_p^j\big)\,, \tag{4.9}$$

$$\frac{\partial F_3^{ij}}{\partial K_q{}^p{}_r} = \frac{1}{2}\big(F_2^{qj}\dot{x}^r\delta_p^i + F_2^{rj}\dot{x}^q\delta_p^i - F_2^{qi}\dot{x}^r\delta_p^j - F_2^{ri}\dot{x}^q\delta_p^j\big)\,, \tag{4.10}$$

Now we have to prove that all solutions of (4.7) – (4.10) are of the form (4.4) where (4.5) and (4.6) are satisfied.

Substituting (4.8) into (4.7), where $\alpha = 1$, we get that

$$F_1^{ij} = \gamma_1^{ij}, \quad \gamma_1^{ij} = -\gamma_1^{ji}\,, \tag{4.11}$$

and γ_1^{ij} satisfies (4.5) and (4.6).

Let us rewrite (4.9) in the form

$$\frac{\partial F_2^{ij}}{\partial K_q{}^p{}_r} = \frac{1}{2}F_1^{ib}\dot{x}^a\delta_p^j(\delta_b^q\delta_a^r + \delta_b^r\delta_a^q)\,,$$

i.e. in the form

$$\frac{\partial F_2^{ij}}{\partial K_q{}^p{}_r} = F_1^{ib}\dot{x}^a\frac{\partial K_a{}^j{}_b}{\partial K_q{}^p{}_r}\,.$$

Integrating this formula and substituting (4.11) we get

$$F_2^{ij} = \gamma_1^{ir}K_r{}^j{}_s\dot{x}^s + \gamma_2^{ij}\,. \tag{4.12}$$

It is easy to see that γ_2^{ij} satisfies (4.5) and (4.6).

Finally (4.10) can be rewritten in the form

$$\frac{\partial F_3^{ij}}{\partial K_q{}^p{}_r} = \frac{1}{2} F_1^{aj}\dot{x}^b\delta^i_p(\delta^q_b\delta^r_a + \delta^r_b\delta^q_a) - \frac{1}{2} F_1^{ai}\dot{x}^b\delta^j_p(\delta^q_b\delta^r_a + \delta^r_b\delta^q_a)\,,$$

i.e. in the form

$$\frac{\partial F_3^{ij}}{\partial K_q{}^p{}_r} = F_2^{aj}\dot{x}^b\frac{\partial K_a{}^i{}_b}{\partial K_q{}^p{}_r} - F_2^{ai}\dot{x}^b\frac{\partial K_a{}^j{}_b}{\partial K_q{}^p{}_r}\,.$$

Integrating this formula and substituting (4.12) we get

$$F_3^{ij} = \gamma_1^{pq} K_p{}^i{}_k K_q{}^j{}_m \dot{x}^k\dot{x}^m + (\gamma_2^{qj} K_q{}^i{}_k - \gamma_2^{qi} K_q{}^j{}_k)\dot{x}^k + \gamma_3^{ij}\,, \tag{4.13}$$

where $\gamma_3^{ij} = -\gamma_3^{ji}$ and γ_3^{ij} satisfies (4.5) and (4.6). □

According to Theorem 4.1 to classify all natural operators from $TM\times_M reg\odot^2 T^*M \times_M C_\tau M$ into $\wedge^2 T^*(TM)$ it is sufficient to classify all operators with equivariant mappings expressed by (4.5) and (4.6), i.e. operators from $TM \times_M reg\odot^2 T^*M$ to $TM\otimes TM$. To classify all such operators we shall use the following Theorem, [5, 4],

Theorem 4.2. *Let (M,g) be an oriented pseudo-Riemannian manifold of dimension $n>3$. Then all natural operators from $TM\times_M reg\odot^2 T^*M$ to $T^*M\otimes T^*M$ (natural F-metrics) are symmetric and are of the form*

$$\beta_u(X,Y) = \mu(h(u))\, g(X,Y) + \nu(h(u))\, g(X,u)g(Y,u) \tag{4.14}$$

where X,Y are vector fields and μ,ν are smooth real functions defined on $\mathbb{R}$.

