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EXISTENCE OF EXTREMAL PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION INCLUSIONS

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ABSTRACT. We consider a nonlinear evolution inclusion defined in the abstract framework of an evolution triple of spaces and we look for extremal periodic solutions. The nonlinear operator is only pseudomonotone coercive. Our approach is based on techniques of multivalued analysis and on the theory of operators of monotone-type. An example of a parabolic distributed parameter system is also presented.

1. INTRODUCTION

In this paper we prove the existence of extremal periodic solutions for nonlinear evolution inclusions defined in the framework of an evolution triple of spaces.

The periodic problem for nonlinear evolution equations with a single-valued pertubation term, was examined recently by Vrabie [10] and Hirano [1]. Vrabie assumes that the nonlinear operator is time-invariant and satisfies the requirement that $A - \lambda I$ is m-accretive for some $\lambda > 0$, while the single-valued perturbation term $f(t,x)$ satisfies a pointwise asymptotic growth condition. Hirano considers an evolution equation defined on a Hilbert space and driven by a time-invariant subdifferential operator which generates a compact semigroup of nonlinear contractions. The single-valued perturbation $f(t,x)$ has sublinear growth and satisfies a unilateral condition (a coercivity-type condition). For evolution inclusions we have the works of Hu-Papageorgiou [2], Lakshmikantham-Papageorgiou [5], Kandilakis-Papageorgiou [4], Papageorgiou-Papalini-Renzacci [9]. All four papers deal with the "convex problem" (i.e. the multivalued perturbation $F(t,x)$ is convex-valued), work (as we do here) within the framework of evolution triples and the nonlinear operator satisfies monotonicity-type hypotheses

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(in Hu-Papageorgiou, Lakshmikantham-Papageorgiou and Papageorgiou-Papalini-Renzacci $A(t,\cdot)$ is maximal monotone and in Kandilakis-Papageorgiou is pseudomonotone). Hu-Papageorgiou work with Hilbert spaces and employ a Nagoumo-type tangential condition and Galerkin approximations. Lakshmikantham-Papageorgiou, prove the multivalued analog of Vrabie's theorem. Finally Kandilakis-Papageorgiou and Papageorgiou-Papalini-Renzacci base their approach on the theory of nonlinear operators of monotone type. Here we go beyond these works and consider "nonconvex" evolution inclusions. In fact we replace the multivalued term $F(t,x)$ by $\operatorname{ext} F(t,x)$(= the extreme points of the set $F(t,x)$). Recall that even if $x \to F(t,x)$ is regular enough (say Hausdorff-continuous (h-continuous), see section 2), $x \to \operatorname{ext} F(t,x)$ need not have any meaningful continuity properties and furthermore, the set $\operatorname{ext} F(t,x)$ need not be even closed (see Hu-Papageorgiou [3]).

2. Preliminaries

In this section we recall from multivalued analysis and from the theory of evolution equations some basic definitions and facts that we will need in the sequel. Our basic references are the books of Hu-Papageorgiou [3] and Zeidler [11].

Let (Ω, Σ) be a measurable space and X a separable Banach space. By $P_{f(c)}(X)$ we denote the collection of all subsets C of X which are nonempty, closed (and convex). Also for $C \subseteq X$ nonempty,

$$|C| = \sup\{\|x\| : \ x \in C\}$$

and for every $x^* \in X^*$,

$$\sigma(x^*, C) = \sup\left[\langle x^*, c\rangle \ : \ c \in C\right] \ \text{(the support function of } C) \, .$$

A multifunction

$$F : \Omega \to P_f(X)$$

is said to be measurable, if for all $x \in X$,

$$\omega \to d(x, F(\omega)) = \inf\{\|x - z\| : \ z \in F(\omega)\}$$

is measurable. A multifunction $F : \Omega \to 2^X \setminus \emptyset$ is said to be graph measurable, if

$$GrF = \{(\omega, x) \in \Omega \times X : \ x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X)$$

with $\mathcal{B}(X)$ being the Borel σ-field of X. For multifunctions with values in $P_f(X)$, measurability implies graph measurability, while the converse is true if there is a σ-finite measure μ on (Ω, Σ) with respect to which Σ is complete. For $1 \le p \le \infty$, by S_F^p we denote the set of all selectors of F which belong to the Lebesgue-Bochner space $L^p(\Omega, X)$, i.e.

$$S_F^p = \{f \in L^p(\Omega, X) : \ f(\omega) \in F(\omega) \ \mu - \text{a.e.}\} \ .$$

It is easy to check that for a graph measurable multifunction $F : \Omega \to 2^X \setminus \emptyset$ the set S_F^p is nonempty if and only if $\omega \to \inf\{\|x\| : \ x \in F(\omega)\}$ is majorized by an $L^p(\Omega)$-function.

On $P_f(X)$ we can define a generalized metric, known as the "Hausdorff metric", by setting

$$h(C,E) = \max\left\{\sup_{c\in C} d(c,E),\ \sup_{e\in E} d(e,C)\right\}.$$

The space $(P_f(X),h)$ is a complete metric space and $P_{fc}(X)$ is a closed subspace of it. If Y is a Hausdorff topological space, a multifunction $F: Y \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from Y into $(P_f(X),h)$.

Let $T=[0,b]$. By $L^1_w(T,X)$ we denote the Lebesgue-Bochner space $L^1(T,X)$ furnished with the weak norm

$$||g||_w = \sup\left\{||\int_s^t g(\tau)d\tau|| :\ 0\le s\le t\le b\right\},\ g\in L^1(T,X).$$

A set $K\subseteq L^p(T,X)$ $(1\le p\le\infty)$ is said to be "decomposable", if for all $g_1,\ g_2\in K$ and all $C\subseteq T$ measurable we have

$$\chi_C g_1 + \chi_{C^c} g_2 \in K$$

(here by χ_C we denote the characteristic function of C).

