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INNER AMENABILITY OF LAU ALGEBRAS

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Abstract. A concept of amenability for an arbitrary Lau algebra called inner amenability is introduced and studied. The inner amenability of a discrete semigroup is characterized by the inner amenability of its convolution semigroup algebra. Also, inner amenable Lau algebras are characterized by several equivalent statements which are similar analogues of properties characterizing left amenable Lau algebras.

1. Introduction and Preliminaries

A complex Banach algebra $\mathcal{A}$ is called *Lau algebra* if it is the (unique) predual of a W^*-algebra $\mathcal{M}$ and the identity element u of $\mathcal{M}$ is a multiplicative linear functional on $\mathcal{A}$. The subject of this class of Banach algebras originated with a paper published in 1983 by Lau [10] in which he referred to them as "F-algebras". Later on, in his useful monograph Pier [24] introduced the name "Lau algebra".

The wide range of Lau algebras includes the Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group algebra $L^1(G)$ of a locally compact group G, and the measure algebra of a locally compact semigroup or hypergroup. In particular, it includes the semigroup algebra $\ell^1(S)$ of a discrete semigroup S.

As pointed out in Lau [10], $\mathcal{M}$ need not be unique. Nevertheless, we shall identify the continuous dual $\mathcal{A}^*$ of $\mathcal{A}$ with $\mathcal{M}$ if there is no confusion. The second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ equipped with the first Arens multiplication $\odot$ is a Lau algebra [10, Prop. 3.2], where $\odot$ is defined by the equations

$$\langle F \odot H, f\rangle = \langle F, Hf\rangle, \quad \langle Hf, a\rangle = \langle H, fa\rangle, \quad \langle fa, b\rangle = \langle f, ab\rangle$$

for all $F, H \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$, and $a, b \in \mathcal{A}$. Let $P(\mathcal{A})$ be the set of all elements a in $\mathcal{A}$ that induce positive functionals on the W^*-algebra $\mathcal{A}^*$, and let $P_1(\mathcal{A})$ be the set of all elements a in $P(\mathcal{A})$ such that $\langle u, a\rangle = 1$; note that $P(\mathcal{A}) = \{a \in \mathcal{A} : \|a\| = \langle u, a\rangle\}$; see [25], 1.5.1 and 1.5.2.

An element M of $P_1(\mathcal{A}^{**})$ is said to be a *topological left* (resp. *two-sided*) *invariant mean* if $a \odot M = M$ (resp. $a \odot M = M \odot a = M$) for all $a \in P_1(\mathcal{A})$. The

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Lau algebra $\mathcal{A}$ is called *left* (resp. *two-sided*) *amenable* if there exists a topological left (resp. two-sided) invariant mean on $\mathcal{A}^*$.

The notion of left amenability of Lau algebras was introduced by Lau in [10]. In the same paper he obtained several characterizations of left amenable Lau algebras. See also Ghahramani and Lau [8], Lau [11], Lau and Wong [14], and the author [19].

The purpose of this paper is to introduce and to study a concept of amenability for Lau algebras. The idea behind our concept of amenability is quite simple. Let us first recall that a discrete semigroup S is called *inner amenable* if there is an element M of $P_1(\ell^\infty(S)^*)$ such that $\langle M, xf\rangle = \langle M, fx\rangle$ for all $x \in S$ and $f \in \ell^\infty(S)$, where $(xf)(y) = f(yx)$ and $(fx)(y) = f(xy)$ for all $y \in S$. Such an M is called an *inner invariant mean* on $\ell^\infty(S)$. Here $\ell^\infty(S)$ denotes the W*-algebra of all bounded complex-valued functions on S which can be realized as the dual of $\ell^1(S)$.

The study of inner amenability is initiated by Effros [7] and pursued by Akeman [1], H. Choda [4], M. Choda [5, 6], Kaniuth and Markfort [9], Paschke [20], Pier [22], and Watatani [27] for discrete groups and recently by Ling [15] for discrete semigroups.

Observe that an element M of $P_1(\ell^\infty(S)^*)$ is inner invariant if and only if $\langle M, af\rangle = \langle M, fa\rangle$ for all $a \in \ell^1(S)$ and $f \in \ell^\infty(S)$; this is because $fa = \sum_{x\in S} a(x)\, fx$ and $af = \sum_{x\in S} a(x)\, xf$.

We say that $\mathcal{A}$ is *inner amenable* if there exists M in $P_1(\mathcal{A}^{**})$ such that $\langle M, fa\rangle = \langle M, af\rangle$ for all $a \in P_1(\mathcal{A})$ and $f \in \mathcal{A}^*$. We call such an M a *topological inner invariant mean* (=TIIM) on $\mathcal{A}^*$ and denote by TIIM$(\mathcal{A}^*)$ the set of topological inner invariant means on $\mathcal{A}^*$.

