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A RESULT ON BEST APPROXIMATION IN p-NORMED SPACES

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Abstract. We study best approximation in p-normed spaces via a general common fixed point principle. Our results unify and extend some known results of Carbone [2], Dotson [3], Jungck and Sessa [6], Singh [14] and many of others.

1. Introduction and Preliminaries

Brosowski [1] and Meinardus [11] established some interesting results on invariant approximations in normed spaces using fixed point theory. Recently Jungck and Sessa [6] have obtained some results on approximation theory in the setting of normed spaces. The purpose of this paper is to study the invariant best approximation in the setting of p-normed spaces. We prove a common fixed point result in p-normed spaces and then obtain a result on best approximation theory. These results unify and extend some recent results in [2], [3], [6], [14] etc.

The following definitions and result will be needed.

Let X be a linear space. A p-norm on X is a real-valued function $||.||_p$ on X with $0 < p \leq 1$ [9], satisfying the following conditions:

(a) $||x||_p \geq 0$ and $||x||_p = 0$ iff $x = 0$

(b) $||\lambda x||_p = |\lambda|^p ||x||_p$

(c) $||x + y||_p \leq ||x||_p + ||y||_p$

for all $x, y \in X$ and all scalars λ. The pair $(X, ||\cdot||_p)$ is called a p-normed space. It is a metric space with $d_p(x, y) = ||x - y||_p$ for all $x, y \in X$, defining a translation-invariant metric d_p on X. If $p = 1$, we obtain the concept of a normed linear space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some p-norm, $0 < p \leq 1$ [9]. The spaces l_p and $L_p[0, 1]$, $0 < p \leq 1$, are p-normed spaces. A p-normed space is not necessarily a locally convex space.

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Let C be a subset of a p-normed space X. Then C is called starshaped with respect to a point $q \in C$ if $\{kx + (1-k)q : 0 \leq k \leq 1\} \subset C$ for each $x \in C$. C is said to be starshaped if it is starshaped with respect to one of its element. Each convex set is necessarily starshaped, but a starshaped set need not be convex. For any $\hat{x} \in X$, let $P_C(\hat{x}) := \{y \in C : d_p(y, \hat{x}) = d_p(\hat{x}, C)\}$, the set of best C-approximants to $\hat{x}$, where $d_p(\hat{x}, C) = \inf_{z \in C} d_p(\hat{x}, z)$. $P_C(\hat{x})$ is always a bounded subset of X and it is closed or convex if C is so [1].

A map $T : C \to C$ is said to be *f-contraction* if there exists a self-map f on C and a real number $k \in (0,1)$ such that

$$\|Tx - Ty\|_p \leq k \, \|fx - fy\|_p \quad \text{for all} \quad x, y \in C\,.$$

If in the above inequality $k = 1$, then T is called *f-nonexpansive*. We denote by $F(T)$ the set of fixed points of T.

In [5] Jungck proved a common fixed point theorem for complete metric spaces:

Theorem 1.1. *Let f be a continuous self map of a complete metric space (X, d). If $T : X \to X$ is a f-contraction map which commutes with f and $T(X) \subseteq f(X)$, then f and T have a unique common fixed point.*

In [6] Jungck and Sessa have obtained the following common fixed point result in the setting of Banach spaces:

Theorem 1.2. *Let C be a nonempty weakly compact subset of a Banach space X which is starshaped with respect to $q \in C$. Let $f : C \to C$ be continuous in the weak and strong topology on C, affine with $q \in F(f)$ and $f(C) = C$. Suppose $T : C \to C$ is a f-nonexpansive map which commutes with f. If $f - T$ is demiclosed or X is an Opial space, then $F(T) \cap F(f) \neq \emptyset$.*

Remark. The weak continuity condition of f in the hypothesis of Theorem 1.2 can be dropped, since continuous affine maps are weakly continuous [3].