Let H be a separable Hilbert space and X a dense subspace carrying the structure of a separable reflexive Banach space, which is embedded continuously into H. Identifying H with its dual (pivot space), we have $X\subseteq H\subseteq X^*$ with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as "evolution triple". We will assume that the embedding of X into H is compact, a situation which is often satisfied in concrete applications. Note that this implies that the embedding of H into X^* is compact too. By $|\cdot|$ (resp. $||\cdot||,\ ||\cdot||_*$) we denote the norm of H (resp. of $X,\ X^*$), by $(\cdot,\cdot)$ the inner product of H and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X,X^*). The two are compatible in the sense that $\langle\cdot,\cdot\rangle|_{H\times X} = (\cdot,\cdot)$. For $1<p,\ q<\infty,\ \frac{1}{p}+\frac{1}{q}=1$, we define

$$W_{pq}(T) = \{x\in L^p(T,X):\ \dot{x}\in L^q(T,X^*)\}.$$

The time derivative involved in this definition is understood in the sense of vector-valued distributions. Equipped with the norm

$$||x||_{W_{pq}} = \left\{||x||_p^2 + ||\dot{x}||_q^2\right\}^{\frac{1}{2}}$$

$W_{pq}(T)$ becomes a separable, reflexive Banach space (a Hilbert space if $p=q=2$, in which case we write $W_{22}(T)=W(T)$). It is well known that $W_{pq}(T)$ is embedded continuously in $C(T,H)$ and since we have assumed that the embedding of X into H is compact, we have that $W_{pq}(T)$ is embedded compactly in $L^p(T,H)$. Also recall that $L^p(T,X)^* = L^q(T,X^*)$ and the duality brackets for the pair $L^p(T,X),\ L^q(T,X^*))$, denoted by $((\cdot,\cdot))$, are given by

$$((u,f)) = \int_0^b \langle u(t), f(t)\rangle\, dt$$

for all $u\in L^q(T,X^*)$ and $f\in L^p(T,X)$.

An operator $A : X \to X^*$ is said to be "demicontinuous", if $x_n \to x$ in X, implies that $A(x_n) \xrightarrow{w} A(x)$ in X^* as $n \to \infty$. Also we say that A is "pseudomonotone", if $x_n \xrightarrow{w} x$ and $\overline{\lim}\langle A(x_n), x_n - x\rangle \leq 0$, imply that for all $y \in X$, $\langle A(x), x - y\rangle \leq \underline{\lim}\langle A(x_n), x_n - y\rangle$. If A is bounded (maps bounded sets to bounded ones), then the above definition is equivalent to saying that if $x_n \xrightarrow{w} x$ in X, $A(x_n) \xrightarrow{w} u$ in X^* and $\overline{\lim}\langle A(x_n), x_n - x\rangle \leq 0$, then $A(x) = u$ and $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$ as $n \to \infty$ (generalized pseudomonotonicity).

3. Existence theorem

Let $T = [0, b]$ and let $X \subseteq H \subseteq X^*$ be an evolution triple of spaces with the embedding of X into H being compact. The problem under consideration is the following:

$$\left\{ \begin{array}{rcl} \dot{x} + A(t, x(t)) & \in & \operatorname{ext} F(t, x(t)) \text{ a.e. on T} \\ x(0) & = & x(b)\,. \end{array} \right\} \tag{1}$$

Our hypotheses on the data of (1) are the following:

$\mathbf{H(A)}$: $A : T \times X \to X^*$ is an operator such that
(i) for every $x \in X$, $t \to A(t, x)$ is measurable;
(ii) for almost all $t \in T$, $x \to A(t, x)$ is demicontinuous, pseudomonotone;
(iii) for almost all $t \in T$ and all $x \in X$, we have

$$\|A(t, x)\|_* \leq a(t) + c\|x\|^{p-1}$$

with $a \in L^q(T)$, $c > 0$, $2 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$;
(iv) for almost all $t \in T$ and all $x \in X$,

$$\langle A(t, x), x\rangle \geq c_1\|x\|^p - a_1(t)$$

with $c_1 > 0$ and $a_1 \in L^1(T)_+$.

$\mathbf{H(F)}$: $F : T \times H \to P_{fc}(H)$ is a multifunction such that
(i) for all $x \in H$, $t \to F(t, x)$ is measurable;
(ii) for almost all $t \in T$, $x \to F(t, x)$ is h-continuous;
(iii) for almost all $t \in T$ and all $x \in H$

$$|F(t, x)| \leq a_2(t) + c_2|x|^{p-1}$$

with $a_2 \in L^q(T)_+$, $c_2 > 0$ and $c_2 \leq c_1\beta^p$ where $\beta > 0$ is such that $\beta|\cdot| \leq \|\cdot\|$;
(iv)

$$\int_0^b \overline{\lim}_{|x|\to\infty} \frac{\sigma(x, F(t, x)) - \langle A(t, x), x\rangle}{|x|^2}\, dt < 0\,.$$

In what follows we will need the following simple observation:

Lemma 1. *If* $\{g_n, g\}_{n\geq 1} \subseteq L^q(T, H)$, $\sup_{n\geq 1}\|g_n\|_q < \infty$ *and* $g_n \xrightarrow{\|\cdot\|_w} g$ *as* $n \to \infty$ *then* $g_n \xrightarrow{w} g$ *in* $L^q(T, H)$ *as* $n \to \infty$.

Proof. Let $s(t) = \sum_{k=1}^{N} \chi_{(t_{k-1},t_k)}(t) v_k$ with $t_0 = 0 < t_1 < ... < t_{N-1} < t_N = b$, $v_k \in H$ and $N \geq 1$. Let $(\cdot,\cdot)_{pq}$ denote the duality brackets for the pair $(L^p(T,H), L^q(T,H))$. We have

$$\begin{aligned} |(s, g_n - g)_{pq}| &\leq \sum_{k=1}^{N} |\int_{t_{k-1}}^{t_k} (g_n(s) - g(s))\, ds|.|v_k| \\ &\leq \|g_n - g\|_w \sum_{k=1}^{N} |v_k| \to 0 \text{ as } n \to \infty\,. \end{aligned}$$

Since step functions are dense in $L^p(T,H)$ and $\sup_{n\geq 1} \|g_n\|_q < \infty$, we conclude that

$$(u, g_n - g)_{pq} \to 0$$

for all $u \in L^p(T,H)$, hence $g_n \xrightarrow{w} g$ in $L^q(T,H)$ as $n \to \infty$. $\square$

By a "solution" of (1), we mean a function $x \in W_{pq}(T)$ such that

$$\left\{ \begin{array}{rcl} \dot{x} \ + \ A(t, x(t)) &=& g(t) \text{ a.e. on } T \\ x(0) &=& x(b)\,, \end{array} \right\}$$

with $g \in S^q_{\text{ext}\, F(\cdot, x(\cdot))}$.