Thus the inner amenability of S is equivalent to the inner amenability of $\ell^1(S)$. In fact, TIIM$(\ell^\infty(S))$ is exactly the set of all inner invariant mean on $\ell^\infty(S)$.

Note that an element M of $P_1(\mathcal{A}^{**})$ is a TIIM on $\mathcal{A}^*$ if and only if $a \odot M = M \odot a$ for all $a \in P_1(\mathcal{A})$ (or equivalently $a \in \mathcal{A}$), and clearly, any topological two-sided invariant mean on $\mathcal{A}^*$ is a TIIM, and therefore two-sided amenable Lau algebras are inner amenable.

We will show that many results concerning left amenability of Lau algebras have similar analogues for inner amenability. We will also give a fixed point property characterizing inner amenable Lau algebras. Finally, we study the relation between inner amenability of two Lau algebras with inner amenability of their direct sum.

2. Elementary Results

Throughout, $\mathcal{A}$ denotes a Lau algebra and u denotes the identity element of the dual W^*-algebra $\mathcal{A}^*$.

Proposition 2.1. *$\mathcal{A}$ is inner amenable if and only if there is a nonzero self-adjoint element F of $\mathcal{A}^{**}$ such that $a \odot F = F \odot a$ and $\|a \odot F\| = \|F\|$ for all $a \in P_1(\mathcal{A})$.*

Proof. The "only if" part is trivial. To prove the converse, we suppose that F is a nonzero self-adjoint element of $\mathcal{A}^{**}$ with $a \odot F = F \odot a$ such that $\|a \odot F\| = \|F\|$ for all $a \in P_1(\mathcal{A})$. Then there exist unique elements F^+ and F^- of $P(\mathcal{A}^{**})$ such that $F = F^+ - F^-$ and $\|F\| = \|F^+\| + \|F^-\|$ for $a \in P_1(\mathcal{A})$ [25, 1.14.3]. Thus $a \odot F^+$ and $a \odot F^-$ are in $P(\mathcal{A}^{**})$, and

$$\begin{aligned} \|a \odot F\| &= \|F^+\| + \|F^-\| = \langle F^+, u\rangle + \langle F^-, u\rangle \\ &= \langle a \odot F^+, u\rangle + \langle a \odot F^-, u\rangle = \|a \odot F^+\| + \|a \odot F^-\|. \end{aligned}$$

Similarly, $\|a \odot F\| = \|F^+ \odot a\| + \|F^- \odot a\|$. These equalities together with

$$a \odot F = a \odot F^+ - a \odot F^- = F^+ \odot a - F^- \odot a$$

imply that $a \odot F^+ = F^+ \odot a$ and $a \odot F^- = F^- \odot a$ [25, 1.14.3]. So, if F^+ is nonzero (say), then $\|F^+\|^{-1} F^+$ lies in $\mathrm{TIIM}(\mathcal{A}^*)$. □

As a consequence of the proof of the above proposition, we have

Corollary 2.2. *$\mathcal{A}^*$ has a* TIIM *in $P_1(\mathcal{A})$ if and only if there is a nonzero self-adjoint element b of $\mathcal{A}$ such that $ab = ba$ and $\|ab\| = \|b\|$ for all $a \in P_1(\mathcal{A})$.*

Observe that for H fixed in $\mathcal{A}^{**}$, the mapping $F \mapsto F \odot H$ is weak*-weak* continuous on $\mathcal{A}^{**}$. For F fixed in $\mathcal{A}^{**}$, the mapping $H \mapsto F \odot H$ is in general not weak*-weak* continuous on $\mathcal{A}^{**}$ unless F is in $\mathcal{A}$. The *topological center* $Z(\mathcal{A}^{**})$ of $\mathcal{A}^{**}$ is defined as the set of all $F \in \mathcal{A}^{**}$ such that the mapping $H \mapsto F \odot H$ is weak*-weak* continuous on $\mathcal{A}^{**}$.

Proposition 2.3. The following statements hold:

(a) *Any $M \in \mathrm{TIIM}(\mathcal{A}^{***})$ restricted to $\mathcal{A}^*$ is a* TIIM *on $\mathcal{A}^*$*;

(b) $\mathrm{TIIM}(\mathcal{A}^*) \cap Z(\mathcal{A}^{**}) \subseteq \mathrm{TIIM}(\mathcal{A}^{***})$.

