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NONZERO AND POSITIVE SOLUTIONS OF A SUPERLINEAR ELLIPTIC SYSTEM

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Abstract. In this paper we consider the existence of nonzero solutions of an undecoupling elliptic system with zero Dirichlet condition. We use Leray-Schauder Degree Theory and arguments of Measure Theory. We will show the existence of positive solutions and we give applications to biharmonic equations and the scalar case.

1. Introduction

In this paper we shall study the existence of nonzero solutions of the elliptic system

$$\text{(S)} \qquad \begin{aligned} -\Delta u &= \lambda u + \delta v + |u|^{r-1} \\ -\Delta v &= \theta u + \gamma v + |v|^{s-1} \end{aligned}$$

in Ω, where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, subject to Dirichlet boundary conditions, $u = v = 0$ on $\partial\Omega$, $\lambda, \delta, \gamma, \theta$ are real numbers and $2 < r, s < 2^* = \frac{2n}{n-2}$, $n \geq 4$.

Using a maximum principle, developed in [5], we can obtain positive solutions of the system (S). Also, we shall give applications to semilinear biharmonic equations of type $-\Delta^2 = \lambda u + |u|^{r-1}$ and the scalar case.

System (S) represents a steady state case of reaction-diffusion systems, that systems have been intensively studied during recent years, see [16] where many references can be found. In particular, existence and multiplicity of positive solutions have been considered because they are only with physical meaning.

Problem (S) has been attacked using a decoupling technique in [2], [4], [8] and [15] and has been used thereafter by many authors. That technique consists in reducing the system (S) to a single nonlinear equation containing an integral and a differential term. If we suppose that the nonlinearity of second equation of (S) is zero and $-\gamma > 0$ then the second equation can be solved for v in terms of u. Let

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us denote $\mathbf{B}$ its solution operator under Dirichlet boundary conditions. For each u we denote $\mathbf{B}(u)$ the solution of the problem $-\Delta v = \theta u + \gamma v$ in Ω, $v = 0$ on $\partial\Omega$. Then our problem becomes the one of finding u such that

$$(P, \lambda, B) \qquad \begin{aligned} -\Delta u - \delta \mathbf{B}(u) &= \lambda u + |u|^{r-1}, \quad \text{on } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

The spectrum of $\mathbf{B}$ and its properties has been studied in [4] where a maximum principle has been developed. The decoupling technique has some obvious shortcomings, for example it is very difficult to apply to systems with three or more equations. Even, in the case of two equations it is too restrictive to give conditions to solve the second equation of (S) for v in terms of u.

Our approach do not make use of that decoupling technique and allow us to give an unified treatment to cooperative and noncooperative cases. For simplicity, we are considering the 2×2 system (S), but our approach applies to more general $n \times n$ systems.

As complement to our results we refer to following papers: In [10] and [11] have been proved nonexistence results of positive super harmonic functions for $\Delta^2 = u^q$, in $\mathbb{R}^n$, $n \geq 3$. Results in the same directions can be found in [18] and [14]. Ni, in [13], has proven that $\Delta^2 = a(x)\, u^q$, in $\mathbb{R}^n$ does not have positive solutions or have positive solutions on depending of $a(x)$. See, also, [9], [12], [17], [18], [19], [20], [21] and references therein.

Letting

$$U = (u, v), \; -\overrightarrow{\Delta} U = \begin{pmatrix} -\Delta u \\ -\Delta v \end{pmatrix}, \; A = \begin{pmatrix} \lambda & \delta \\ \theta & \gamma \end{pmatrix}, \; G(U) = \begin{pmatrix} |u|^{r-1} \\ |v|^{s-1} \end{pmatrix}$$

we can write (S) as

$$\text{(P)} \qquad \begin{aligned} -\overrightarrow{\Delta} U &= A(U) + G(U), \quad \text{on } \Omega \\ U &= (0,0) = \Theta, \quad \text{on } \partial\Omega. \end{aligned}$$

Costa and Magalheães, [3], give a precise description of kernel of $\overrightarrow{\Delta} - A$. They proved that the kernel of $\overrightarrow{\Delta} - A$ is nonzero if and only if $A - \lambda_j \mathbf{I}$ is a singular matrix for some $\lambda_j \in \sigma(-\Delta)$.

Letting $\mathcal{L} = -\overrightarrow{\Delta} - A$ we can write, if it is possible, (P) as follows $U = \mathcal{L}^{-1}(G(U))$, on Ω, $U = (0,0)$, on $\partial\Omega$. Then the solutions of (S) are the fixed points of $\mathcal{L}^{-1} G$.