Recall that $W_{pq}(T) \subseteq C(T,H)$ and so the periodic boundary conditions make sense. In the next proposition we derive a uniform a priori bound for the solutions of (1), which are known as "extremal periodic solutions".

Proposition 1. *If hypotheses $H(A)$, $H(F)$ hold, then there exists $M_1 > 0$ such that for all $x \in W_{pq}(T)$ solution of* (1) *and all $t \in T$ we have $|x(t)| \leq M_1$.*

Proof. Suppose not. Then we can find solutions $x_n \in W_{pq}(T)$, $n \geq 1$, of (1) such that

$$\|x_n\|_{C(T,H)} \geq n\,.$$

By definition we have

$$\left\{ \begin{array}{rcl} \dot{x}_n \ + \ A(t, x_n(t)) &=& g_n(t) \text{ a.e. on } T \\ x_n(0) &=& x_n(b), \ \ g_n \in S^q_{\text{ext}\, F(\cdot, x_n(\cdot))}\,. \end{array} \right\}$$

We take the duality brackets of this equation with $x_n(t)$. We obtain:

$$\langle \dot{x}_n(t), x_n(t) \rangle + \langle A(t, x_n(t)), x_n(t) \rangle = (g_n(t), x_n(t)) \text{ a.e. on } T\,,$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} |x_n(t)|^2 = (g_n(t), x_n(t)) - \langle A(t, x_n(t)), x_n(t) \rangle \text{ a.e. on } T\,.$$

Extend $x_n(\cdot)$, $g_n(\cdot)$, $A(\cdot, x)$, $a_2(\cdot)$, $c_2(\cdot)$ (see hypothesis $H(F)(iii)$) by b-periodicity on all of $\mathbb{R}$. Then divide by $1 + |x_n(t)|^2$ and integrate over $[s,t]$, $s \in$

$\mathbb{R}$, $t \in [s, s+b]$. We obtain

$$\begin{aligned}
&\frac{1}{2}\ln(1+|x(t)|^2) - \frac{1}{2}\ln(1+|x(s)|^2) \\
&\qquad \le \int_s^t \frac{(g_n(\tau), x_n(\tau)) - \langle A(\tau, x_n(\tau)), x_n(\tau)\rangle}{1+|x_n(\tau)|^2}\, d\tau \\
&\qquad \le \int_s^t \frac{a_2(\tau)|x_n(\tau)| + c_2|x_n(\tau)|^p - c_1\beta^p|x_n(\tau)|^p + a_1(\tau)}{1+|x_n(\tau)|^2}\, d\tau \\
&\qquad \le 2(\|a_2\|_1 + \|a_1\|_1) = M_2 < \infty \text{ (since } c_2 \le c_1\beta^p) .
\end{aligned}$$

So for $s \in \mathbb{R}$, $t \in [s, s+b]$, we have

$$\begin{aligned}
\frac{1}{2}\ln(1+|x(t)|^2) &\le \frac{1}{2}\ln(1+|x(s)|^2) + M_2 \\
\Rightarrow \frac{1}{2}\max_{t\in T}\ln(1+|x(t)|^2) &\le \frac{1}{2}\min_{s\in T}\ln(1+|x(s)|^2) + M_2 .
\end{aligned}$$

From this last inequality and since by hypothesis $\|x_n\|_{C(T,H)} \to \infty$, we infer that

$$\min_{s\in T}|x_n(s)| \to \infty \text{ as } n \to \infty .$$

Therefore for $n \ge 1$ large enough, say $n \ge n_0$, we will have that $|x_n(t)|$ are bounded by a positive constant from below for all $t \in T$. Now return to the equation

$$\frac{1}{2}\frac{d}{dt}|x_n(t)|^2 + \langle A(t, x_n(t)), x_n(t)\rangle = (g_n(t), x_n(t)) \text{ a.e. on } T .$$

Divide by $|x_n(t)|^2$, $n \ge n_0$, and integrate over T. Using the fact that $x_n(0) = x_n(b)$, we obtain

$$\begin{aligned}
0 &= \frac{1}{2}\ln|x_n(b)|^2 - \frac{1}{2}\ln|x_n(0)|^2 \\
&= \int_0^b \frac{(g_n(t), x_n(t)) - \langle A(t, x_n(t)), x_n(t)\rangle}{|x_n(t)|^2}\, dt \\
&\le \int_0^b \frac{\sigma(x_n(t), F(t, x_n(t))) - \langle A(t, x_n(t)), x_n(t)\rangle}{|x_n(t)|^2}\, dt \\
\Rightarrow 0 &\le \overline{\lim}_{n\to\infty}\int_0^b \frac{\sigma(x_n(t), F(t, x_n(t))) - \langle A(t, x_n(t)), x_n(t)\rangle}{|x_n(t)|^2}\, dt
\end{aligned}$$

$$\le \int_0^b \overline{\lim}_{n\to\infty}\frac{\sigma(x_n(t), F(t, x_n(t))) - \langle A(t, x_n(t)), x_n(t)\rangle}{|x_n(t)|^2}\, dt < 0 \tag{2}$$

the last two inequalities being a consequence respectively of Fatou's lemma and of hypothesis $H(F)(iv)$. From (2) we have a contradiction, which proves the proposition. $\square$

Using this proposition, we can now state and prove an existence theorem for problem (1) (i.e. we establish the existence of extremal periodic solutions).

