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ON OSCILLATORY LINEAR DIFFERENTIAL EQUATIONS OF THIRD ORDER

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Abstract. Sufficient conditions are obtained in terms of coefficient functions such that a linear homogeneous third order differential equation is strongly oscillatory.

1. Introduction.

In this paper we consider

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0\,, \tag{1}$$

where a and $b \in C^1((0,\infty), R)$, $c \in C((0,\infty), R)$. The adjoint of (1) is given by

$$\big((z' - a(t)z)' + b(t)z\big)' - c(t)z = 0\,. \tag{1*}$$

If $a(t)$, $b(t)$ and $c(t)$ are constants a, b and c ($c \neq 0$), respectively, then (1) takes the form

$$y''' + ay'' + by' + cy = 0\,. \tag{2}$$

It is well-known that Eq. (2) always admits a non-oscillatory solution. In the literature, we usually come across following two types of definitions for oscillation of a solution of (1) $\{(1^*)\}$:

Definition 1. A nontrivial solution $y(t)$ of (1) $\{(1^*)\}$ is said to be oscillatory on $[T_y, \infty)$, $T_y > 0$, if it has arbitrarily large zeros in $[T_y, \infty)$, that is, there exists a sequence $\langle t_n \rangle \subset [T_y, \infty)$ such that $t_n \to \infty$ as $n \to \infty$ and $y(t_n) = 0$ for $n = 1, 2, \dots$

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Definition 2. A nontrivial solution $y(t)$ of (1) $\{(1^*)\}$ is said to be oscillatory on $[T_y, \infty)$, if it has infinite number of zeros in $[T_y, \infty)$.

These two definitions are equivalent for (1) $\{(1^*)\}$. However, if we consider (1) $\{(1^*)\}$ on $(0, d)$, where $d < \infty$, then Definition 1 has no meaning. A nontrivial solution $y(t)$ of (1) $\{(1^*)\}$ is said to be nonoscillatory if it is not oscillatory. If Eq. (1) $\{(1^*)\}$ has a nontrivial oscillatory solution, then it is said to be oscillatory; otherwise, Eq. (1) $\{(1^*)\}$ is said to be nonoscillatory.

In [2], Greguš has obtained sufficient conditions on q such that every solution of

$$y''' + q(t)y' + q'(t)y = 0\,, \qquad t \in (-\infty, \infty)\,,$$

has infinitely many zeros in $(-\infty, \infty)$.

Let $\mathscr{S}$ and $\mathscr{S}^*$ denote the solution spaces of (1) and (1^*), respectively. Thus each of them is a three dimensional vector space over the field of real numbers. Let $\mathscr{S}_1\{\mathscr{S}_1^*\}$ denote a nontrivial subspace of $\mathscr{S}\{\mathscr{S}^*\}$. Then $\mathscr{S}_1\{\mathscr{S}_1^*\}$ is said to be nonoscillatory if every nonzero member of $\mathscr{S}_1\{\mathscr{S}_1^*\}$ is nonoscillatory, $\mathscr{S}_1\{\mathscr{S}_1^*\}$ is said to be weakly oscillatory if it contains a nontrivial oscillatory and a nonoscillatory solution. $\mathscr{S}_1\{\mathscr{S}_1^*\}$ is said to be strongly oscillatory if every nonzero member of $\mathscr{S}_1\{\mathscr{S}_1^*\}$ oscillates and $\mathscr{S}_1\{\mathscr{S}_1^*\}$ is said to be oscillatory if $\mathscr{S}_1\{\mathscr{S}_1^*\}$ is either weakly oscillatory or strongly oscillatory. It may be noted that weakly oscillatory definition applies only to subspaces of dimension greater than or equal to two. If $\mathscr{S}\{\mathscr{S}^*\}$ is nonoscillatory, weakly oscillatory or strongly oscillatory, then Eq. (1)$\{(1^*)\}$ is said to be nonoscillatory, weakly oscillatory or strongly oscillatory, respectively. In [1], Dolan has established following results:

Theorem A. *If Eq.* (1) $\{(1^*)\}$ *is weakly oscillatory, then Eq.* (1^*) $\{(1)\}$ *is oscillatory.*

Theorem B. *If* $\mathscr{S}\{\mathscr{S}^*\}$ *contains a nonoscillatory two-dimensional subspace, then* $\mathscr{S}^*(\mathscr{S}\}$ *is either nonoscillatory or strongly oscillatory.*

Following two questions were raised by Dolan [1]:

(i) Does there exist an example of a linear third order differential equation with the property that every two dimensional subspace of the solution space is weakly oscillatory?

