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ON QUADRATICALLY INTEGRABLE SOLUTIONS OF THE SECOND ORDER LINEAR EQUATION

T. CHANTLADZE, N. KANDELAKI AND A. LOMTATIDZE

Abstract. Integral criteria are established for $\dim V_i(p) = 0$ and $\dim V_i(p) = 1, i \in \{0,1\}$, where $V_i(p)$ is the space of solutions u of the equation

$$u'' + p(t)u = 0$$

satisfying the condition

$$\int^{+\infty} \frac{u^2(s)}{s^i} ds < +\infty .$$

1. Main Results

Consider the equation

$$u'' + p(t)u = 0 , \tag{1}$$

where $p : [0,+\infty[\to]-\infty,+\infty[$ is a locally integrable function.

Under a solution of equation (1) is understood a locally absolutely continuous together with its first derivative function $u : [0,+\infty[\to]-\infty,+\infty[$ satisfying (1) almost everywhere.

Denote by $V_i(p)$ $(i = 0,1)$ the set of solutions u of equation (1) satisfying the condition

$$\int^{+\infty} \frac{u^2(s)}{s^i} ds < +\infty , \tag{2}$$

and denote the set of solutions u satisfying

$$\lim_{t\to+\infty} u(t) = 0$$

by $Z(p)$.

Below we give some new results on the interlocation as well as on the dimensionality of the above sets.

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Theorem 1. *Let* $i \in \{0,1\}$, $p(t) \le 0$ *in some neighbourhood of* $+\infty$, *and*

$$\int^{+\infty} \frac{ds}{s^i \left[\int_0^s \eta |p(\eta)|\, d\eta\right]^2} < +\infty \,. \tag{3}$$

Then $V_i(p) = Z(p)$ *and* $\dim V_i(p) = 1$.

Suppose that there exists a finite limit

$$\lim_{t\to+\infty} \frac{1}{t} \int_1^t \int_1^s p(\eta)\, d\eta\, ds = c_p$$

and put

$$p_* = \liminf_{t\to+\infty} t \left(c_p - \int_1^t p(s)\, ds \right), \qquad p^* = \limsup_{t\to+\infty} t \left(c_p - \int_1^t p(s)\, ds \right).$$

Theorem 2. *Let* $p_* \le -\frac{3}{4}$ *and*

$$p^* < p_* - 1 + \frac{1}{2}\sqrt{1 - 4p_*}\,. \tag{4}$$

Then $\dim V_0(p) = 1$. *If, moreover,* $p(t) \le 0$ *in some neighbourhood of* $+\infty$, *then* $V_0(p) = Z(p)$.

Theorem 3. *Let* $p_* < 0$ *and*

$$p^* < p_* - \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4p_*}\,.$$

Then $\dim V_1(p) = 1$. *If, moreover,* $p(t) \le 0$ *in some neighbourhood of* $+\infty$, *then* $V_1(p) = Z(p)$.

It is proved in [1] that if $p_* > -\frac{1}{2}$ and $p^* < \frac{1}{4}$, then $V_0(p) = \{0\}$. The following theorem makes this result more complete.

Theorem 4. *Let* $p_* < -\frac{1}{2}$ *and*

$$p^* < -\sqrt{p_*^2 - p_* - \frac{3}{4}}\,. \tag{5}$$

Then $V_0(p) = \{0\}$.

2. Proof

Proof of Theorem 1. From (3) it follows that $\int^{+\infty} s|p(s)|ds = +\infty$. However this condition is necessary and sufficient for $Z(p) \ne \{0\}$ (see [2] and [3]). On the other hand, obviously $\dim Z(p) = 1$ and $V_i(p) \subset Z(p)$. Thus it is sufficient to show that $Z(p) \subset V_i(p)$.

