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DIRECT PRODUCT DECOMPOSITIONS OF PSEUDO MV–ALGEBRAS

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Dedicated to Professor František Šik on the occasion of his 80. anniversary

Abstract. In this paper we deal with the relations between the direct product decompositions of a pseudo MV-algebra and the direct product decomposicitons of its underlying lattice.

I. Introduction

Direct product decompositions of MV-algebras have been investigated in [8]. It is well-known that for each MV-algebra $\mathcal{A}$ there exists an abelian lattice ordered group G with a strong unit u such that $\mathcal{A}$ can be obtained by a well-defined construction from G; in accordance with the notation from the monograph by Cignoli, D'Ottaviano and Mundici [2] we write $\mathcal{A} = \Gamma(G, u)$. One of the items dealt with in [8] was the relation between direct product decompositions of $\mathcal{A}$ and direct product decompositions of G.

The MV-algebra is an algebraic structure of type $(2, 1, 0, 0)$ (cf. [2]); the binary operation is denoted by the symbol $\oplus$ and it is assumed to be commutative.

The notion of MV-algebra can be generalized in such a way that the assumption of the commutativity of the operation $\oplus$ is omitted (cf. Georgescu and Iorgulescu [5], [6], and Rachůnek [11]).

The results of [6] were used by Dvurečenskij and Pulmannová [3]; further, the results of [11] were applied by Chajda, Halaš and Rachůnek [1].

For further results on pseudo MV-algebras cf. Dvurečenskij [4], Leustean [10], Rachůnek [12] and the author [9].

In the present paper we apply the terminology and the notation from [6]. Thus we deal with an algebra of type $(2, 1, 1, 0, 0)$ which is called a pseudo MV-algebra; for the definition, cf. Section 2 below. (I remark that the substantial part of this paper has been finished before I was acquainted with [11] and [1].)

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If G is a lattice ordered group (which need not be abelian) and if $0 \leqq u \in G$, then by similar construction as in the abelian case we can construct the algebraic structure $\mathcal{A} = \Gamma(G, u)$. It turns out that $\mathcal{A}$ is a pseudo MV-algebra.

In [3] it was proposed the problem whether for each pseudo MV-algebra $\mathcal{A}_1$ there exists a lattice ordered group G_1 with a strong unit u_1 such that $\mathcal{A}_1 = \Gamma(G_1, u_1)$. Dvurečenskij [4] proved that the answer is positive.

Let A be the underlying set of the pseudo MV-algebra $\mathcal{A}$. By applying the basic pseudo MV-operations we can define a partial order $\leqq$ on the set A such that $L(\mathcal{A}) = (A; \leqq)$ is a bounded distributive lattice. (Cf. [6].)

Let $\mathcal{A}$ a pseudo MV-algebra. In the present paper we describe a construction showing that to each element e of A which has a complement in the lattice $L(\mathcal{A})$ there corresponds a direct product decomposition of $\mathcal{A}$ with two direct factors.

Conversely, we prove that each two-factor direct product decomposition of $\mathcal{A}$ can be obtained, up to isomorphism, by the mentioned construction.

Further, we show that each direct product decompostion of the lattice $L(\mathcal{A})$ induces a direct product decomposition of $\mathcal{A}$, and conversely.

This implies that any two direct product decompositions of $\mathcal{A}$ have isomorphic refinements.

Let us also remark that if there exists a lattice ordered group G with an element $0 \leqq u \in G$ such that $\mathcal{A} = \Gamma(G, u)$, then the assertion of Theorem 2.5 from [8] concerning internal direct product decompositions of an MV-algebra holds also in the case of pseudo MV-algebras; it suffices to use the same proof which has been applied in [8].

2. Preliminaries

We start by recalling the definition of pseudo MV-algebra (cf. [6], [3]).

Let $\mathcal{A} = (A; \oplus, {}^-, {}^\sim, 0, 1)$ be an algebraic structure, where A is a nonempty set, $\oplus$ is a binary operation, ${}^-$ and ${}^\sim$ are unary operations, 0 and 1 are nulary operations on A. For each $x, y \in A$ we put

$$y \odot x = (x^- \oplus y^-)^\sim .$$

The algebraic structure $\mathcal{A}$ is called a pseudo MV-algebra if the following conditions are satisfied for each $x, y, z \in A$:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

We define $x \leqq y$ iff $x^- \oplus y = 1$. Then $\leqq$ is a partial order on A. Put $L(\mathcal{A}) = (A; \leqq)$.