2. Preliminaries and Notations

Let $L^r(\Omega)$, $L^s(\Omega)$ be the Banach spaces with norms

$$\|u\|_r = \left(\int_\Omega |u|^r \right)^{\frac{1}{r}}, \quad \|u\|_s = \left(\int_\Omega |u|^s \right)^{\frac{1}{s}}$$

respectively. We call $\mathbf{L}^{r,\,s}(\Omega) = L^r(\Omega) \times L^s(\Omega)$ with the following norm $\|(u,v)\|_{r,s} = \|u\|_r + \|v\|_s$. We shall assume the following conditions:

$(C.1)$ $2 < r,\ s < \frac{2n}{n-2}$.

$(C.2)$ $n \geq 4$ and $n > \max\{2r^*, 2s^*\}$, where $\frac{1}{r} + \frac{1}{r^*} = 1$ and $\frac{1}{s} + \frac{1}{s^*} = 1$.

The $(C.i)$ conditions tell us that

$$r^* < r < \frac{nr^*}{n - 2r^*} \quad \text{and} \quad s^* < s < \frac{ns^*}{n - 2s^*}$$

and then the embedding

$$W^{2,\ r^*,\ s^*}(\Omega) \hookrightarrow \mathbf{L}^{r,\ s}(\Omega) \tag{2.1}$$

is compact, where $W^{2,\ r^*,\ s^*}(\Omega) = W^{2,\ r^*}(\Omega) \times W^{2,\ ,\ s^*}(\Omega)$ (cf. [1], p. 97). Then we have the following

Lemma 2.1. *The operator $\mathcal{L}:\ D(\mathcal{L}) \subset \mathbf{L}^{r^*,\ s^*}(\Omega) \to \mathbf{L}^{r^*,\ s^*}(\Omega)$, where $D(\mathcal{L}) = W^{2,\ r^*,\ s^*}(\Omega) \cap W_0^{1,\ r^*,\ s^*}(\Omega)$, is a linear and bijective operator if for all $\lambda_j \in \sigma(-\Delta)$ the matrix $A - \lambda_j \mathbf{I}$ is regular.*

Proof. Costa and Magalhães in [3] proved that $\mathcal{L}$ is an injective operator. Now, for $f \in \mathbf{L}^{r^*,\ s^*}(\Omega)$ the equation

$$\mathcal{L}(U) = f \tag{2.2}$$

is equivalent to $\left\{\mathbf{I}-\left(-\overrightarrow{\Delta}\right)^{-1} A\right\}(U) = \left(-\overrightarrow{\Delta}\right)^{-1}(f)$. Since $\left(-\overrightarrow{\Delta}\right)^{-1} A$ is compact and $\mathbf{I}-\left(-\overrightarrow{\Delta}\right)^{-1} A$ is injective, Fredholm's alternative tell us that (2.2) has a solution for all $f \in \mathbf{L}^{r^*,\ s^*}(\Omega)$. $\square$

By Lemma 2.1 and the embedding (2.1) we can consider : $\mathcal{L}^{-1} : \mathbf{L}^{r,\ s}(\Omega) \to \mathbf{L}^{r,\ s}(\Omega)$ as a linear, injective and compact operator.

Finally, the function G defines a Nemitsky's operator (denoted in the same form) $G : \mathbf{L}^{r,\ s}(\Omega) \to \mathbf{L}^{r^*,\ s^*}(\Omega)$ which is continuous and bounded, see [6], p. 26. So, the solutions of system (S) are the fixed points of $\mathcal{L}^{-1}G : \mathbf{L}^{r,\ s}(\Omega) \to \mathbf{L}^{r,\ s}(\Omega)$ which is bounded and compact as well.

3. Main Results

Throughout this paper we shall suppose that $(C.1)$ and $(C.2)$ conditions hold.

Proposition 3.1. *If, for all $\lambda_j \in \sigma(-\Delta)$, the matrix $A - \lambda_j I$ is regular, then there exists $\alpha > 0$ such that*

$$d_{LS}\left[\mathbf{I}-\mathcal{L}^{-1}G, B(\Theta, \alpha), \Theta\right] = 0 \tag{3.1}$$

Proof. It is sufficient to show the existence of $\alpha > 0$ such that for all $U = (u, v) \in \mathbf{L}^{r,\ s}(\Omega)$ and $\|U\|_{r,s} = \alpha$ we have

$$\left\|\mathcal{L}^{-1}\left(|u|^{r-1}, |v|^{s-1}\right)\right\|_{r,s} > \|U\|_{r,s}\ . \tag{3.2}$$

See [7], p. 104.