Theorem 1. *If hypotheses $H(A)$, $H(F)$ hold, then problem* (1) *has a solution* $x \in W_{pq}(T)$.

Proof. By virtue of Proposition 2, without any loss of generality we may assume that for almost all $t \in T$ and all $x \in H$,

$$|\widehat{F}(t,x)| \le \varphi(t)$$

with $\varphi \in L^q(T)_+$. Otherwise we replace $F(t,x)$ by

$$\hat{F}(t,x) = F(t, \rho_{M_1}(x))$$

with

$$\rho_{M_1} : H \to H$$

being the M_1-radial retraction on H. Indeed note that for all $x \in H$, $t \to \hat{F}(t,x)$ is measurable, for almost all $t \in T$, $x \to \hat{F}(t,x)$ is h-continuous (since ρ_{M_1} is nonexpansive), for almost all $t \in T$ and all $x \in H$

$$|F(t,x)| \le a_2(t) + c_2(t) M_1 = \varphi(t)$$

and

$$\begin{aligned} \overline{\lim}_{|x|\to\infty} \frac{\sigma(x, F(t,x)) - \langle A(t,x), x\rangle}{|x|^2} &\le \overline{\lim}_{|x|\to\infty} \frac{|x|\varphi(t) - c_1\|x\|^p + a_1(t)}{|x|^2} \\ &\le \overline{\lim}_{|x|\to\infty} \frac{-c_1\beta^p|x|^p}{|x|^2} \end{aligned}$$

where recall $\beta > 0$ is such that $\beta|\cdot| \le \|\cdot\|$. Then the last $\limsup$ is $-\infty$ if $p > 2$ and $-c_1\beta < 0$ if $p = 2$. Hence hypothesis $H(F)$ are satisfied by the modified multifunction $\hat{F}$.

Let

$$V = \{u \in L^q(T,H) : |u(t)| \le \varphi(t) \text{ a.e. on } T\} .$$

For every $u \in V$, the periodic problem

$$\text{(3)} \qquad \left\{ \begin{array}{rcl} \dot{x}(t) + A(t,x(t)) &=& u(t) \text{ a.e. on } T \\ x(0) &=& x(b)\,, \end{array} \right\}$$

has a solution $x \in W_{pq}(T)$ (see for example Kandilakis-Papageorgiou [4]). Let

$$K = \{x \in W_{pq}(T) : x \text{ is a solution of (3) with } u \in V\} .$$

We will show that K is compact in $C(T,H)$. To this end let $\{x_n\}_{n\ge1} \subseteq K$. Then x_n is a solution of (3) corresponding to $u_n \in V$, $n \ge 1$. The set V is weakly compact in $L^q(T,H)$ and so by passing to a subsequence if necessary, we may assume that $u_n \xrightarrow{w} u$ in $L^q(T,H)$, $u \in V$. We have

$$\langle \dot{x}_n(t), x_n(t)\rangle + \langle A(t,x_n(t)), x_n(t)\rangle = (u_n(t), x_n(t))$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}|x_n(t)|^2 + c_1\|x_n(t)\|^p \le |u_n(t)|\frac{1}{\beta}\|x_n(t)\| + a_1(t) \text{ a.e. on } T .$$

Integrating over T and because $x_n(0) = x_n(b)$, we obtain for some $c_3,\ c_4 > 0$

$$\|x_n\|^p_{L^p(T,X)} \le c_3\|x_n\|_{L^p(T,X)} + c_4$$

$$\Rightarrow \|x_n\|_{L^p(T,X)} \le c_5 \tag{4}$$

for some $c_5 > 0$ and all $n \ge 1$.

Also directly from the equation (3), we have

$$\|\dot{x}_n\|_* \le a(t) + c\|x_n(t)\|^{p-1} + \beta_1|u_n(t)| \text{ a.e. on } T$$

with $\beta_1 > 0$ such that $\|\cdot\|_* \le \beta_1|\cdot|$. From the last inequality it follows that

$$\|\dot{x}\|_{L^q(T,X^*)} \le c_6 \text{ for some } c_6 > 0 \text{ and all } n \ge 1\,. \tag{5}$$

From (4) and (5), we deduce that $\{x_n\}_{n\ge1} \subseteq W_{pq}(T)$ is bounded. Recall that $W_{pq}(T)$ is embedded compactly in $L^p(T,H)$ and continuously in $C(T,H)$. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_{pq}(T)$, $x_n \to x$ in $L^p(T,H)$, $x_n(t) \to x(t)$ in H for all $t \in T\setminus N_1$, $\lambda(N_1) = 0$ (λ being the Lebesgue measure on $\mathbb{R}$) and $x_n(t) \xrightarrow{w} x(t)$ in H for all $t \in T$. This last convergence follows from continuous embedding of $W_{pq}(T)$ into $C(T,H)$, hence $x_n \xrightarrow{w} x$ in $C(T,H)$. Note that the sequence

$$\{\langle\dot{x}_n(\cdot), x_n(\cdot) - x(\cdot)\rangle\}_{n\ge1}$$

is uniformly integrable. Thus given $\varepsilon > 0$, we can find $s,\ t \in T\setminus N_1,\ s \le t$, such that

$$\begin{aligned}\int_t^b |\langle\dot{x}_n(\tau), x_n(\tau) - x(\tau)\rangle|\,d\tau &\le \frac{\varepsilon}{2}\\ \int_0^s |\langle\dot{x}_n(\tau), x_n(\tau) - x(\tau)\rangle| d\tau &\le \frac{\varepsilon}{2}\end{aligned} \tag{6}$$