Proposition 2.1. (Cf. [6].) *$L(\mathcal{A})$ is a lattice with the least element 0 and with the greatest element 1. Moreover, for each $x, y \in A$ we have*

$$x \vee y = x \oplus (x^{\sim} \odot y), \quad x \wedge y = x \odot (x^{-} \oplus y).$$

Lemma 2.2 (Cf. [6]). *If $\{y_i\}_{i\in I} \subseteq A$ and if $\bigvee_{i\in I} y_i$ exists in $L(\mathcal{A})$, then the the lattice $L(\mathcal{A})$ satisfies the condition*

$$x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I} (x \wedge y_i).$$

Hence, in particular, $L(\mathcal{A})$ is a distributive lattice.

Let G be a lattice ordered group and let $0 \leqq u \in G$. The group operation in G is denoted by $+$, though we do not assume that this operation is commutative. Further, let A be the inverval $[0, u]$ of G. For $x, y \in A$ we put

$$\begin{aligned} x \oplus y &= (x + y) \wedge u\,, \\ x^{-} &= u - x\,, \\ x^{\sim} &= -x + u\,. \end{aligned}$$

Proposition 2.3 (Cf. [6]). *The algebraic structure $\mathcal{A} = (A; \oplus, ^{-}, ^{\sim}, 0, u)$ is a pseudo MV-algebra.*

If $\mathcal{A}$ is as in 2.3, then we denote $\mathcal{A} = \Gamma(G, u)$.

3. Auxiliary results

Again, let $\mathcal{A}$ be a pseudo MV-algebra and $L = L(\mathcal{A})$. Assume that e is an element of A which has a complement e' in the lattice $L(\mathcal{A})$. In view of 2.2, e' is uniquely determined.

Consider the intervals

$$X_1 = [0, e], \quad X_2 = [0, e']$$

of the lattice L. For $a \in A$ we put

$$a_1 = e \wedge a, \quad a_2 = e' \wedge a\,.$$

Lemma 3.1 (Cf. [6]). *Let $p, q \in A$, $p \wedge q = 0$. Then*

$$p \oplus q = p \vee q = q \oplus p\,.$$

Corollary 3.2. *Let $x^1 \in X_1$, $x^2 \in X_2$. Then $x^1 \oplus x^2 = x^2 \oplus x^1$.*

Lemma 3.3. *$a = a_1 \oplus a_2 = a_1 \vee a_2$ for each $a \in A$.*

Proof. The distributivity of L yields

$$a = a \wedge 1 = a \wedge (e \vee e') = (a \wedge e) \vee (a \wedge e') = a_1 \vee a_2 \,.$$

Hence in view of 3.1, $a = a_1 \oplus a_2$. □

Lemma 3.4. *Let $a \in A$, $x^1 \in X_1$, $x^2 \in X_2$, $a = x^1 \oplus x^2$. Then $x^1 = a_1$ and $x^2 = a_2$.*

Proof. By applying 3.1, 3.2 and the distributivity of L we obtain

$$x^1 = x^1 \wedge (x^1 \vee x^2) = x^1 \wedge (x^1 \oplus x^2) = x^1 \wedge a = x^1 \wedge (a_1 \vee a_2) = (x^1 \wedge a_1) \vee (x^1 \wedge a_2) \,.$$

We have $x^1 \wedge a_2 = 0$, whence $x^1 = x^1 \wedge a_1$ and thus $x^1 \leqq a_1$. Similarly we verify that $a_1 \leqq x^1$, thus $a_1 = x^1$. Analogously, $a_2 = x^2$. □

Lemma 3.5 (Cf. [6], Propos. 1.20)**.** *If $x \wedge y = 0$, then $x \wedge (y \oplus z) = x \wedge z$.*

Lemma 3.6. *The set X_1 is closed with respect to the operation $\oplus$.*

Proof. Let $a, b \in X_1$. Then we have $a \wedge e' = 0$. Hence according to 3.5 we obtain

$$e' \wedge (a \oplus b) = e' \wedge b = 0 \,,$$

whence $(a \oplus b)_2 = 0$. Thus 3.3 yields $(a \oplus b)_1 = a \oplus b$. Therefore $a \oplus b \in X_1$. □