(ii) Does there exist an example of a linear third order differential equation such that the solution space $\mathscr{S}$ and $\mathscr{S}^*$ are strongly oscillatory?

In [4], Neuman has provided answers to above two questions. He has shown that there does not exist a linear third order differential equation of the form (1) with the property that every two-dimensional subspace of its solution space is weakly oscillatory. Further, he has constructed an example of a strongly oscillatory Eq. (1) whose adjoint (1^*) is also strongly oscillatory.

In this paper we have obtained easily verifiable sufficient conditions in terms of coefficient functions a, b and c so that Eq. (1) is strongly oscillatory.

2. Equation (1) may be written as

$$(r(t)y'')' + q(t)y' + p(t)y = 0\,, \tag{1}$$

where $r(t) = \exp\left(\int_0^t a(s)\,ds\right), q(t) = b(t)\,r(t)$ and $p(t) = c(t)\,r(t)$. We assume that

$$(\mathrm{H}_1) \qquad a(t) \le 0\,, \quad b(t) \ge 0\,, \quad c(t) < 0\,, \quad t > 0\,,$$

and

$$(\mathrm{H}_2) \quad \begin{cases} \text{Second order linear homogeneous equation} \\ (r(t)z')' + q(t)z = 0 \\ \text{is nonoscillatory.} \end{cases} \tag{3}$$

Remark. Clearly, $p(t) < 0$, $q(t) \ge 0$, $r(t) > 0$, $r' \le 0$ and hence

$$\int_0^\infty \frac{dt}{r(t)} = \infty\,.$$

In view of (H_2) and Leighton's oscillation criteria [5, p. 70] we have

$$\int_0^\infty q(t)\,dt < \infty\,.$$

We have the following result due to Keener [3] for our work.

Theorem 1 [3, p. 62]. *If* (H_2) *holds, then every solution of*

$$(r(t)z')' + q(t)z = f(t)$$

is nonoscillatory, where f is a real-valued continuous function on $(0,\infty)$ such that $f(t) \ge 0$, $t > 0$.

Lemma 2. *Suppose that* (H_1) *and* (H_2) *hold. If $y(t)$ is a nonoscillatory solution of* (1), *then $y(t)\,y'(t) > 0$ for large t.*

Proof. We may assume, without any loss of generality, that $y(t) > 0$ for $t \ge t_0 > 0$. Since $y'(t)$ is a solution of

$$(r(t)z')' + q(t)z = -p(t)\,y(t)\,, \quad t \ge t_0$$

then from Theorem 1 it follows that $y'(t) > 0$ or < 0 for $t \ge t_1 \ge t_0$. If possible, let $y'(t) < 0$ for $t \ge t_1$. As $(r(t)\,y''(t))' > 0$ for $t \ge t_1$, then $y''(t) > 0$ or < 0 for $t \ge t_2 \ge t_1$. However, $y''(t) < 0$ for $t \ge t_2$ implies that $y(t) < 0$ for large t, a contradiction. Thus $y''(t) > 0$ for $t \ge t_2$. This implies, due to (1), that $y'''(t) > 0$ for large t. Hence $y'(t) > 0$ for large t, contradicting our assumption that $y'(t) < 0$ for $t \ge t_1$. Thus $y'(t) > 0$ for $t \ge t_1$.
This completes the proof of the lemma. $\square$

Theorem 3. *Let* (H$_1$) *and* (H$_2$) *hold. If*

$$(\mathrm{H}_3)\quad\begin{cases}\text{(i)} & -\infty < \liminf\limits_{t\to\infty} a(t) \le 0;\\ \text{(ii)} & \frac{1}{3}a^2(t) - b(t) + a'(t) > 0 \quad \textit{and}\\ \text{(iii)} & \int\limits_{\sigma}^{\infty}\Big[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \frac{a(t)a'(t)}{3} + c(t) - \\ & -\frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{\frac{3}{2}}\Big]\,dt = \infty, \quad \sigma > 0\end{cases}$$

then Eq. (1) *is strongly oscillatory.*

Proof. If possible, let (1) admit a nonoscillatory solution $y(t)$. Then $y(t)\,y'(t) > 0$ for $t \ge t_0 > 0$ by Lemma 2. Clearly, $z(t) = y'(t)/y(t)$, $t \ge t_0$, is a positive solution of the second order Riccati equation.

