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THE VALUATED RING OF THE ARITHMETICAL FUNCTIONS AS A POWER SERIES RING

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Abstract. The paper examines the ring A of arithmetical functions, identifying it to the domain of formal power series over $\mathbf{C}$ in a countable set of indeterminates. It is proven that A is a complete ultrametric space and all its continuous endomorphisms are described. It is also proven that A is a quasi-noetherian ring.

In [2], E. D. Cashwell and C. J. Everett have proved that the set of all arithmetical functions A constitutes a unique factorization domain under ordinary addition and Dirichlet product defined by:

$$(f * g)(n) = \sum_{d|n} f(d) g(\frac{n}{d}).$$

This domain is isomorphic to the domain A' of formal power series over $\mathbf{C}$ in a countable set of indeterminates. The authors of [2] have proved that the theorem on unique factorization into primes, up to ordering and units, holds in A' and hence must hold in A.

Let the primes of $\mathbf{N}^*$ be listed in any definite order $p_1,\ p_2, \dots,\ p_s, \dots$ and let

$$A' = \{F = \sum_{k=0}^{\infty} a_{k+1} X_1^{\alpha_1(k+1)} X_2^{\alpha_2(k+1)} \dots X_s^{\alpha_s(k+1)} \dots , \tag{1}$$

$$\text{where } a_{k+1} \in \mathbf{C} \text{ and } p_1^{\alpha_1(k+1)} \dots p_s^{\alpha_s(k+1)} \dots = k+1 \} .$$

We emphasize that the only restriction of these series is that only a finite number of X_i actually appear (i. e. have $\alpha_i(k+1) > 0$) in any term. Then $\varphi : A \to A'$ defined by:

$$\varphi(f) = \sum_{k=0}^{\infty} f(k+1) X_1^{\alpha_1(k+1)} X_2^{\alpha_2(k+1)} \dots X_s^{\alpha_s(k+1)} \dots \quad (\forall) f \in A \tag{2}$$

is a ring isomorphism ([2]).

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We choose $\alpha \in (0,1)$ and we define a discrete nonarchimedian valuation of rank 1, $v' : A' \to \mathbf{R} \cup \{\infty\}$, such that

$$v'(F) = \begin{cases} -\log_\alpha(\min\{k+1 | a_{k+1} \neq 0\}) & \text{if} \quad F \in A' \setminus \{0\} \\ \infty & \text{if} \quad F = 0\,. \end{cases} \tag{3}$$

Then, putting

$$|F|' = \alpha^{v'(F)} = \begin{cases} \frac{1}{\min\{k+1 | a_{k+1} \neq 0\}} & \text{if} \quad F \in A' \setminus \{0\} \\ 0 & \text{if} \quad F = 0 \end{cases} \tag{4}$$

we get a nonarchimedian norm on A', and if we put

$$d'(F,G) = |F - G|' \quad (\forall) F, G \in A' \tag{5}$$

the pair (A', d') becomes an ultrametric space.

We may now define a nonarchimedian valuation $v : A \to \mathbf{R} \cup \{\infty\}$ as follows:

$$v(f) = v'(\varphi(f)) \quad (\forall) f \in A\,. \tag{6}$$

Consequently, we obtain a nonarchimedian norm on A

$$|f| = \alpha^{v(f)} \quad (\forall) f \in A \tag{7}$$

and also a distance

$$d(f,g) = \alpha^{v(f-g)} \quad (\forall) f, g \in A\,. \tag{8}$$

With respect to the distance d, A becomes an ultrametric space. Moreover, φ is an isometry between (A, d) and (A', d'). The topology notions relative to A will always refer to the canonically defined ones, by the ultrametric d.

Theorem 1. *A is a complete ultrametric space. Moreover, there exists in A a countable set $\{\pi_k\}_{k \in \mathbf{N}^*}$ such that the $\mathbf{C}$-algebra generated by these elements $\mathbf{C}[\pi_1, \pi_2, \ldots, \pi_n, \ldots]$ is dense in A, and the set*

$$\{\pi_1^{\alpha_1(k+1)} \pi_2^{\alpha_2(k+1)} \ldots \pi_s^{\alpha_s(k+1)} \ldots, k \in \mathbf{N}, k+1 = p_1^{\alpha_1(k+1)} p_2^{\alpha_2(k+1)} \ldots p_s^{\alpha_s(k+1)} \ldots\}$$

represents a Schauder base in the $\mathbf{C}$-algebra A, that is, every $f \in A$ may be written as a convergent series

$$f = \sum_{k=0}^{\infty} f(k+1) \pi_1^{\alpha_1(k+1)} \pi_2^{\alpha_2(k+1)} \ldots \pi_s^{\alpha_s(k+1)}\,, \tag{9}$$
$$\text{where } k+1 = p_1^{\alpha_1(k+1)} p_2^{\alpha_2(k+1)} \ldots p_s^{\alpha_s(k+1)}\,.$$