Lemma 3.7. *Let $a, b \in A$. Then*

$$(a \oplus b)_1 = a_1 \oplus b_1 \,, \quad (a \oplus b)_2 = a_2 \oplus b_2 \,.$$

Proof. In view of 3.2 and 3.3 we have

$$\begin{aligned} a \oplus b = (a_1 \oplus a_2) \oplus (b_1 \oplus b_2) &= a_1 \oplus (a_2 \oplus b_1) \oplus b_2 = \\ &= a_1 \oplus (b_1 \oplus a_2) \oplus b_2 = (a_1 \oplus b_1) \oplus (a_2 \oplus b_2) \,. \end{aligned}$$

According to 3.6, $a_1 \oplus b_1 \in X_1$. Similarly, $a_2 \oplus b_2 \in X_2$. Thus 3.4 yields

$$(a \oplus b)_1 = a_1 \oplus b_1 \,, \quad (a \oplus b_2)_2 = a_2 \oplus b_2 \,.$$

□

Proposition 3.8 (Cf. [3], 4.4.3, Exercise 7.6.4.5)**.** *Let $\mathcal{A}$ be a pseudo MV-algebra, $A \neq \{0\}$. Then there exists a lattice ordered group G with an element $0 < u \in G$ such that $\mathcal{A}$ can be embedded (as a pseudo MV-algebra) into the pseudo MV-algebra $\Gamma(G, u)$.*

Let $x \in A$. Denote

$$P_x = \{p \in A : p \oplus x = 1\} \,,$$

$$Q_x = \{q \in A : x \oplus q = 1\} \,.$$

Lemma 3.9. *Let $x \in A$. Then*

$$x^- = \min P_x , \quad x^\sim = \min Q_x .$$

Proof. In view of 1.5 in [6] we have $x^- \oplus x = 1$, whence $x^- \in P_x$. Let G and u be as in 3.8. Thus $u = 1$ and $x^- + x = u$. Let $p \in P_x$. Hence

$$u = 1 = p \oplus x = (p + x) \wedge u ,$$

which yields

$$p + x \geqq u = x^- + x .$$

Therefore $p \geqq x^-$. Hence $x^- = \min P_x$. Analogously we verify that $x^\sim = \min Q_x$. □

Remark 3.9.1. From 3.9 we conclude that unary operations $^-$ and $^\sim$ are uniquely determined by the operation $\oplus$ and by the partial order $\leqq$ on A.

For $x \in X_1$ we denote

$$P_x^1 = \{p \in X_1 : p \oplus x = e\} ,$$
$$Q_x^1 = \{q \in X_1 : x \oplus q = e\} .$$

Lemma 3.10. *Let $x \in X_1$. Then*

$$(x^-)_2 = (x^\sim)_2 = e' ,$$
$$(x^-)_1 = \min P_x^1 , \quad (x^\sim)_1 = \min Q_x^1 .$$

Proof. Clearly $(x^-)_2 \leqq e'$. By way of contradiction, suppose that $(x^-)_2 < e'$. Then

$$1 = x^- \oplus x = (x^-)_1 \oplus (x^-)_2 \oplus x .$$

Since $(x^-)_2 \in X_2$ and $x \in X_1$, in view of 3.2 we have

$$(x^-)_2 \oplus x = x \oplus (x^-)_2 .$$

Therefore

$$1 = (x^-)_1 \oplus x \oplus (x^-)_2 .$$

In view of 3.6, $(x^-)_1 \oplus x \in X_1$. Thus $(x^-)_1 \oplus x \leqq e$, whence

$$1 \leqq e \oplus (x^-)_2 = e \vee (x^-)_2 .$$

In view of the distributivity and according to the relation $(x^-)_2 < e'$ we get $e \vee (x^-)_2 < 1$, which is a contradiction. Thus $(x^-)_2 = e'$.

Further, by way of contradiction, assume that the relation

$$(x^-)_1 = \min P^1_x$$

fails to hold.