In what follows and in accordance with our previously introduced notation, for any $s,\ t \in T$, by $((\cdot,\cdot))_{st}$ we denote the duality brackets for the pair $(L^p([s,t],X)$, $L^q([s,t],X^*))$. Using the integration by parts formula for functions in $W_{pq}(T)$ (see Zeidler [11], Proposition 23.23, pp. 422-423), we have

$$\begin{aligned}&((\dot{x}_n, x_n - x))_{st} = \frac{1}{2}|x_n(t) - x(t)|^2 - \frac{1}{2}|x_n(s) - x(s)|^2 + ((\dot{x}, x_n - x))_{st} \to 0\\ &\text{as } n \to \infty\end{aligned} \tag{7}$$

since $s,\ t \in T\setminus N_1$, by $((\cdot,\cdot))_{st}$. Then we have

$$\begin{aligned}|((\dot{x}_n, x_n - x))| &= \int_0^b |\langle\dot{x}_n(\tau), x_n(\tau) - x(\tau)\rangle| d\tau\\ &= \int_0^s |\langle\dot{x}_n(\tau), x_n(\tau) - x(\tau)\rangle| d\tau\\ &\quad + \int_t^b |\langle\dot{x}_n(\tau), x_n(\tau) - x(\tau)\rangle| d\tau + |((\dot{x}_n, x_n - x))_{st}|\end{aligned}$$

From (6) and (7) it follows that

$$((\dot{x}_n, x_n - x)) \to 0 \text{ as } n \to \infty. \tag{8}$$

But note that if

$$\hat{A} : L^p(T, X) \to L^q(T, X^*)$$

is the Nemitsky operator corresponding to A (i.e. $\hat{A}(x)(\cdot) = A(\cdot, x(\cdot))$), then

$$\begin{aligned} ((\hat{A}(x_n), x_n - x)) =& ((u_n, x_n - x)) - ((\dot{x}_n, x_n - x)) \\ =& (u_n, x_n - x)_{pq} - ((\dot{x}_n, x_n - x)) \to 0 \quad \text{as } n \to \infty\,. \end{aligned}$$

From this last convergence and Proposition 1 of Papageorgiou [8] we have that

$$\hat{A}(x_n) \xrightarrow{w} \hat{A}(x) \text{ in } L^q(T, X^*) \text{ and } ((\hat{A}(x_n), x_n)) \to ((\hat{A}(x), x)) \text{ as } n \to \infty.$$

Finally since $x_n(t) \xrightarrow{w} x(t)$ in H as $n \to \infty$ and $x_n(0) = x_n(b)$, $n \geq 1$, we have that $x(0) = x(b)$. Therefore in the limit as $n \to \infty$, we obtain

$$\left\{ \begin{array}{rcl} \dot{x}(t) + A(t, x(t)) &=& u(t) \text{ a.e. on } T \\ x(0) &=& x(b)\,, \quad u \in V \end{array} \right\}$$

$$\Rightarrow x \in K\,.$$

From the above argument, it is clear that if s', $t' \in T \setminus N_1$, $s' \leq t'$, we have

$$((\hat{A}(x_n), x_n))_{s't'} \to ((\hat{A}(x), x))_{s't'} \text{ as } n \to \infty\,.$$

As before because

$$\{\langle \dot{x}_n(\cdot), x_n(\cdot) - x(\cdot)\rangle\}_{n \geq 1}$$

is uniformly integrable, given s, $t \in T$, $s \leq t$ and $\varepsilon \geq 0$, we can find s', $t' \in T \setminus N_1$, $s' \leq t'$ such that

$$((\hat{A}(x_n), x_n - x))_{s't'} - \varepsilon \leq ((\hat{A}(x_n), x_n - x))_{st} \leq ((\hat{A}(x_n), x_n - x))_{s't'} + \varepsilon$$

$$\Rightarrow -\varepsilon \leq \overline{\lim}((\hat{A}(x_n), x_n - x))_{st} \leq \varepsilon \text{ and } -\varepsilon \leq \underline{\lim}((\hat{A}(x_n), x_n - x))_{st} \leq \varepsilon\,.$$

Let $\varepsilon \downarrow 0$ to conclude that for all s, $t \in T$, $s \leq t$, we have

$$((\hat{A}(x_n), x_n))_{st} \to ((\hat{A}(x), x))_{st} \text{ as } n \to \infty.$$

For all $n \geq 1$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x_n(t)|^2 - \frac{1}{2}\frac{d}{dt}|x(t)|^2 + \langle A(t, x_n(t)), x_n(t)\rangle - \langle A(t, x(t)), x(t)\rangle \\ &\qquad = (u_n(t), x_n(t)) - (u(t), x(t)) \text{ a.e. on } T\,. \end{aligned}$$

Integrating the above inequality on $[s, t]$ we obtain

$$\begin{aligned} \Rightarrow \frac{1}{2}\left[\,|x_n(t)|^2 - |x(t)|^2\,\right] - \frac{1}{2}\left[\,|x_n(s)|^2 - |x(s)|^2\right] + ((\hat{A}(x_n), x_n))_{st} - ((\hat{A}(x), x))_{st} \\ = ((u_n, x_n))_{st} - ((u, x))_{st}\,. \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain

$$\overline{\lim}(\left[\,|x_n(t)|^2 - |x(t)|^2\,\right] - \left[\,|x_n(s)|^2 - |x(s)|^2\,\right]) = 0$$

from which it follows that

$$\begin{aligned}\overline{\lim}\,\left[\,|x_n(t)|^2-|x(t)|^2\right] &\le \overline{\lim}\,\left[\,|x_n(s)|^2-|x(s)|^2\right]\\ \underline{\lim}\,\left[\,|x_n(t)|^2-|x(t)|^2\right] &\le \underline{\lim}\,\left[\,|x_n(s)|^2-|x(s)|^2\right].\end{aligned}$$

Let $t=b$. Then for $0\le s\le b$, we have

$$\overline{\lim}\,\left[\,|x_n(b)|^2-|x(b)|^2\right] \le \overline{\lim}\,\left[\,|x_n(s)|^2-|x(s)|^2\right] \tag{9}$$

$$\underline{\lim}\,\left[\,|x_n(b)|^2-|x(b)|^2\right] \le \underline{\lim}\,\left[\,|x_n(s)|^2-|x(s)|^2\right]. \tag{10}$$

Also let $s=0$. Then for $0\le t\le b$, we have

$$\overline{\lim}\,\left[\,|x_n(t)|^2-|x(t)|^2\right] \le \overline{\lim}\,\left[\,|x_n(0)|^2-|x(0)|^2\right] \tag{11}$$

$$\underline{\lim}\,\left[\,|x_n(t)|^2-|x(t)|^2\right] \le \underline{\lim}\,\left[\,|x_n(0)|^2-|x(0)|^2\right]. \tag{12}$$