With arguments of Measure Theory we can show the existence of $a > 0$ such that

$$\left\| \mathcal{L}^{-1}\left(|u|^{r-1}, |v|^{s-1} \right) \right\|_{r,s} > a \tag{3.3}$$

for all $U = (u, v) \in \mathbf{L}^{r,\ s}(\Omega)$, $\|U\|_{r,s} = 1$. In fact: Suppose, on the contrary, that

$$\inf_{\|U\|_{r,s}=1} \left\{ \left\| \mathcal{L}^{-1}\left(|u|^{r-1}, |v|^{s-1} \right) \right\|_{r,s} \right\} = 0\,. \tag{3.4}$$

Let $\left\{ \left(|u_n|^{r-1}, |v_n|^{s-1} \right) \right\} \subset \mathbf{L}^{r^*,\ s^*}(\Omega)$, $\|u_n\|_r + \|v_n\|_s = 1$, a minimizing sequence of (3.4) . Since

$$\left\| |u_n|^{r-1} \right\|_{r^*} = \|u_n\|_r^{r-1} \leq 1$$
$$\left\| |v_n|^{s-1} \right\|_{s^*} = \|v_n\|_s^{s-1} \leq 1\,,$$

there exists $(u, v) \in \mathbf{L}^{r^*,\ s^*}(\Omega)$ such that $|u_n|^{r-1} \rightharpoonup u$ and $|v_n|^{s-1} \rightharpoonup v$. Now, since $\mathcal{L}^{-1}$ is an injective and compact operator then $u = 0$ and $v = 0$. So we have

$$\left(\int_\Omega |u_n|^r \right)^{\frac{1}{r}} + \left(\int_\Omega |v_n|^s \right)^{\frac{1}{s}} = 1 \tag{3.5}$$

and

$$\int_\Omega |u_n|^{r-1} \to 0 \quad \text{and} \quad \int_\Omega |v_n|^{s-1} \to 0\,. \tag{3.6}$$

By Egorov's Theorem, for all $\mu > 0$ there exists $\Omega_\mu \subset \Omega$, $|\Omega_\mu| > |\Omega| - \mu$, such that $(u_n, v_n) \to \Theta$ uniformly on Ω_μ. Then, for all $\epsilon > 0$ there exists $N = N(\epsilon, \mu) \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|u_n(x)| \leq \frac{\epsilon}{|\Omega_\mu|} \quad \text{and} \quad |v_n(x)| \leq \frac{\epsilon}{|\Omega_\mu|}\,, \tag{3.7}$$

for all $x \in \Omega_\mu$.

If we choose ϵ sufficiently small, we get

$$|u_n(x)|^r < |u_n(x)|^{r-1} < \frac{\epsilon}{|\Omega_\mu|} \tag{3.8}$$

and

$$|v_n(x)|^s < |v_n(x)|^{s-1} < \frac{\epsilon}{|\Omega_\mu|}\,,$$

for all $x \in \Omega_\mu$.

Using (3.7) we obtain $\int_\Omega |u_n|^r < \epsilon$ and $\int_\Omega |v_n|^s < \epsilon$. So we have

$$\left(\int_{\Omega_\mu} |u_n|^r \right)^{\frac{1}{r}} + \left(\int_{\Omega_\mu} |v_n|^s \right)^{\frac{1}{s}} < \epsilon^{\frac{1}{r}} + \epsilon^{\frac{1}{s}}\,. \tag{3.9}$$

From (3.9) and (3.5) we get

$$\left(\int_{\Omega_\mu^c} |u_n|^r\right)^{\frac{1}{r}} + \left(\int_{\Omega_\mu^c} |v_n|^s\right)^{\frac{1}{s}} > 1 - \left\{\epsilon^{\frac{1}{r}} + \epsilon^{\frac{1}{s}}\right\}. \tag{3.10}$$

Let $n \geq N$ and suppose that

$$\int_\Omega |u_n|^{r-1} < \frac{\epsilon}{2} \quad \text{and} \quad \int_\Omega |v_n|^{s-1} < \frac{\epsilon}{2}. \tag{3.11}$$