Recall that if X is a topological linear space, then its continuous dual space X' is said to separate the points of X if for each $x \neq 0$ in X, there exists an $f \in X'$ such that $fx \neq 0$. In this case the weak topology on X is well-defined (see for example, Rudin [13, §3.11]). We mention that if X is not locally convex, then X' need not separate the points of X. For example, if $X = L_p[0,1], 0 < p < 1$, then $X' = \{0\}$. However, there are some non-locally convex spaces (such as the p-normed space $l_p, 0 < p < 1$) whose dual separates the points [9]).

Let X be a complete p-normed space whose dual X' separates the points of X. A map $T : C \subseteq X \to X$ is said to be *demiclosed* if for every sequence $\{x_n\} \subset C$ such that x_n converges weakly to $x \in C$ and Tx_n converges strongly to $y \in X$, then $y = Tx$.

The space X is said to be an *Opial space* [12, 10] (also see [7, p. 498]) if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n \to \infty} \|x_n - x\|_p < \liminf_{n \to \infty} \|x_n - y\|_p$$

holds for all $y \neq x$.

2. Main Results

Throughout this section, X denotes a complete p-normed space whose dual separates the points of X. We have the following common fixed point result for such spaces. The proof is an appropriate modification of the one given in [6].

Theorem 2.1. *Let C be a nonempty weakly compact subset of X which is star-shaped with respect to $q \in C$. Let f be a continuous and affine self map of C with $q \in F(f)$ and $f(C) = C$. Suppose $T : C \to C$ is a f-nonexpansive map which commutes with f. If $f - T$ is demiclosed or X is an Opial space, then $F(T) \cap F(f) \neq \emptyset$.*

Proof. Let $\{k_n\}$ be a sequence of real numbers such that $0 < k_n < 1$ and $k_n \to 1$. Define a sequence $\{T_n\}$ of maps on C by putting

$$T_n x = k_n T x + (1 - k_n) q$$

for all $x \in C$. Then for each $n \geq 1$, a map T_n does carry C into C since C is starshaped with respect to $q \in C$ and $T(C) \subseteq C$. Also, note that $T_n(C) \subseteq f(C)$. Since T commutes with f and f is affine map, for each $x \in C$,

$$\begin{aligned} T_n f x &= k_n T f x + (1 - k_n) f q \\ &= k_n f T x + (1 - k_n) f q \\ &= f(k_n T x + (1 - k_n) q) = f T_n x \,. \end{aligned}$$

Thus each T_n commutes with f. Further, we observe that for each $n \geq 1$,

$$||T_n x - T_n y||_p \leq (k_n)^p ||fx - fy||_p$$

for all $x, y \in C$. Hence, by Theorem 1.1, there is a unique $x_n \in C$ such that

$$x_n = T_n x_n = f x_n$$

for all $n \geq 1$. Since C is weakly compact, there is a subsequence $\{x_{n_i}\}$ of sequence $\{x_n\}$ which converges weakly to some $x_0 \in C$. Since f is a continuous affine map, it is also weakly continuous, and so we have

$$f x_0 = \lim_i f x_{n_i} = \lim_i x_{n_i} = x_0 \,.$$

Also note that

$$(f - T) x_{n_i} = (\frac{1}{k_{n_i}} - 1)(q - x_{n_i}) \,.$$

Thus

$$||(f - T) x_{n_i}||_p = (\frac{1}{k_{n_i}} - 1)^p ||q - x_{n_i}||_p \leq (\frac{1}{k_{n_i}} - 1)^p [||x_{n_i}||_p + ||q||_p] \,.$$