In coordinates

$$\beta_{ij} = \mu(h(u))\, g_{ij} + \nu(h(u))\, g_{ip}g_{jq}\dot{x}^p\dot{x}^q\,.$$ □

Now we can prove the inverse version of the above Theorem 4.2.

Theorem 4.3. *Let (M,g) be an oriented pseudo-Riemannian manifold of dimension $n>3$. Then all natural operators from $TM\times_M reg\odot^2 T^*M$ to $TM\otimes TM$ are symmetric and are of the form*

$$\gamma_u(\rho,\sigma) = \gamma_1(h(u))\,\tilde{g}(\rho,\sigma) + \gamma_2(h(u))\,\rho(u)\sigma(u) \tag{4.15}$$

where ρ,σ are 1-form fields and γ_1,γ_2 are smooth real functions defined on $\mathbb{R}$.

In coordinates

$$\gamma^{ij} = \gamma_1(h(u))\, g^{ij} + \gamma_2(h(u))\,\dot{x}^i\dot{x}^j\,. \tag{4.16}$$

Proof. Any natural operator $TM\times_M reg\odot^2 T^*M$ to $TM\otimes TM$ can be interpreted as a natural real function on $TM\times_M T^*M\times_M T^*M\times_M reg\odot^2 T^*M$ bilinear on T^*M. Similarly any natural F-metric of Theorem 4.2 is a natural real function on $TM\times_M TM\times_M TM\times_M reg\odot^2 T^*M$ bilinear on TM. Theorem 4.3 now follows from the classification of Theorem 4.2 and the fact, [2], that all natural isomorphisms $T^*M\to TM$ induced by the metric g are of the form

$$X = \kappa(h(u))\, g^\sharp(\rho)\,,\quad X^i = \kappa(h(u))\, g^{ip}\rho_p\,, \tag{4.17}$$

where ρ is a 1-form field and κ is a smooth real function defined on $\mathbb{R}$ such that $\kappa(t) \neq 0$ for any $t \in \mathbb{R}$. The operator γ is now a composition of the operator (4.17) and natural F-metric β, i.e. in coordinates

$$\begin{aligned}\gamma^{ij} &= \kappa^2(h(u))\, g^{ip} g^{jq} \big(\mu(h(u))\, g_{pq} + \nu(h(u))\, g_{pr} g_{ps} \dot{x}^r \dot{x}^s\big) \\ &= \gamma_1(h(u))\, g^{ij} + \gamma_2(h(u))\, \dot{x}^i \dot{x}^j\,,\end{aligned} \tag{4.18}$$

where $\gamma_1 = \kappa^2 \mu, \gamma_2 = \kappa^2 \nu$. (4.18) is just the equivariant mapping corresponding to (4.15). □

Remark 4.4. In Theorems 4.2 and 4.3 we have restricted the dimension of the underlying manifold on $n > 3$ because our standard model is an Lorentzian manifold of dimension 4 (spacetime). But both Theorems 4.2 and 4.3 are correct for non-oriented manifolds if $n \geq 2$. For oriented manifolds in dimensions 2 and 3 there are also antisymmetric natural F-metrics and so antisymmetric natural operators γ (see [5]).

Now we can classify all natural 2-vector fields. We have

Theorem 4.5. *Let (M, g) ($\dim M > 3$) be an oriented pseudo-Riemannian manifold endowed with a symmetric linear connection K. Then all natural operators from $TM \times_M \operatorname{reg} \odot^2 T^*M \times_M C_\tau M$ into $\wedge^2 T(TM)$ projectable over the identity of TM are of the form*

$$\Lambda(\gamma, K) = K \bar{\wedge}_\gamma \vartheta\,, \tag{4.19}$$

where $\bar{\wedge}_\gamma$ denotes the wedge product followed by the contraction through the operator γ of Theorem 4.3. In coordinates

$$\Lambda(\gamma, K) = \big(\gamma_1(h(u))\, g^{ij} + \gamma_2(h(u))\, \dot{x}^i \dot{x}^j\big)(\partial_i + K_i{}^m{}_k \dot{x}^k \dot{\partial}_m) \wedge \dot{\partial}_j\,, \tag{4.20}$$

where γ_1, γ_2 are smooth real functions defined on $\mathbb{R}$.