Proof. Because φ is an isometry and it is also an isomorphism of $\mathbf{C}$-algebras, it is sufficient to prove the statements for the $\mathbf{C}$-algebra A'. We will prove first that A' is a complete ultrametric space. Let

$$\{G_n = \sum_{k=0}^{\infty} a_{k+1,n} X_1^{\alpha_1(k+1)} X_2^{\alpha_2(k+1)} \ldots X_s^{\alpha_s(k+1)}\}_{n \in \mathbf{N}}$$

be a Cauchy sequence. Then for every $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbf{N}$ such that $|G_m - G_n|' < \varepsilon$ for every $m, n \geq n_0(\varepsilon)$. Thus $(\forall) k \in \mathbf{N}$ $(\exists) n_0(k) \in \mathbf{N}$ such that $a_{l+1,m} = a_{l+1,n}$ $(\forall) l \in \{0, 1, \ldots, k\}$, $(\forall) m, n \geq n_0(k)$. We may assume that for every $k \in \mathbf{N}$, $n_0(k)$ is the smallest natural number with the before-mentioned property. Then

$$n_0(0) \le n_0(1) \le n_0(2) \le \dots .$$

We consider

$$G = \sum_{k=0}^{\infty} a_{k+1,n_0(k)} X_1^{\alpha_1(k+1)} X_2^{\alpha_2(k+1)} \dots X_s^{\alpha_s(k+1)} , \tag{10}$$

and it is obvious that G_n converges to G. So, A' is a complete ultrametric space and the same is also true for A.

Now, for every $F \in A'$ we may write

$$\begin{aligned} F &= \sum_{k=0}^{\infty} a_{k+1} X_1^{\alpha_1(k+1)} X_2^{\alpha_2(k+1)} \dots X_s^{\alpha_s(k+1)} \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} a_{k+1} X_1^{\alpha_1(k+1)} X_2^{\alpha_2(k+1)} \dots X_s^{\alpha_s(k+1)} . \end{aligned}$$

Denoting by $\pi_i = \varphi^{-1}(X_i)$ $(\forall) i \in \mathbf{N}^*$, we observe that $\pi_i(n) = \delta_{p_i,n}$ $(\forall) i, n \in \mathbf{N}^*$, that the set $\{\pi_1, \pi_2, \dots, \pi_i, \dots\}$ is contained in the maximal ideal of the local ring A, and that $|\pi_i| = \frac{1}{p_i}$ $(\forall) i \in \mathbf{N}^*$. From these remarks, it follows that the general term of the series (9) converges to zero and therefore this series is a convergent one.

Keeping in mind that φ is an isometry of $\mathbf{C}$-algebras, Theorem 1 is completely proved. $\square$

It is well known that A is not a noetherian ring. We will prove that A is, however, a quasi-noetherian ring. We need some definitions.

Definition 1. *A valuated ring A is called B-ring if*
i) $|x| \le 1$, $(\forall) x \in A$ *;*
ii) $(\forall) x \in A$ *with* $|x| = 1$*, it results that x is a unit in A.*

Definition 2. *A B-ring A is called quasi-noetherian if every ideal $I \subset A$ is quasi-finite, that is $(\exists) a_k \in A$ $(k \in \mathbf{N}^*)$ with $\lim_{k\to\infty} a_k = 0$ such that every $a \in I$ can be written as a sum of a convergent series*

$$a = \sum_{k=1}^{\infty} c_k a_k , \quad c_k \in A . \tag{11}$$

Theorem 2. *A is a quasi-noetherian ring.*

Proof. It is obvious that A is a B-ring. It is known ([1], p. 56) that a B-ring A is quasi-noetherian if $\sup_{f\in M} |f| < 1$ and M, the maximal ideal of A, is quasi-finite.

If $f \in M$, then $|f| \le \frac{1}{2} < 1$. From Theorem 1 we deduce that M is quasi-finite. Hence A is a quasi-noetherian ring. $\square$

As a last result, we will describe the shape of the continuous $\mathbf{C}$-endomorphisms of A.

Theorem 3. *Every continuous endomorphism θ of the* **C***-algebra A is defined by:*

$$\theta(\pi_i) = \gamma_i, \ i \in \mathbf{N}^*, \ \textit{where} \tag{12}$$

$$\lim_{k+1=p_1^{\alpha_1(k+1)}p_2^{\alpha_2(k+1)}\dots p_s^{\alpha_s(k+1)}\to\infty} \gamma_1^{\alpha_1(k+1)}\gamma_2^{\alpha_2(k+1)}\dots\gamma_s^{\alpha_s(k+1)} = 0\,.$$

Proof. In order to define a continuous endomorphism θ of the **C**-algebra A, it is sufficient to define θ on the set $\{\pi_k\}_{k\in\mathbf{N}^*}$.

The statement of Theorem 3 results now from Theorem 1. □

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