Since C is bounded, $x_{n_i} \in C$ implies $\{||x_{n_i}||_p\}$ is bounded and so by the fact that $k_{n_i} \to 1$, we have

$$||(f - T) x_{n_i}||_p \to 0$$

Now, if $f-T$ is demiclosed, then $(f-T)x_0 = 0$ and thus $fx_0 = x_0 = Tx_0$. If X is an Opial space and $x_0 \neq Tx_0$, then

$$\begin{aligned}\liminf_i \|x_{n_i} - x_0\|_p &< \liminf_i \|x_{n_i} - Tx_0\|_p \\ &\leq \liminf_i \|Tx_{n_i} - Tx_0\|_p + \liminf_i \|(f-T)x_{n_i}\|_p\end{aligned}$$

and thus

$$\liminf_i \|x_{n_i} - x_0\|_p < \liminf_i \|Tx_{n_i} - Tx_0\|_p \,.$$

But on the other hand we have

$$\liminf_i \|Tx_{n_i} - Tx_0\|_p \leq \liminf_i \|fx_{n_i} - fx_0\|_p = \liminf_i \|x_{n_i} - x_0\|_p \,.$$

which is a contradiction. Hence $x_0 \in F(T) \cap F(f)$.

Remark 2.2. Theorem 2.1 extends Theorem 1.2 for p-normed spaces and if $f = I$, the identity map on C and $p = 1$, then it becomes Theorem 2 of Dotson [3].

In what follows we shall need the following lemma [8] (see also [4]). For the sake of completeness we give its proof.

Lemma 2.3. *Let C be a subset X. Then, for any $\hat{x} \in X$, $P_C(\hat{x}) \subseteq \partial C$ (the boundary of C).*

Proof. Let $y \in P_C(\hat{x})$, and suppose that y is in the interior of C. Then there exists an $\epsilon > 0$ such that $B_\epsilon(y) \subset C$. For each $n \geq 1$, let $z_n = \frac{1}{n}\hat{x} + (1 - \frac{1}{n})y$. Then

$$\|z_n - y\|_p = \|\frac{1}{n}(\hat{x} - y)\|_p = (\frac{1}{n})^p \|\hat{x} - y\|_p .$$

For sufficiently large $N \geq 1$, we have $\|z_N - y\|_p < \epsilon$, and so $z_N \in B_\epsilon(y) \subset C$. Now

$$\|\hat{x} - z_N\|_p = (1 - \frac{1}{N})^p \|\hat{x} - y\|_p < \|\hat{x} - y\|_p = d_p(\hat{x}, C)$$

which contradicts the fact that $y \in P_C(\hat{x})$ and hence $y \in \partial C$.

As an application of Theorem 2.1, we have the following result on best approximation.

Theorem 2.4. *Let T and f be self maps of an Opial space X, C be a subset of X such that $T(\partial C) \subseteq C$ and $\hat{x} \in F(T) \cap F(f)$. Let $P_C(\hat{x})$ be nonempty, weakly compact and starshaped with respect to $q \in F(f)$ and let T be a f-nonexpansive map on $P_C(\hat{x}) \cup \{\hat{x}\}$, where f is affine, continuous on $P_C(\hat{x})$, $f(P_C(\hat{x})) = P_C(\hat{x})$ and commutes with T on $P_C(\hat{x})$. Then $P_C(\hat{x}) \cap F(T) \cap F(f) \neq \emptyset$.*

Proof. Let $y \in P_C(\hat{x})$. Since $f(P_C(\hat{x})) = P_C(\hat{x})$, so $fy \in P_C(\hat{x})$. Also, by Lemma 2.3 $y \in \partial C$. As $T(\partial C) \subseteq C$, so $Ty \in C$. Since $T\hat{x} = \hat{x}$ and T is a f-nonexpansive map,

$$\|Ty - \hat{x}\|_p = \|Ty - T\hat{x}\|_p \leq \|fy - f\hat{x}\|_p.$$

As $f\hat{x} = \hat{x}$ and $fy \in P_C(\hat{x})$,

$$\|Ty - \hat{x}\|_p \leq \|fy - \hat{x}\|_p = d_p(\hat{x}, C).$$

Since $Ty \in C$ we have, $Ty \in P_C(\hat{x})$. Thus T maps $P_C(\hat{x})$ into $P_C(\hat{x})$ and hence the result follows by Theorem 2.1 with assumption that X is an Opial space.

Remark 2.5. Theorem 2.4 extends Theorem 7 of Jungck and Sessa [6] to p-normed spaces and contains Theorem 3 of Carbone [2] as a special case for $p = 1$.

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