Proof. By Theorem 4.3 we have $\gamma_1^{ij} = \gamma_3^{ij} = 0$ in Theorem 4.1. So we have the equivariant mappings $F : Q \to S$ corresponding to our operators in the form

$$F_1^{ij} = 0\,, \qquad F_2^{ij} = \gamma_2^{ij}\,, \qquad F_3^{ij} = (\gamma_2^{qj} K_{qk}^i - \gamma_2^{qi} K_q{}^i{}_k)\dot{x}^k\,, \tag{4.21}$$

where

$$\gamma_2^{ij} = \gamma_1(h(u))\, g^{ij} + \gamma_2(h(u))\, \dot{x}^i \dot{x}^j\,.$$

It is easy to see that it is just the mapping corresponding to (4.19). □

Remark 4.6. The canonical 2-vector field from Section 3 corresponds to $\gamma_1 \equiv 1,\ \gamma_2 \equiv 0$. From (2.1) and (2.2) it follows that the 2-vector field corresponding to $\gamma_1 \equiv 0,\ \gamma_2 \equiv 1$ is $u^H \wedge u^V$. Then (4.19) can be written in the form

$$\Lambda(\gamma, K) = \gamma_1(h(u))\, \Lambda(g, K) + \gamma_2(h(u))\, u^H \wedge u^V \tag{4.22}$$

which is much more convenient for our further purposes.

Remark 4.7. The 2-vector field $\Lambda(\gamma, K)$ can be characterized by

$$\Lambda(\gamma,K)(\rho^V,\sigma^V)=0\,,\quad \Lambda(\gamma,K)(\rho^V,\sigma^H)=\gamma(\rho,\sigma)\,,$$
$$\Lambda(\gamma,K)(\rho^H,\sigma^V)=-\gamma(\rho,\sigma)\,,\quad \Lambda(\gamma,K)(\rho^H,\sigma^H)=0\,,$$

where ρ,σ are 1-form fields on M, ρ^V,σ^V are their vertical lifts (pullbacks) and ρ^H,σ^H are horizontal lifts with respect to K.

5. Natural Poisson structures

In this Section we shall recall, [3], conditions for $\Lambda(\gamma, K)$ to define a Poisson structure on TM, i.e. we have to find conditions for $\Lambda(\gamma, K)$ to satisfy (1.2).

Lemma 5.1. *The 2-vector field $\Lambda(\gamma, K)$ is of constant maximal rank if and only if $\gamma_1(t)\neq 0$ and $\gamma_1(t)+2t\gamma_2(t)\neq 0$ for any $t\in\mathbb{R}$.*

Proof. $\Lambda(\gamma, K)$ is of constant maximal rank if and only if the matrix $\gamma_1(h)\,g^{ij}+\gamma_2(h)\,\dot x^i\dot x^j$ is regular. It is easy to see that it is so if and only if $\gamma_1(t)$ and $\gamma_1(t)+2t\gamma_2(t)$ are everywhere nonvanishing functions. □

Lemma 5.2. *We have*

$$\begin{aligned}[\Lambda(\gamma,K),\Lambda(\gamma,K)] = \gamma_1^2(h)\,&[\Lambda(g,K),\Lambda(g,K)]\\ &+\gamma_1(h)\dot\gamma_1(h)\,((\nabla g)(u,u)^\sharp)^V\wedge\Lambda(g,K)\\ &+\gamma_1(h)\dot\gamma_2(h)\,((\nabla g)(u,u)^\sharp)^V\wedge u^H\wedge u^V\\ &+2\big(\gamma_1(h)\gamma_2(h)-\gamma_1(h)\dot\gamma_1(h)-2h\gamma_2(h)\dot\gamma_1(h)\big)\,u^H\wedge\Lambda(g,K)\\ &+\gamma_2(h)\dot\gamma_1(h)\,(\nabla g)(u,u)(u)\,u^V\wedge\Lambda(g,K)\\ &+2\gamma_1(h)\gamma_2(h)\,[\Lambda(g,K),u^H]\wedge u^V\,.\end{aligned}$$

Proof. It is easy to prove it by using (4.22) and properties of the Schouten-Nijenhuis bracket. □

Theorem 5.3. *The nondegenerate 2-vector field $\Lambda(\gamma, K)$ defines a Poisson structure on TM if and only if the conditions* (A), (B) *and*

$$\text{(C)}\qquad \gamma_1(t)\gamma_2(t)-\gamma_1(t)\dot\gamma_1(t)-2t\gamma_2(t)\dot\gamma_1(t)=0$$

are satisfied for any $t\in\mathbb{R}$.