Recalling that $x_n(0)=x_n(b)$, $n\ge 1$, and $x(0)=x(b)$, from (9)$\to$(12) we deduce that for all $t\in T$ we have

$$\begin{aligned}\underline{\lim}\,\left[\,|x_n(t)|^2-|x(t)|^2\right] &= \underline{\lim}\,\left[\,|x_n(0)|^2-|x(0)|^2\right]\\ \overline{\lim}\,\left[\,|x_n(t)|^2-|x(t)|^2\right] &= \overline{\lim}\,\left[\,|x_n(0)|^2-|x(0)|^2\right].\end{aligned}$$

But for $t\in T\setminus N_1$, we know that $x_n(t)\to x(t)$ in H. So

$$|x_n(0)|\to|x(0)| \text{ as } n\to\infty.$$

Also we have $x_n(0)\xrightarrow{w}x(0)$ in H as $n\to\infty$. Since a Hilbert space has the Kadec-Klee property (see Hu-Papageorgiou [3], Definition I.1.72(d) and Lemma I.174, p.28), we have that $x_n(0)\to x(0)$ in H and of course $x_n(b)\to x(b)$ as $n\to\infty$.

For any $t\in T$, via the integration by parts formula, we have

$$\frac12|x_n(t)-x(t)|^2+((\hat A(x_n)-\hat A(x),x_n-x))_{0t}=\frac12|x_n(0)-x(0)|^2+((u_n-u,x_n-x))_{0t}$$

$$\begin{aligned}\Rightarrow \frac12|x_n(t)-x(t)|^2\le &\int_0^b|(u_n(s)-u(s),x_n(s)-x(s))|\,ds\\ &+\int_0^b|\langle A(s,x_n(s)),x_n(s)-x(s)\rangle|\,ds\\ &+((\hat A(x),x_n-x))_{0t}+\frac12|x_n(0)-x(0)|^2.\end{aligned} \tag{13}$$

We examine the terms in the right hand side of inequality (13) above. We already know that

$$\int_0^b|(u_n(s)-u(s),x_n(s)-x(s))|\,ds\le 2\|\varphi\|_q\|x_n-x\|_{L^p(T,H)}\to 0 \tag{14}$$

as $n\to\infty$, since $x_n\xrightarrow{w}x$ in $W_{pq}(T)$ and

$$|x_n(0)-x(0)|\to 0 \text{ as } n\to\infty. \tag{15}$$

Let

$$\xi_n(t)=\langle A(t,x_n(t)),x_n(t)-x(t)\rangle.$$

Let $N_2 \subseteq T$ be a Lebesgue-null set such that $N_2 \supseteq N_1$ and hypotheses $H(A)$(ii), (iii), (iv) hold. For $t \in T \setminus N_2$, we have

$$\xi_n(t) \geq \psi_n(t) = c_1\|x_n(t)\|^p - (a(t) + c\|x_n(t)\|^{p-1})\|x(t)\| - a_1(t)\,. \tag{16}$$

Set

$$C = \{t \in T :\ \underline{\lim}\,\xi_n(t) < 0\}\,.$$

Observe that $C \subseteq T$ is measurable and for the moment suppose $\lambda(C) > 0$. Then $\lambda(C \cap (T \setminus N_2)) > 0$ and for every $t \in C \cap (T \setminus N_2)$ from (16) we have that $\{x_n(t)\}_{n\geq 1}$ is bounded in X. Since X is separable, reflexive and we already know that $x_n(t) \xrightarrow{w} x(t)$ in H, we deduce that $x_n(t) \xrightarrow{w} x(t)$ in X for all $t \in C \cap (T \setminus N_2)$. Since by hypothesis $H(A)(ii)$, $A(t,\cdot)$ is pseudomonotone, we have $\xi_n(t) \to 0$ as $n \to \infty$ for all $t \in C \cap (T \setminus N_2)$, a contradiction to the definition of C. So $\lambda(C) = 0$ and $\underline{\lim}\,\xi_n(t) \geq 0$ a.e. on T. Then by Fatou's lemma we have

$$0 \leq \int_0^b \underline{\lim}\,\xi_n(t)\,dt \leq \underline{\lim}\int_0^b \xi_n(t)\,dt \leq \overline{\lim}\int_0^b \xi_n(t)\,dt = \overline{\lim}((\hat{A}(x_n), x_n - x)) = 0$$

$$\Rightarrow \int_0^b \xi_n(t)\,dt \to 0 \text{ as } n \to \infty\,.$$

Note that $|\xi_n(t)| = \xi_n^+(t) + \xi_n^-(t) = \xi_n(t) + 2\xi_n^-(t)$. Since $\underline{\lim}\,\xi_n(t) \geq 0$ a.e. on T, we have that $\xi_n^-(t) \to 0$ a.e. on T as $n \to \infty$. Also from (16), we see that

$$0 \leq \xi_n^-(t) \leq \psi_n^-(t) \text{ a.e. on } T$$

and $\{\psi_n^-\}_{n\geq 1}$ is uniformly integrable (since $\{\psi_n\}_{n\geq 1}$ is). So from the generalized dominated convergence theorem (see for example Hu-Papageorgiou [3], Theorem A.2.54, p.907), we have that $\int_0^b \xi_n^-(t)\,dt \to 0$. Thus finally

$$\int_0^b |\xi_n(t)|\,dt = \int_0^b \xi_n(t)\,dt + 2\int_0^b \xi_n^-(t)\,dt \to 0$$

$$\Rightarrow \xi_n \to 0 \text{ in } L^1(T)\,,$$

$$\Rightarrow \int_0^b |\langle A(t, x_n(t)), x_n(t) - x(t)\rangle|\,dt \to 0 \text{ as } n \to \infty\,. \tag{17}$$