According to 3.7 we have

$$(x^-)_1 \oplus x_1 = (x^- \oplus x)_1 = 1_1 = 1 \wedge e = e\,,$$

whence $(x^-)_1 \in P^1_x$. Thus in view of the assumption there is $z \in P^1_x$ such that $(x^-)_1 \nleq z$. Denote

$$t = (x^-)_1 \wedge z\,.$$

Then $t < (x^-)_1 \wedge z$. Then $t < (x^-)_1$ and hence $t \in X_1$, yielding $t_1 = t$. We have

$$t \oplus x = ((x^-)_1 \wedge z) \oplus x = ((x^-)_1 \oplus x) \wedge (z \oplus x)$$

(in view of 1.16, [5]). Since $z \oplus x = e$, we get $t \oplus x = e$, whence $t \in P^1_x$.

In view of the distributivity of L we obtain

$$t \oplus e' = t \vee e' < (x^-)_1 \vee e' = (x^-)_1 \vee (x^-)_2 = x^-\,.$$

Further,

$$\begin{aligned}(t \oplus e') \oplus x = t \oplus (e' \oplus x) = t \oplus (x \oplus e') = (t \oplus x) \oplus e' =\\ = e \oplus e' = e \vee e' = 1\,.\end{aligned}$$

Since $t \oplus e' < x^-$, in view of 3.9 we arrived at a contradiction. Therefore $(x^-)_1 = \min P^1_x$.

The remaining relations concerning $x^\sim$ can be proved analogously. □

Lemma 3.11. *Let $x \in X_1$ and let G be as in* 3.8. *Let $b^1 \in G$, $b^1 + x = e$. Then $b^1 = \min P^1_x$.*

Proof. It suffices to apply analogous steps as in the proof of 3.9. □

Analogously we have

Lemma 3.11.1. *Let $y \in X_2$ and let $b^2 \in G$, $y + b^2 = e'$. Then $b^2 = \min Q^1_y$.*

It is obvious that if b^1 and b^2 are as in 3.11 and 3.11.1, then $b^1 \in X_1$ and $b^2 \in X_2$.

Put $b = b^1 \oplus b^2$. In view of 3.4 we have

$$b_1 = b^1, \quad b_2 = b^2\,.$$

Lemma 3.12. *Let $a \in A$. Denote $a_1 = x$, $a_2 = y$ and let b^1, b^2 be as above. Then*

$$(a^-)_1 = b^1\,.$$

Proof. We have $b^1 \wedge b^2 = 0$. Hence $b = b_1 + b_2$; similarly, $a = a_1 + a_2$. Thus

$$\begin{aligned}b + a = (b_1 + b_2) + (a_1 + a_2) = b_1 + (b_2 + a_1) + a_2 = b_1 + (a_1 + b_2) + a_2 =\\ = (b_1 + a_1) + (b_2 + a_2) = e + e' = e \vee e' = 1\,.\end{aligned}$$

Therefore $b = a^-$. Hence

$$(a^-)_1 = b_1 = b^1\,.$$ □

4. The pseudo MV-algebra $\mathcal{X}_1$

Assume that $\mathcal{A}$ is an MV-algebra and let e, e', X_1 and X_2 be as in the previous section.

We have already observed (cf. 3.6) that X_1 is closed with respect to the operation $\oplus$. Further, it is obvious that X_1 is also closed with respect to the operations $\wedge$ and $\vee$.

In view of 3.9 and 3.10 we define the unary operations ${}^{-(e)}$ and ${}^{\sim(e)}$ on X_1 by putting

$$x^{-(e)} = \min P_x^1, \quad x^{\sim(e)} = \min Q_x^1$$

for each $x \in X_1$.

Now, we define a binary operation $\odot_e$ on X_1 by

$$y \odot_e x = (x^{-(e)} \oplus y^{-(e)})^{\sim(e)} .$$

Let us consider the algebraic structure

$$\mathcal{X}_1 = (X_1; \oplus, {}^{-(e)}, {}^{\sim(e)}, 0, e) .$$

For $a \in A$ let a_1 be as in Section 3; let us now apply the notation

$$a_1 = \varphi_1(a) .$$

Then the mapping $\varphi_1 : A \to X_1$ is surjective. We have clearly $\varphi_1(0) = 0$, $\varphi_1(1) = e$. Moreover, in view of 3.7, φ_1 is a homomorphism with respect to the operation $\oplus$.