Suppose $u \in Z(p)$. Without loss of generality we can assume that for some $t_0 > 0$,

$$p(t) \le 0 \qquad \text{for } t > t_0\,, \tag{6}$$

$$u(t) > 0, \qquad u'(t) < 0 \qquad \text{for } t > t_0\,. \tag{7}$$

Multiplying both sides of (1) by t and integrating from t_0 to t, we obtain

$$\int_{t_0}^{t} sp(s)u(s)\,ds = tu'(t) - t_0u'(t_0) - u(t) + u(t_0) \qquad \text{for } t > t_0\,.$$

Hence on account of (6) and (7), we easily conclude that for some $r > 0$,

$$u(t)\int_0^t s|p(s)|\,ds < M \qquad \text{for } t > t_0\,.$$

Therefore, by virtue of (3) condition (2) holds. Thus the theorem is proved. □

Proof of Theorem 2. According to (4), we can find $\varepsilon > 0$ such that $p^* < -\frac{1}{2} - 2\varepsilon$ and

$$p^* < p_* - 1 + \frac{1}{2}\sqrt{1 - 4(p_* - \varepsilon)} - 3\varepsilon\,. \tag{8}$$

Suppose

$$Q(t) = t\left(c_p - \int_1^t p(s)\,ds\right) \qquad \text{for } t > 0 \tag{9}$$

and choose $t_\varepsilon > 1$ such that

$$p_* - \varepsilon < Q(t) < p^* + \varepsilon \qquad \text{for } t > t_\varepsilon\,. \tag{10}$$

Let $\alpha = -\frac{1}{2} - p^* - 2\varepsilon$. It is evident that $\alpha > 0$ and

$$\alpha - \sqrt{\alpha} + p^* + \varepsilon < 0\,. \tag{11}$$

Due to (8), it is easy to see that $\alpha + \sqrt{\alpha} + p_* - \varepsilon < 0$. If along with this we take into account (11), then from (10) we get

$$-\alpha - \sqrt{\alpha} < Q(t) < -\alpha + \sqrt{\alpha} \qquad \text{for } t > t_\varepsilon\,,$$

and therefore, $Q^2(t) + 2\alpha Q(t) + \alpha(\alpha - 1) < 0$ for $t > t_\varepsilon$. In view of this, it is clear that the function $w(t) = t^\alpha$ satisfies the inequality

$$w''(t) \le -\frac{Q^2(t)}{t^2}w(t) - \frac{2Q(t)}{t}w'(t) \qquad \text{for } t > t_\varepsilon\,.$$

Consequently, the equation

$$v'' = -\frac{Q^2(t)}{t^2}v - \frac{2Q(t)}{t}v' \tag{12}$$

has a solution v satisfying the inequalities

$$0 < v(t) < t^\alpha \qquad \text{for } t > t_\varepsilon \tag{13}$$

(see, e.g., [4]).

It can be directly checked that the function

$$u(t) = v(t)\exp\left(\int_1^t \frac{Q(s)}{s}\,ds\right) \qquad \text{for } t > t_\varepsilon \tag{14}$$

is a solution of equation (1). By (10) and (13), there are $M > 0$ and $t_1 > t_\varepsilon$ such that

$$0 < u(t) < Mt^{\alpha+p^*+\varepsilon} \qquad \text{for } t > t_1\,.$$

Hence, taking into consideration how α is, we conclude that $u \in V_0(p)$. Therefore we have proved that $V_0(p) \neq \{0\}$.

Since $u(t) > 0$ for $t > t_1$, we have $\dim V_0(p) \le 1$ (see, e.g., [1]). However $\dim V_0(p) = 1$, since $V_0(p) \neq \{0\}$.

Let us now suppose that $p(t) \le 0$ in some neighbourhood of $+\infty$. Then it is obvious that $\dim Z(p) \le 1$ and $V_0(p) \subset Z(p)$. Hence in view of the fact that $\dim V_0(p) = 1$, we obtain $V_0(p) = Z(p)$. This completes the proof of the theorem. □

The proof of Theorem 3 is omitted, since it is analogous to that of Theorem 2 with the only difference $\alpha = -p^* - \varepsilon$.