Finally we examine the sequence $\left\{((\hat{A}(x), x_n - x))_{0t}\right\}_{n\geq 1}$. Set

$$\gamma_n(t) = \int_0^t \langle A(s, x(s)), x_n(s) - x(s)\rangle\,ds = ((\hat{A}(x), x_n - x))_{0t}\,, \quad n \geq 1\,.$$

Let $t_n \in T$ such that $\gamma_n(t_n) = \max_{t\in T}\gamma_n(t)$. We may assume that $t_n \to t$ in T. We have

$$\begin{aligned}
\gamma_n(t_n) &= \int_0^{t_n} \langle A(s, x(s)), x_n(s) - x(s)\rangle\,ds \\
&= \int_0^b \langle \chi_{[0,t_n]}(s)A(s, x(s)), x_n(s) - x(s)\rangle\,ds \\
&= ((\chi_{[0,t_n]}\hat{A}(x), x_n - x))\,.
\end{aligned}$$

Note that $\chi_{[0,t_n]}\hat{A}(x) \to \chi_{[0,t]}\hat{A}(x)$ in $L^q(T,X^*)$. Indeed

$$\|\chi_{[0,t_n]}\widehat{A}(x) - \chi_{[0,t]}\widehat{A}(x)\|_{L^q(T,X^*)} = \int_{\min\{t,t_n\}}^{\max\{t,t_n\}} \|A(s,x(s))\|_* \, ds \longrightarrow 0$$

as $n \to \infty$ since $t_n \to t$. Because $x_n \xrightarrow{w} x$ in $L^p(T,X)$, we infer that

$$\gamma_n(t_n) = ((\chi_{[0,t_n]}\hat{A}(x), x_n - x)) \to 0 \text{ as } n \to \infty\,. \tag{18}$$

Using (14), (15), (17) and (19) in (13), we see that

$$\max_{t\in T} |x_n(t) - x(t)| \to 0 \text{ as } n \to \infty$$

$$\Rightarrow x_n \to x \text{ in } C(T,H)\,, \quad x \in K\,.$$

Thus we have proved that K is compact in $C(T,H)$. Then by Mazur's theorem

$$W = \overline{\operatorname{conv}}\, K$$

is compact and convex in $C(T,H)$. Consider the multifunction

$$G : W \to P_{fc}(L^q(T,H))$$

defined by $G(x) = S^q_{F(\cdot,x(\cdot))}$. We can apply Theorem II.8.31, p.260, of Hu-Papageorgiou [3], and obtain

$$r : W \to L^1_w(T,H)$$

a continuous map such that

$$r(x) \in \operatorname{ext} G(x) \text{ for all } x \in W$$

From Theorem II.4.6, p. 192, of Hu-Papageorgiou [3], we know that

$$\operatorname{ext} S^q_{F(\cdot,x(\cdot))} = S^q_{\operatorname{ext} F(\cdot,x(\cdot))}.$$

Since $C(T,H)$ is embedded continuously in $L^q(T,H)$, we see that W viewed as a subset of $L^q(T,H)$ is compact and convex. The Lebesgue-Bochner space $L^q(T,H)$ is uniformly convex. So the metric projection map

$$p_W : L^q(T,H) \to W$$

(i.e. $\|x - p_W(x)\|_{L^q(T,H)} = d_{L^q(T,H)}(x,W)$) is well-defined, single-valued and continuous. Consider the following periodic problem:

$$\left\{ \begin{array}{c} \dot{x}(t) + A(t,x(t)) = r(p_W(x))(t) \text{ a.e. on } T \\ x(0) = x(b)\,. \end{array} \right\} \tag{19}$$

Let

$$V : L^p(T,X) \to L^q(T,X^*)$$

be defined by

$$V(x) = \hat{A}(x) - r(p_W(x))\,.$$

We will show that for any sequence $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ with $\overline{\lim}((V(x_n, x_n - x)) \le 0$ we have $V(x_n) \xrightarrow{w} V(x)$ in $L^q(T,X^*)$ and $((V(x_n), x_n)) \to ((V(x), x))$.

So let $x_n \xrightarrow{w} x$ in $W_{pq}(T)$ and assume that $\overline{\lim}((V(x_n), x_n - x)) \leq 0$. Then $x_n \to x$ in $L^p(T,H)$, hence $x_n \to x$ in $L^q(T,H)$ since $q < p$. Thus $p_W(x_n) \to p_W(x)$ in $L^q(T,H)$ and so

$$r(p_W(x_n)) \xrightarrow{\|\cdot\|_w} r(p_W(x)) .$$

Therefore by virtue of Lemma 1, we have $r(p_W(x_n)) \xrightarrow{w} r(p_W(x))$ in $L^q(T,H)$. Moreover,

$$((r(p_W(x_n)), x_n - x)) = (r(p_W(x_n)), x_n - x)_{pq} \to 0 .$$

Thus we obtain that

$$\overline{\lim}((\hat{A}(x_n), x_n - x)) \leq 0$$

which by virtue of Proposition 1 of Papageorgiou [8], implies that $\hat{A}(x_n) \xrightarrow{w} \hat{A}(x)$ in $L^q(T,X^*)$ and $((\hat{A}(x_n), x_n)) \to ((\hat{A}(x), x))$ as $n \to \infty$. So we can apply Theorem 1.2, p. 319, of Lions [7] (see also Theorem 2.1 of Papageorgiou-Papalini-Renzacci [9]), with

$$L : D \subseteq L^p(T,X) \to L^q(T,X^*)$$

defined by $L(x) = \dot{x}$ for all $x \in D = \{x \in W_{pq}(T) : x(0) = x(b)\}$ (which is maximal monotone and densely defined, see Hu-Papageorgiou [3], Proposition II.9.4, p.419), to deduce that V is surjective. So problem (19) has a solution $x \in W_{pq}(T)$ and clearly from the definitions of W and r, we have that $x \in W$. So $r(p_W(x)) = r(x)$ and we conclude that $x \in W_{pq}(T)$ is a solution of problem (1). □

If we consider the convexified problem

$$\left\{ \begin{array}{rcl} \dot{x}(t) + A(t,x(t)) & \in & F(t,x(t)) \text{ a.e. on } T \\ x(0) & = & x(b) \end{array} \right\} \tag{20}$$

then an immediate byproduct of the proof of Theorem 1, is the following corollary:

Corollary 1. *If hypotheses $H(A)$, $H(F)$ hold, then problem (20) has a nonempty solution set $S \subseteq W_{pq}(T)$ which is compact in $C(T,H)$.*

4. An application

In this section we present an example of a quasilinear periodic distributed parameter control system, with a priori feedback (i.e. state dependent control constraint set).