Proof. $\Leftarrow$ It follows from Theorem 3.4, Lemma 3.7 and Lemma 3.10.

$\Rightarrow$ All 2-vector fields on the right hand side of Lemma 5.2 are independent, i.e. $[\Lambda(\gamma,K),\Lambda(\gamma,K)]=0$ if and only if all terms on the right hand side vanish. Since $\gamma_1(h)\neq 0$ the first term vanishes (by Theorem 3.4) if and only if the condition (A) is satisfied. The second and the third terms vanish if and only if the condition (B) is satisfied (Lemma 3.7) and the fourth term vanishes if and only if the condition (C) is satisfied. The others terms vanish because of the conditions (A) and (B) (Lemma 3.10). □

Now, let us compare the conditions of Theorem 5.3 for natural Poisson structures with conditions for natural symplectic structures given in [1]. First we recall that any 2-form field $\Omega(\beta, K)$ on TM, $\dim M > 3$, naturally given by g and K can be characterized by

$$\begin{aligned} \Omega(\beta,K)(X^H,Y^H) &= 0, & \Omega(\beta,K)(X^H,Y^V) &= -\beta(X,Y), \\ \Omega(\beta,K)(X^V,Y^H) &= \beta(Y,X), & \Omega(\beta,K)(X^V,Y^V) &= 0. \end{aligned} \tag{5.1}$$

where X, Y are vector fields on M and β is a natural F-metric given by Theorem 4.2. In coordinates

$$\Omega(\beta, K) = \big(\mu(h(u))\, g_{mj} + \nu(h(u))\, g_{mr} g_{js} \dot{x}^r \dot{x}^s\big)\big(\dot{d}^m - K_i{}^m{}_k \dot{x}^k d^i\big) \wedge d^j\,.$$

Then we have, [1],

Theorem 5.4. *$\Omega(\beta, K)$ is a symplectic form on TM if and only if $\nu = \dot{\mu}$ and the real smooth function μ satisfies*

$$\mu(t) \neq 0, \quad \mu(t) + 2t\dot{\mu}(t) \neq 0 \tag{5.2}$$

for all $t \in \mathbb{R}$. Moreover g and K have to satisfy conditions (A) *and* (B). $\square$

Let us suppose that the 2-vector field $\Lambda(\gamma, K)$ is the Poisson 2-vector field given by the symplectic 2-form field $\Omega(\beta, K)$. It is easy to see that in this case γ (given by (4.16)) is the inverse of the natural F-metric β (given by Theorem 4.2), i.e. $\beta_{ik}\gamma^{kj} = \delta_i^j$. Then

$$\mu(t)\gamma_1(t) = 1\,, \quad \mu(t)\gamma_2(t) + \nu(t)\gamma_1(t) + 2t\nu(t)\gamma_2(t) = 0\,. \tag{5.3}$$

If the conditions of Theorem 5.4 are satisfied, then $\mu = \frac{1}{\gamma_1}$ and $\nu = -\frac{\dot{\gamma}_1}{\gamma_1^2}$ (from the condition $\nu = \dot{\mu}$ for symplectic forms). Substituting these equalities into second term of (5.3) we get the condition (C) of Theorem 5.3. On the other hand if $\Omega(\beta, K)$ is a symplectic 2-form field given by the nondegenerate Poisson 2-vector field $\Lambda(\gamma, K)$, then the conditions of Theorem 5.3 induce in the same way conditions of Theorem 5.4. So the conditions for natural symplectic and natural Poisson structures on TM are equivalent.