Proof of Theorem 4. Assume the contrary. Let u be a nontrivial solution of equation (1) and $u \in V_0(p)$. According to (5) and applying Theorem 1.6 from [5], equation (1) is nonoscillatory. Thus without loss of generality we can assume that $u(t) > 0$ for $t > t_0$. Choose $\varepsilon \in \,]0, \frac{1}{2}[$ and $t_\varepsilon > t_0$ such that (10) holds and

$$p_* - \varepsilon < -\frac{1}{2}\,, \tag{15}$$

$$p^* + \varepsilon < -\sqrt{(p_* - \varepsilon)^2 - (p_* - \varepsilon) - \frac{3}{4}}\,. \tag{16}$$

It is evident that the function v defined by (14) and (9) is a solution of equation (12). According to our assumption,

$$v(t) > 0 \qquad \text{for } t > t_0\,, \tag{17}$$

$$\int^{+\infty} v^2(s)\exp\left(2\int_1^s \frac{Q(\eta)}{\eta}\,d\eta\right) ds < +\infty\,. \tag{18}$$

Let us show that for some $t_1 > t_0$,

$$v'(t) > 0 \qquad \text{for } t > t_1\,. \tag{19}$$

Indeed, if there exists $t_* > t_1$ such that $v'(t_*) \le 0$, then by virtue of the equality

$$\left(v'(t)\exp\left(2\int_1^t \frac{Q(\eta)}{\eta}\,d\eta\right)\right)' = -\frac{Q^2(t)}{t^2}\exp\left(2\int_1^t \frac{Q(\eta)}{\eta}\,d\eta\right) v(t) \qquad \text{for } t > 0\,,$$

we have

$$v'(t) < 0 \qquad \text{for } t > t_* . \tag{20}$$

Then in view of (15), from (12) we find $v''(t) < 0$ for $t > t_*$. But this together with (20) contradicts inequality (17).

By (10), (17), and (19), from (12) we get

$$v''(t) \leq -\frac{(p^* + \varepsilon)^2}{t^2} v(t) - \frac{2(p_* - \varepsilon)}{t} v'(t) \qquad \text{for } t > t_1 .$$

Consequently, the equation

$$w'' = -\frac{(p^* + \varepsilon)^2}{t^2} w - \frac{2(p_* - \varepsilon)}{t} w' \tag{21}$$

has a solution w satisfying the inequalities

$$0 < w(t) < v(t) \qquad \text{for } t > t_1 .$$

Hence due to (10) and (18), we have

$$\int^{+\infty} w^2(s) s^{2(p_* - \varepsilon)}\, ds < +\infty . \tag{22}$$

On the other hand, we can easily check that the functions $w_k(t) = t^{\lambda_k}$, $k = 1, 2$, where

$$\lambda_k = \frac{1}{2}\left[1 - 2(p_* - \varepsilon) - (-1)^k \sqrt{(1 - 2(p_* - \varepsilon))^2 - 4(p^* + \varepsilon)^2}\right] , \qquad k = 1, 2,$$

are linearly independent solutions of equation (21). By (16), $2\lambda_1 + 2(p_* - \varepsilon) > -1$ and therefore,

$$\int^{+\infty} w_k^2(s) s^{2(p_* - \varepsilon)}\, ds = +\infty , \qquad k = 1, 2 .$$

Thus neither of nontrivial solutions of equation (21) satisfies condition (22). This concludes the proof of the theorem. □

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T. Chantladze and N. Kandelaki
N. Muskhelishvili Institute of Computational Mathematics
Georgian Academy of Sciences
8, Akuri St., 380093 Tbilisi, GEORGIA

A. Lomtatidze
Department of Mathematical Analysis, Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, CZECH REPUBLIC
E-mail: bacho@math.muni.cz