From (3.11) and (3.5) we get the existence of $c > 0$ such that

$$\begin{aligned} &\int_\Omega |u_n|^{r-1}(|u_n| - 1) + \int_\Omega |v_n|^{s-1}(|v_n| - 1) \\ &\geq \int_\Omega |u_n|^r + \int_\Omega |v_n|^s - \epsilon \geq c - \epsilon. \end{aligned} \tag{3.12}$$

From (3.12) we conclude that $W_n = \{x \in \Omega;\ |u_n(x)| > 1 \ \text{ or } \ |v_n(x)| > 1\}$ satisfies $|W_n| > 0$. Let $W = \bigcup W_n$, then by (3.8) and for ϵ small enough we see that $W \subset \Omega_\mu^c$. Now,

$$\int_{\Omega_\mu^c - W} |u_n|^r + \int_{\Omega_\mu^c - W} |v_n|^s \leq \left|\Omega_\mu^c - W\right| < \left|\Omega_\mu^c\right|. \tag{3.13}$$

By (3.10) and (3.13) we see that

$$\int_W |u_n|^r + \int_W |v_n|^s > 1 - \left\{\epsilon^{\frac{1}{r}} + \epsilon^{\frac{1}{s}}\right\} - |\Omega_\mu| \tag{3.14}$$

and

$$\int_W |u_n|^{r-1} \to 0 \quad \text{and} \quad \int_W |v_n|^{s-1} \to 0. \tag{3.15}$$

From (3.14), (3.15) and Egorov's Theorem we can conclude the existence of $W_t \subset W$ such that $|W_t| > |W| - t$ and $N_0 = N_0(\epsilon, t) \in \mathbb{N}$ such that

$$|u_n(x)|^r < |u_n(x)|^{r-1} < \frac{\epsilon}{|W_t|} \tag{3.16}$$

and

$$|v_n(x)|^s < |v_n(x)|^{s-1} < \frac{\epsilon}{|W_t|},$$

for all $x \in W_t$ and $n \geq N_0$.

If we choose ϵ small enough such that $\frac{\epsilon}{|W_t|} < 1$ we get a contradiction between (3.16) and our definition of W. So we prove that (3.3) holds.

We can suppose that $r \geq s$, then from (3.3) we obtain

$$\left\|\mathcal{L}^{-1}\left(\frac{|u|^{r-1}}{\|U\|_{r,s}^{r-s}}, |v|^{s-1}\right)\right\|_{r,s} > a\,\|U\|_{r,s}^{s-1}, \tag{3.17}$$

for all $U \neq \Theta$. Now, since $s > 2$ then $a\left\|U\right\|_{r,s}^{s-1} > \left\|U\right\|_{r,s}$, for $\left\|U\right\|_{r,s}$ large enough. Letting

$$w = \frac{u}{\left\|U\right\|_{r,s}^{\frac{r-s}{r-1}}},$$

(3.17) become, for $\left\|U\right\|_{r,s}$ large enough,

$$\left\|\mathcal{L}^{-1}\left(\left|w\right|^{r-1}, \left|v\right|^{s-1}\right)\right\|_{r,s} > \left\|U\right\|_{r,s}. \tag{3.18}$$

Now, if we call $V = (w, v)$ we see that $\left\|U\right\|_{r,s} > \left\|V\right\|_{r,s}$ and then, by (3.18), we get

$$\left\|\mathcal{L}^{-1}\left(\left|w\right|^{r-1}, \left|v\right|^{s-1}\right)\right\|_{r,s} > \left\|V\right\|_{r,s}. \tag{3.19}$$

The proof is completed. □

Theorem 3.1. *If for all $\lambda_j \in \sigma(-\Delta)$, $A - \lambda_j I$ is a regular matrix then System* (S) *has at least a nonzero solution. In addition, if we assume that*

(H.1) $\lambda,\ \gamma < \lambda_1$ *and* $\delta,\ \theta \geq 0$

(H.2) $\det(\lambda_1 \mathbf{I} - \mathbf{A}) > 0$

then System (S) *has a positive solution.*

Proof. It is clear that

$$\operatorname{Ind}\ \left[\mathbf{I} - \mathcal{L}^{-1}(G(.)),\ \Theta\right] = 1. \tag{3.20}$$

By (3.20), Proposition 3.1 and making use of the domain decomposition property of Degree Theory we get the existence of a nonzero solution of (S).