Remark 5.5. In Theorem 5.3 we have supposed $\Lambda(\gamma, K)$ to be nondegenerate. If we admit also 2-vector fields $\Lambda(\gamma, K)$ which are not of maximal rank we get from Lemma 5.2 that all such 2-vector fields are characterized by $\gamma_1 \equiv 0$, i.e. they are of the form $\Lambda(\gamma, K) = \gamma_2(h(u))\, u^H \wedge u^V$, where γ_2 is arbitrary smooth real function on $\mathbb{R}$.

6. Natural Jacobi structures

In this Section we shall recall, [3], conditions for $E, \Lambda(\gamma, K)$ to define a Jacobi structure on TM, i.e. we have to find conditions for $E, \Lambda(\gamma, K)$ to satisfy (1.3).

Lemma 6.1. *We have*

$$
\begin{aligned}
[E,\Lambda(\gamma,K)] &= \alpha(h)\gamma_1(h)\,[u^H,\Lambda(g,K)]\\
&+\Big(\frac{1}{2}\alpha(h)\dot\gamma_1(h)\,\nabla g(u,u)(u)+2h\beta(h)\dot\gamma_1(h)-\beta(h)\gamma_1(h)\Big)\,\Lambda(g,K)\\
&+\Big(\frac{1}{2}\alpha(h)\dot\gamma_2(h)\,\nabla g(u,u)(u)+2h\beta(h)\dot\gamma_2(h)+\beta(h)\gamma_2(h)\\
&\quad -2h\dot\beta(h)\gamma_2(h)-\frac{1}{2}\dot\alpha(h)\gamma_2(h)\,\nabla g(u,u)(u)-\dot\beta(h)\gamma_1(h)\Big)\,u^H\wedge u^V\\
&+\frac{1}{2}\dot\alpha(h)\gamma_1(h)(\nabla g(u,u)^\sharp)^V\wedge u^H+\frac{1}{2}\dot\beta(h)\gamma_1(h)\,(\nabla g(u,u)^\sharp)^V\wedge u^V\,.
\end{aligned}
$$

Proof. It is easy to prove Lemma 6.1 by using properties of the Schouten-Nijenhuis bracket. □

Lemma 6.2. *Let $\Lambda(\gamma,K)$ be of maximal rank. We have*

$$[E,\Lambda(\gamma,K)]=0$$

if and only if one of the following groups of conditions is satisfied:

I. $\alpha\equiv 0$, $\beta\equiv 0$;

II. $\nabla g=0$ *and the equalities*

$$\beta(t)\big(\gamma_1(t)-2t\dot\gamma_1(t)\big)=0\,, \tag{6.1}$$

$$\beta(t)\big(\gamma_2(t)+2t\dot\gamma_2(t)\big)-\dot\beta(t)\big(\gamma_1(t)+2t\gamma_2(t)\big)=0 \tag{6.2}$$

are satisfied for all $t\in\mathbb{R}$.

Proof. All terms on the right hand side in Lemma 6.1 have to vanish. The first term vanishes if and only if $\alpha\equiv 0$ or $[u^H,\Lambda(g,K)]=0$.

First let us suppose $[u^H,\Lambda(g,K)]=0$ which is equivalent by Lemma 3.8 with $\nabla g=0$, i.e. K is the metric connection. The last two terms in Lemma 6.1 vanish and the second and third terms vanish if and only if (6.1) and (6.2) are satisfied and we have II.

Now, let us suppose $\alpha\equiv 0$ and $\nabla g\neq 0$. Then the second and the third terms vanish if and only if (6.1) and (6.2) are satisfied and the last term vanishes if and only if $\dot\beta\equiv 0$, i.e. β is a constant function. Then for $\beta\not\equiv 0$ (6.1) is equivalent with

$$\gamma_1(t)=2t\dot\gamma_1(t),\quad \text{i.e.}\quad \gamma_1(t)=\sqrt{t} \tag{6.3}$$

and similarly (6.1) is equivalent with

$$\gamma_2(t)=-2t\dot\gamma_2(t),\quad \text{i.e.}\quad \gamma_2(t)=\frac{1}{\sqrt{t}} \tag{6.4}$$

which is in the contradiction with the globality of γ_1,γ_2. Hence (6.1) and (6.2) are satisfied only for $\beta\equiv 0$ and we have I. □