For proving that φ_1 is a homomorphism with respect to the operation ${}^-$ we have to verify that the relation

$$\varphi_1(a^-) = \varphi_1(a)^{-(e)}$$

is valid for each $a \in A$.

Let $a \in A$. Denote $x = a_1$. Under the notation as in 3.12 we have

$$\varphi_1(a^-) = (a^-)_1 = b^1 .$$

In view of 3.11, $b^1 = \min P_x^1$. Thus

$$b_1 = x^{-(e)} = a_1^{-(e)} = \varphi_1(a)^{-(e)} .$$

Analogously we verify that φ_1 is a homomorphism with respect to the operation ${}^\sim$.

Summarizing, we have

Lemma 4.1. *φ_1 is a homomorphism of the pseudo MV-algebra $\mathcal{A}$ onto the algebraic structure $\mathcal{X}$.*

In view of 4.1 we conclude that $\mathcal{X}_1$ satisfies all the identities (A1) - (A8) (under the notation modified in the obvious way). Hence we obtain

Corollary 4.2. *$\mathcal{X}_1$ is a pseudo MV-algebra.*

Analogous consideration can be performed for X_2; we apply the symbols φ_2 and $\mathcal{X}_2$.

For each $a \in A$ we put

$$\varphi(a) = (\varphi_1(a), \varphi_2(a)) .$$

The direct product of pseudo MV-algebras is defined in the usual way; cf. e.g., [5].

Proposition 4.3. *φ is an isomorphism of the pseudo MV-algebra $\mathcal{A}$ onto the direct product $\mathcal{A}_1 \times \mathcal{A}_2$.*

Proof. Under the notation as in Section 3 we have

$$\varphi(a) = (a_1, a_2) = (a \wedge e, a \wedge e') .$$

Since L is a distributive lattice φ is a bijection. Then 4.1 yields that φ is an isomorphism. $\square$

Let $B(\mathcal{A})$ be the set of all elements of L which have a complement. Then $B(\mathcal{A})$ is a Boolean algebra; it was dealt with in [5], Section 4.

By a direct product decomposition of $\mathcal{A}$ we understand an isomorphism of $\mathcal{A}$ onto a direct product of pseudo MV-algebras.

In view of 4.3, to each element e of $B(\mathcal{A})$ there corresponds a uniquely determined two-factor direct product decomposition of $\mathcal{A}$.

5. Two-factor direct product decompositions of $\mathcal{A}$

In this section we show that each two-factor direct product decomposition of the pseudo MV-algebra $\mathcal{A}$ is constructed, up to isomorphism, by the method described in Section 4.

Assume that we have a two-factor direct product decomposition of $\mathcal{A}$, i.e., an isomorphism

$$\psi : \mathcal{A} \to \mathcal{A}_1 \times \mathcal{A}_2 , \tag{1}$$

such that ψ is a bijection.

Let L_1 and L_2 be the lattices corresponding to $\mathcal{A}_1$ or to $\mathcal{A}_2$, respectively. Since the lattice operations in $L = L(\mathcal{A})$ are defined by means of the operations $\oplus, ^-$ and $^\sim$, we conclude that the mapping

$$\psi : L \to L_1 \times L_2 \tag{2}$$

determines a direct product decomposition of the lattice L.

The lattices L_1 and L_2 must be bounded; let 0^i and 1^i be the least or the greatest element of L_i, respectively $(i = 1, 2)$.

Put $\mathcal{A}_1 \times \mathcal{A}_2 = \mathcal{A}_0$ and let A_0 be the underlying set of $\mathcal{A}_0$. Further, let A_i be the underlying set of $\mathcal{A}_i$ $(i = 1, 2)$. Denote

$$\begin{aligned} A_1^* &= \{(a^1, 0^2) : a^1 \in A_1\}, & A_2^* &= \{(0^1, a^2) : a^2 \in A_2\}, \\ X_1 &= \psi^{-1}(A_1^*), & X_2 &= \psi^{-1}(A_2^*), \\ e &= \psi^{-1}((1^1, 0^2)), & e' &= \psi^{-1}((0^1, 1^2)). \end{aligned}$$

Then we clearly have

Lemma 5.1. (i) $e \vee e' = 1$, $e \wedge e' = 0$; (ii) $X_1 = [0, e]$, $X_2 = [0, e']$.