$$u'' + 3uu' + a(t)u' = -[z^3(t) + a(t)z^2(t) + b(t)\,z(t) + c(t)]\,. \tag{4}$$

Integrating (4) from t_0 to t $(t > t_0)$ we obtain

$$\begin{aligned} z'(t) =& z'(t_0) + \frac{3}{2}z^2(t_0) + a(t_0)\,z(t_0) - \frac{3}{2}z^2(t) - a(t)\,z(t) \\ &- \int_{t_0}^{t}\left[z^3(s) + a(s)\,z^2(s) + (b(s) - a'(s))\,z(s) + c(s)\right]\,ds\,. \end{aligned} \tag{5}$$

If

$$H\,(z(t)) = z^3(t) + a(t)\,z^2(t) + (b(t) - a'(t))\;z(t) + c(t)\,,$$

then $H(z(t))$ attains its minimum value for $z(t) > 0$ at

$$z(t) = \frac{1}{3}\left[-a(t) + \left(a^2(t) - 3b(t) + 3a'(t)\right)^{\frac{1}{2}}\right]$$

and the minimum value is given by

$$\frac{2}{27}a^3(t) - \frac{1}{3}a(t)\,b(t) + \frac{1}{3}a(t)\,a'(t) + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{\frac{3}{2}}\,.$$

Further, if

$$F(z(t)) = \frac{3}{2}z^2(t) + a(t)\,z(t)\,,$$

then $F(z(t))$ attains its minimum value for $z(t) > 0$ at $z(t) = -a(t)/3$ and the minimum value is given by $-a^2(t)/6$. Hence (5) yields

$$\begin{aligned} z'(t) \le & z'(t_0) + \frac{3}{2}z^2(t_0) + a(t_0)\,z(t_0) + \frac{a^2(t)}{6} \\ -\int_{t_0}^{t}&\left[\frac{2a^3(s)}{27} - \frac{a(s)\,b(s)}{3} + \frac{a(s)\,a'(s)}{3} + c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s) + a'(s)\right)^{\frac{3}{2}}\right]\,dt \end{aligned}$$

From (H_3) it follows that $\lim_{t\to\infty} z'(t) = -\infty$. Thus $z(t) < 0$ for large t, a contradiction. The proof of the theorem is complete. $\square$

Remark. Theorem 3 fails to hold for Euler's equation

$$y''' + \frac{a_0}{t} y'' + \frac{b_0}{t^2} y' + \frac{c_0}{t^3} y = 0\,,$$

where $a_0 < 0$, $b_0 > 0$, $c_0 < 0$, because (H_3) (iii) is not satisfied.

Following result due to Potter [5, Theorem 2.36] is needed.

Theorem 4. *Suppose that r and $q \in C^1((0,\infty), R)$ such that r is positive and q is nonnegative in $(0,\infty)$ and*

$$\int_1^\infty \frac{dt}{r(t)} = \infty\,.$$

If $L = \lim_{t\to\infty} r(t) \left\{ [r(t)\, q(t)]^{\frac{-1}{2}} \right\}'$ exists and $L > 2$, then (3) is nonoscillatory.

Remark. Theorem 3 does not hold for (2), the third order equation with constant coefficients, with $a < 0$, $b > 0$, $c < 0$ because

$$L = \lim_{t\to\infty} \mathrm{e}^{at} \left\{ \left[\mathrm{e}^{2at} b\right]^{\frac{-1}{2}} \right\}' > 2$$

if and only if $a^2 > 4b$. Further,

$$\frac{2a^3}{27} - \frac{ab}{3} - \frac{2}{3\sqrt{3}} \left(\frac{a^2}{3} - b \right)^{\frac{3}{2}} > 0$$

if and only if $a^2 < 4b$. Thus (H_3) (iii) and $L > 2$ do not hold simultaneously, where L is defined in Theorem 4.

The following example illustrates Theorem 3.

Example. Consider

$$y''' - y'' + \left(\frac{1}{4.0000004} + \frac{1}{t} \right) y' - \frac{k}{t^2} y = 0\,, \quad t \geq 12, \tag{6}$$

where $k > 0$ is a constant. In this case $L = 2.0000001 > 2$. Then (H_2) holds by Theorem 4. The calculation shows that

$$\frac{2a^3(t)}{27} - \frac{a(t)\, b(t)}{3} + \frac{a(t)\, a'(t)}{3} + c(t) - \frac{2}{3\sqrt{3}} \left(\frac{a^2(t)}{3} - b(t) + a'(t) \right)^{\frac{3}{2}}$$
$$= 0.00000005 + \frac{1}{3t} + \frac{0.1666664}{t} + \cdots - \frac{k}{t^2}$$

and

$$\frac{a^2(t)}{3} - b(t) + a'(t) = \frac{1.0000004}{12.000001} - \frac{1}{t} > 0$$

for $t \geq 12$. Hence (H_3) is satisfied. From Theorem 3 it follows that (6) is strongly oscillatory.

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