So let $T = [0,b]$ and $Z \subseteq \mathbb{R}^N$ a bounded domain with C^1-boundary Γ. We consider the following control system, with $2 \leq p < \infty$:

(21)

$$\left\{ \begin{array}{c} \dfrac{\partial x}{\partial t} - \operatorname{div}(a(t,x)\|Dx\|^{p-2}Dx) = (g(t,z,x(t,z)), u(t,z))_{\mathbb{R}^m} \text{ a.e. on } T \times Z \\ x|_{T\times\Gamma} = 0, x(0,z) = x(b,z) \text{ a.e. on } Z \\ u(t,z) \in \operatorname{ext} U(t,z,x(t,z)) \text{ a.e. on } T \times Z \end{array} \right\}$$

In what follows λ_1 denotes the first eigenvalue of the negative p-Laplacian

$$-\Delta_p x = -\operatorname{div}(\|Dx\|^{p-2}Dx)$$

with Dirichlet boundary conditions (i.e. of $(-\Delta_p, W_0^{1,p}(Z))$. It is well known (see for example Lindqvist [6]), that $\lambda_1 > 0$ is simple and isolated. Our hypotheses on the data of (21) are the following:

H(a) : $a : T \times \mathbb{R} \to \mathbb{R}_+$ is a Caratheodory function such that for almost all $t \in T$ and all $x \in \mathbb{R}$

$$0 < c_1 \leq a(t, x) \leq \theta\,.$$

H(g) : $g : T \times Z \times \mathbb{R} \to \mathbb{R}^m$ is a function such that

(i) for all $x \in \mathbb{R}$, $(t, z) \to g(t, z, x)$ is measurable;

(ii) for all $(t, z) \in T \times Z$, $x \to g(t, z, x)$ is continuous;

(iii) for almost all $(t, z) \in T \times Z$ and all $x \in \mathbb{R}$, we have

$$||g(t, z, x)|| \leq \gamma_1(t, z) + \gamma_2(t, z)|x|$$

with $\gamma_1 \in L^q(T, L^2(Z))$, $\gamma_2 \in L^\infty(T \times Z)$ and $||\gamma_2||_\infty < c_1\lambda_1^p$.

H(U) : $U : T \times Z \times \mathbb{R} \to P_{fc}(\mathbb{R}^m)$ is a multifunction such that

(i) for all $x \in \mathbb{R}$, $(t, z) \to U(t, z, x)$ is measurable;

(ii) for all $(t, z) \in T \times Z$, $x \to U(t, z, x)$ is h-continuous;

(iii) for almost all $(t, z) \in T \times Z$ and all $x \in \mathbb{R}$, $|U(t, z, x)| \leq 1$.

Let $X = W_0^{1,p}$, $H = L^2(Z)$, $X^* = W^{-1,q}(Z)$. Then (X, H, X^*) is an evolution triple with compact embeddings. On X we consider the norm $||Dx||_p$ for $x \in W_0^{1,p}(Z)$ (Poincare's inequality). It is well-known that

$$\lambda_1 = \inf \left[\frac{||Dx||_p^p}{||x||_p^p} : \ x \in W_0^{1,p}(Z)\,, \quad x \neq 0 \right]$$

(Rayleigh quotient; see Lindqvist [6]).

Let $A : T \times X \to X^*$ be defined by

$$\langle A(t, x), y\rangle = \int_Z a(t, x(z))||Dx||^{p-2}(Dx, Dy)_{\mathbb{R}^N} dz \text{ for all } y \in W_0^{1,p}(Z)\,.$$

From hypothesis $H(a)$, we see that for all $x \in X$, $t \to A(t, x)$ is measurable, for all $t \in T$, $x \to A(t, x)$ is demicontinuous, pseudomonotone (in fact of type (S_+), see Hu-Papageorgiou [3]),

$||A(t, x)||_* \leq \theta||x||^{p-1}$ and $\langle A(t, x), x\rangle \geq c_1||x||^p$ for almost all $t \in T$ and all $x \in X$.

Also let $F : T \times H \to P_{fc}(H)$ be defined by

$$F(t, x) = \{y \in H : \ y(z) = (g(t, z, x(z)), u(z))_{\mathbb{R}^m},\ u(z) \in U(t, z, x(z)),\ \text{a.e. on } Z\}$$

Using hypotheses $H(g)$ and $H(U)$, it is straightforward to check that F satisfies hypotheses $H(F)$. Note that when $p > 2$, hypothesis $H(F)(iv)$ is trivially satisfied since we have

$$\overline{\lim}_{|x|\to\infty} \frac{\sigma(x, F(t, x)) - \langle A(t, x), x\rangle}{|x|^2} = -\infty\,,$$

while for $p = 2$ is satisfied since by virtue of $H(g)(iii)$ $||\gamma_2||_\infty < c_1\lambda_1^p$.

Rewrite problem (21) in the following equivalent evolution inclusion form:

$$(22) \qquad \left\{ \begin{array}{rcl} \dot{x} + A(t, x(t)) & \in & \operatorname{ext} F(t, x(t)) \text{ a.e. on T} \\ x(0) & = & x(b)\,. \end{array} \right\}$$

We can apply Theorem 1 on problem (22) and obtain:

Proposition 2. *If hypotheses $H(a)$, $H(g)$, $H(U)$ hold then problem* (21) *has a solution $x \in L^p(T, W_0^{1,p}(Z)) \cap C(T, L^2(Z))$ with $\frac{\partial x}{\partial t} \in L^q(T, W^{-1,q}(Z))$.*

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