Thus in view of 4.3 we have a direct product decomposition

$$\varphi : \mathcal{A} \to \mathcal{X}_1 \times \mathcal{X}_2, \tag{3}$$

where

$$\varphi(a) = (e \wedge a, e' \wedge a) \tag{4}$$

for each $a \in A$.

Further, for each $a^1 \in A_1$ and each $a^2 \in A_2$ we put

$$\varphi_1(a^1) = (a^1, 0^2), \quad \varphi_2(a^2) = (0^1, a^2).$$

Both the mappings $\varphi_1 : A_1 \to A_1^*$ and $\varphi_2 : A_2 \to A_2^*$ are bijections. Therefore there exist pseudo MV-algebras $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ such that for $i \in \{1, 2\}$ we have

(i) A_i^* is the underlying set of $\mathcal{A}_i^*$;

(ii) φ_i is an isomorphism of $\mathcal{A}_i$ onto $\mathcal{A}_i^*$.

For $i \in \{1, 2\}$ we denote by ψ_i the mapping ψ reduced to the set X_i.

In view of 3.6, the set X_1 is closed with respect to the operation $\oplus$; the same is valid for the set X_2. Further, according to 5.1, both X_1 and X_2 are closed with respect to the lattice operations $\vee$ and $\wedge$. From this we obtain

Lemma 5.2. *Let $i \in \{1, 2\}$. Then ψ_i is an isomorphism with respect to the operation $\oplus$ and with respect to the lattice operations $\vee, \wedge$.*

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be as in (3). From 5.2 and 3.9.1 we conclude

Lemma 5.3. *Let $i \in \{1, 2\}$. Then ψ_i is an isomorphism of the pseudo MV-algebra $\mathcal{X}_i$ onto the pseudo MV-algebra $\mathcal{A}_i^*$.*

For $i \in \{1, 2\}$ and $x^i \in X_i$ put

$$\psi_i^0(x^i) = \varphi_i^{-1}(\psi_i(x^i)).$$

From the properties of φ_i and from 5.3 we get

Proposition 5.4. *Let $i \in \{1, 2\}$. Then ψ_i^0 is an isomorphism of the pseudo MV-algebra $\mathcal{X}_i$ onto the pseudo MV-algebra $\mathcal{A}_i$.*

In other words, the direct factors standing in (1) are, up to isomorphism, the same as the direct factors standing in (3), and the direct product decomposition (3) is constructed by the procedure from Section 4.

Moreover, we show that the mapping ψ is uniquely determined by the mappings φ, ψ_1 and ψ_2. In fact, let $a \in A$ and

$$\varphi(a) = (a_1, a_2), \quad \psi(a) = (a^1, a^2).$$

Since $a_1 \in X_1$, there is $p \in A_1$ with

$$\psi_1(a_1) = \psi(a_1) = (p, 0^2).$$

Similarly, there is $q \in A_2$ such that

$$\psi_2(a_2) = \psi(a_2) = (0^1, q).$$

Then we have

Proposition 5.5. *Under the notation as above, $a^1 = p$ and $a^2 = q$.*

Proof. In view of (4) we have $a_1 = a \wedge e$. Thus according to (2) we get

$$\psi(a_1) = \psi(a) \wedge \psi(e),$$
$$(p, 0^2) = (a^1, a^2) \wedge (1^1, 0^2) = (a^1, 0^2)$$

whence $p = a^1$. Similarly we obtain $q = a^2$. □

6. Direct product decompositions of $L(\mathcal{A})$

In the present section we apply the previous results for dealing with the direct product decompositions having an arbitrary number of direct factors.

We investigate the relations between the direct product decompositions of $\mathcal{A}$ and those of $L(\mathcal{A})$.