To see our second affirmation we proceed as follows: As in Theorem 1 of [5], we multiply the first and second equation of (S) by $\widetilde{u} = \max\{0, -u\}$ and $\widetilde{v} = \max\{0, -v\}$ respectively and we get

$$\int_\Omega -\Delta u\ u = \int_\Omega \nabla u\ \nabla\widetilde{u} = -\int_\Omega \left|\nabla\widetilde{u}\right|^2 = -\lambda\int_\Omega \left|\widetilde{u}\right|^2 + \delta\int_\Omega v\widetilde{u} + \int_\Omega \left|u\right|^{r-1}\widetilde{u}$$

which produces

$$\lambda_1\int_\Omega \left|\widetilde{u}\right|^2 \leq \int_\Omega \left|\nabla\widetilde{u}\right|^2 = \lambda\int_\Omega \left|\widetilde{u}\right|^2 - \delta\int_\Omega v\widetilde{u} - \int_\Omega \left|u\right|^{r-1}\widetilde{u} \leq \lambda\int_\Omega \left|\widetilde{u}\right|^2 + \delta\int_\Omega \widetilde{v}\widetilde{u}.$$

By Cauchy-Schwarz inequality combined with (H .1) we have

$$(\lambda_1 - \lambda)\int_\Omega \left|\widetilde{u}\right|^2 \leq \delta\left\|\widetilde{u}\right\|_2\left\|\widetilde{v}\right\|_2.$$

And similarly

$$(\lambda_1 - \gamma) \int_\Omega |\widetilde{v}|^2 \leq \theta \, \|\widetilde{u}\|_2 \, \|\widetilde{v}\|_2 \, .$$

So that

$$\{(\lambda_1 - \lambda)(\lambda_1 - \gamma) - \delta\theta\} \, \|\widetilde{u}\|_2^2 \, \|\widetilde{v}\|_2^2 \leq 0 \, . \tag{3.21}$$

Condition (H.2) together (3.21) implies $\widetilde{u} = 0$ or $\widetilde{v} = 0$ and then $\widetilde{u} = 0$ and $\widetilde{v} = 0$. By regularity we conclude that $u, \; v \geq 0$. □

Remark 1. With aid of Theorem 3.1 we can study the scalar case, that is to say

$$\begin{aligned} -\Delta v &= \gamma v + |v|^{s-1} \, , \quad \text{on} \quad \Omega \\ v &= 0 \, , \quad \text{on} \quad \partial\Omega \, . \end{aligned} \tag{3.22}$$

Let

$$\begin{aligned} -\Delta u &= \lambda u + 0v \\ -\Delta v &= \theta u + \gamma v + |v|^{s-1} \end{aligned} \tag{3.23}$$

on Ω, $u = v = 0$ on $\partial\Omega$.

If we assume that $\lambda \notin \sigma(-\Delta)$ it is well known that only the zero function is a solution of the first equation of (3.23) and then (3.23) is reduced to (3.22). We summarize all this in the following

Theorem 3.2. *Suppose that there exists $\lambda \in R$ such that $\lambda \notin \sigma(-\Delta)$ and $\frac{\lambda+\gamma}{2} \pm \sqrt{\left(\frac{\lambda-\gamma}{2}\right)^2} \notin \sigma(-\Delta)$, then (3.22) has at least a nonzero solution. If, in addition, $\lambda, \; \gamma < \lambda_1$ then (3.22) has a positive a positive solution.*

Proof. It is only necessary to observe that the eigenvalues of matrix A associated to (3.23) are $\frac{\lambda+\gamma}{2} \pm \sqrt{\left(\frac{\lambda-\gamma}{2}\right)^2}$. Now, we apply Theorem 3.1 and we get our assertion. □

Remark 2. Theorem 3.1 can be applied to the following biharmonic equation under Navier and Dirichlet conditions:

$$-\Delta^2 u = \theta u + |u|^{r-1} \, , \quad \text{on} \quad \Omega \, , \tag{3.24}$$

and $u = \Delta u = 0$ on $\partial\Omega$. In fact (3.24) can be put as

$$\begin{aligned} -\Delta u &= 0u + (-1)\, v \\ -\Delta v &= \theta u + 0v + |v|^{s-1} \, . \end{aligned} \tag{3.25}$$

Then we have the following

Theorem 3.3. *Suppose that $\pm\sqrt{-\theta} \notin \sigma(-\Delta)$ then (3.24) has at least a nonzero solution.*

Proof. The proof is a straightforward application of Theorem 3.1.

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