Lemma 6.3. *Let $\Lambda(\gamma,K)$ be of maximal rank. We have*

$$[\Lambda(\gamma,K),\Lambda(\gamma,K)]=2E\wedge\Lambda(\gamma,K)$$

if and only if the conditions (A), (B) *and*

$$\alpha(t)\gamma_1(t) = \gamma_1(t)\gamma_2(t) - \gamma_1(t)\dot{\gamma}_1(t) - 2t\gamma_2(t)\dot{\gamma}_1(t)\,, \qquad \beta(t) = 0 \tag{6.5}$$

are satisfied for any $t \in \mathbb{R}$.

Proof. We have

$$2E \wedge \Lambda(\gamma, K) = 2\alpha\gamma_1\, u^H \wedge \Lambda(g, K) + 2\beta\gamma_1\, u^V \wedge \Lambda(g, K)\,.$$

Comparing this with expression of $[\Lambda(\gamma, K), \Lambda(\gamma, K)]$ from Lemma 5.2 we get

$$[\Lambda(g, K), \Lambda(g, K)] = 0\,, \tag{6.6}$$

$$((\nabla g)(u, u)^\sharp)^V \wedge \Lambda(g, K) = 0\,, \tag{6.7}$$

$$\dot{\gamma}_2((\nabla g)(u, u)^\sharp)^V \wedge u^H \wedge u^V = 0\,, \tag{6.8}$$

$$\gamma_1(t)\gamma_2(t) - \gamma_1(t)\dot{\gamma}_1(t) - 2t\gamma_2(t)\dot{\gamma}_1(t) = \alpha(t)\gamma_1(t)\,, \tag{6.9}$$

$$\big(2\beta\gamma_1 - \gamma_2\dot{\gamma}_1(\nabla g)(u, u)(u)\big)u^V \wedge \Lambda(g, K) = 0\,, \tag{6.10}$$

$$\gamma_2[\Lambda(g, K), u^H] \wedge u^V = 0\,. \tag{6.11}$$

By Theorem 3.4 (6.6) holds if and only if (A) is satisfied, (6.7) is equivalent with (B). (A) and (B) imply (6.8) and (6.11) and (6.10) is reduced by Lemma 3.10 to $\beta \equiv 0$. □

Theorem 6.4. *Let* $\Lambda(\gamma, K)$ *be of maximal rank and* E *be a non-zero vector field.* E *and* $\Lambda(\gamma, K)$ *define a Jacobi structure on* TM *if and only if* K *is the metric connection and* $E = \alpha(h)\, u^H$*, where*

$$\alpha(t)\gamma_1(t) = \gamma_1(t)\gamma_2(t) - \gamma_1(t)\dot{\gamma}_1(t) - 2t\gamma_2(t)\dot{\gamma}_1(t) \tag{D}$$

for any $t \in \mathbb{R}$.

Proof. Theorem 6.4 follows from Lemma 6.1 and Lemma 6.3. □

Remark 6.5. If E is the zero vector field then Theorem 6.4 reduces to Theorem 5.3 and the Jacobi structures reduces to the Poisson structure.

Remark 6.6. In Theorem 6.4 we have supposed $\Lambda(\gamma, K)$ to be of maximal rank. If we admit also 2-vector fields which are not of maximal rank we get from Lemma 6.3 that all such 2-vectors are characterized by $\gamma_1 \equiv 0$. Lemma 6.1 then implies that the functions α, β and γ_2 have to satisfy

$$\begin{aligned} &\frac{1}{2}\nabla g(u, u)(u)\big(\alpha(h(u))\dot{\gamma}_2(h(u)) - \dot{\alpha}(h(u))\gamma_2(h(u))\big) \\ &\quad + 2h(u)\big(\beta(h(u))\dot{\gamma}_2(h(u)) - \dot{\beta}(h(u))\gamma_2(h(u))\big) + \beta(h(u))\gamma_2(h(u)) = 0\,. \end{aligned}$$

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