Theorem 6.1. *Suppose that*

$$\varphi : \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i$$

is a direct product decomposition of a pseudo MV-algebra $\mathcal{A}$. Then, at the same time, we have a direct product decomposition

$$\varphi : L(\mathcal{A}) \to \prod_{i \in I} L(\mathcal{A}_i).$$

Proof. The underlying sets of $\mathcal{A}$ and of $L(\mathcal{A})$ coincide; a similar situation occurs for $\mathcal{A}_i$ and $L(\mathcal{A}_i)$. Now it suffices to apply the fact that the operations $\vee$ and $\wedge$ are defined by means of the basic operations of MV-algebra $\mathcal{A}$ (cf. 2.1). □

Now let us assume that we are given a direct product decomposition of the lattice $L(\mathcal{A}) = L$ of the form

(1) $$\psi_1 : L \to \prod_{i \in I} L_i.$$

If there exists a direct product decomposition

(2) $$\psi_2 : \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i$$

such that $\psi_2 = \psi_1$ and $L(\mathcal{A}_i) = L_i$ for each $i \in I$, then we say the direct product decomposition (1) induces the direct product decomposition (2).

The following assertion is obvious.

Lemma 6.2. *Let $\mathcal{A}$ be a pseudo MV-algebra, $L = L(\mathcal{A})$. Further, let L' be a lattice and let φ be an isomorphism of L onto L'. Then there exists a pseudo MV-algebra $\mathcal{A}'$ with $L(\mathcal{A}') = L'$ such that φ is an isomorphism of $\mathcal{A}$ onto $\mathcal{A}'$.*

Let $\mathcal{A}$ and $L(\mathcal{A})$ be as above. Assume that we have a direct product decomposition

$$\psi : L \to \prod_{i\in I} L_i \,. \tag{3}$$

For $i \in I$ we denote by 0^i and 1^i the least and the greatest element of L_i, respectively.

There exist elements e_i and e_i' in L such that

$$\begin{aligned} \psi(e_i)_i &= 1^i, \quad \psi(e_i)_j = 0^j \quad \text{for } j \in I \setminus \{i\}\,, \\ \psi(e_i')_i &= 0^i, \quad \psi(e_i')_j = 1^j \quad \text{for } j \in I \setminus \{i\}\,. \end{aligned}$$

Then we have

$$e_i \wedge e_i' = 0, \quad e_i \vee e_i' = 1\,.$$

Hence in view of 4.3 there exists a direct product decomposition

$$\varphi_i : \mathcal{A} \to \mathcal{X}_{i1} \times \mathcal{X}_{i2},$$

where (under the usual notation) we have

$$X_{i1} = [0, e_i], \quad X_{i2} = [0, e_i']$$

and for each $a \in A$,

$$\varphi_i(a)_1 = a \wedge e_i, \quad \varphi_i(a)_2 = a \wedge e_i'\,.$$

According to the isomorphism ψ we infer that the mapping

$$\psi_i : X_{i1} \to L_i$$

defined by

$$\psi_i(x) = \psi(x)_i \quad \text{for each } x \in X_{i1}$$

is an isomorphism of X_{i1} onto L_i.

Therefore in view of 6.2 we obtain

Lemma 6.3. *There is a pseudo MV-algebra $\mathcal{B}_i$ such that*

(i) *$L(\mathcal{B}_i) = L_i$;*

(ii) *ψ_i is an isomorphism of $\mathcal{X}_i$ onto $\mathcal{B}_i$.*

Consider the pseudo MV-algebra

$$\prod_{i\in I} \mathcal{B}_i = \mathcal{B}\,.$$

Then in view of 6.3 we have $L(\mathcal{B}) = \prod_{i \in I} L_i$.

Since all basic pseudo MV-operations in $\mathcal{B}$ are calculated component-wise, from 6.3 and from the direct product decomposition φ_i we infer that the mapping ψ is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$.

From this and from the fact that ψ is a bijection we conclude that ψ is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. In other words, we obtained a direct product decomposition

$$\psi : \mathcal{A} \to \prod_{i \in I} \mathcal{B}_i$$

which is induced by the direct product decomposition (3) of the lattice L.

Summarizing, we have

Theorem 6.4. *Let $\mathcal{A}$ be a pseudo MV-algebra and $L = L(\mathcal{A})$. Then each direct product decomposition of L induces a direct product deomposition of $\mathcal{A}$.*

According to [7], any two direct product decompositions of a lattice have isomorphic refinements. From this and from 6.4 we conclude

Theorem 6.5. *Any two direct product decompositions of a pseudo MV-algebra have isomorphic refinements.*

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