Proof. (a) is clear. To prove (b), suppose that $M \in \mathrm{TIIM}(\mathcal{A}^*) \cap Z(\mathcal{A}^{**})$. Since $P_1(\mathcal{A})$ is weak* dense in $P_1(\mathcal{A}^{**})$ [11, Lemma 2.1], for any $N \in P_1(\mathcal{A}^{**})$, there exists a net (a_γ) in $P_1(\mathcal{A})$ with weak* limit N, and therefore

$$\begin{aligned} N \odot M &= \text{weak}^* - \lim_\gamma a_\gamma \odot M \\ &= \text{weak}^* - \lim_\gamma M \odot a_\gamma = M \odot N \end{aligned}$$

by the definition of $Z(\mathcal{A}^{**})$. That is $M \in \mathrm{TIIM}(\mathcal{A}^{***})$. □

Recall that an element E of $\mathcal{A}^{**}$ is called *mixed identity* if $a \odot E = E \odot a = a$ for all $a \in \mathcal{A}$. It is easy to see that an element E of $\mathcal{A}^{**}$ is a mixed identity if and only if it is a weak* cluster point of a bounded approximate identity in $\mathcal{A}$ [3, p. 146]. Furthermore, since we endowed $\mathcal{A}^{**}$ with the first Arens product $\odot$, any mixed identity is a right identity of $\mathcal{A}^{**}$ but not a left identity in general. It follows from the following generalization of [8, Prop. 2.3] that any mixed identity with norm one of $\mathcal{A}^{**}$ is in $\mathrm{TIIM}(\mathcal{A}^*)$.

Proposition 2.4. *Suppose that $\mathcal{A}^{**}$ has a right identity E. Then the following statements are equivalent:*

(a) $\mid E \mid$ *is a right identity;*
(b) $\|E\| = 1$;
(c) $E \in P(\mathcal{A}^{**})$;

here $\mid E \mid$ *denotes the absolute value of E regarded as an element in the predual of the W^*-algebra $\mathcal{A}^{***}$* [26, p. 143].

Proof. Fix $a \in P_1(\mathcal{A})$, and use the multiplicativity of u on $\mathcal{A}^{**}$ to conclude that

$$\langle E, u\rangle = \langle a, u\rangle\langle E, u\rangle = \langle a \odot E, u\rangle = \langle a, u\rangle = 1.$$

Now the result follows immediately. □

When $\mathcal{A}^{**}$ has a mixed identity with norm one, we say that $\mathcal{A}$ is *strictly inner amenable* if there is a TIIM on $\mathcal{A}^*$ that is not a mixed identity.

Let us recall that a locally compact group G is *amenable* if there is a *left invariant mean* on the dual W^*-algebras $L^\infty(G)$ of $L^1(G)$; that is an element M of $P_1(L^\infty(G)^*)$ such that $\langle M, fx\rangle = \langle M, f\rangle$ for all $f \in L^\infty(G)$ and $x \in G$. Examples of amenable locally compact groups include all solvable groups, abelian groups and all compact groups; see Pier [23] for details.

Example 2.5. (a) If $\mathcal{A}^{**}$ has a mixed identity E with $\|E\| = 1$, then E is a topological left invariant mean on $\mathcal{A}^*$ if and only if $\mathcal{A}$ is one dimensional; it follows that if $\mathcal{A}$ is two-sided amenable and has dimension more than one, then $\mathcal{A}$ is strictly inner amenable.

In particular, if G is a non-trivial amenable locally compact group, then $L^1(G)$ is strictly inner amenable; this follows from the fact that G is amenable if and only if $L^1(G)$ is two-sided (or left) amenable [23, Theorem 4.19]. For example, the group algebra $\ell^1(F_2)$ of the free group on two generators F_2, is not strictly inner amenable [7], and hence it is not two-sided amenable (see [23, Prop. 14.1] for a direct proof of this fact).

(b) Let P be the discrete group of all permutations of $\mathbf{N}$ leaving all but a finite number of elements unchanged. Then $P \times F_2$ is a non-amenable discrete group and has an inner invariant mean M not equal to the Dirac measure at the identity element of $P \times F_2$ [23, Prop. 22.37]. Therefore $\ell^1(P \times F_2)$ is a strictly inner amenable Lau algebra that is not left amenable.

Before stating the following consequence of Proposition 2.3, let us recall that $\mathcal{A}$ is called *Arens regular* if $Z(\mathcal{A}^{**})$ coincides with $\mathcal{A}^{**}$.

Corollary 2.6. *Suppose that $\mathcal{A}$ is Arens regular. Then*

(a) *$\mathcal{A}^{**}$ is inner amenable if and only if $\mathcal{A}$ is inner amenable.*
(b) *If $\mathcal{A}^{**}$ has a mixed identity with norm one and is not strictly inner amenable, then $\mathcal{A}$ is not strictly inner amenable.*

Note that $P_1(\mathcal{A})$ endowed with induced norm topology of $\mathcal{A}$ and the product of $\mathcal{A}$ is a topological semigroup. Let $C_b(P_1(\mathcal{A}))$ denote the Banach space of all bounded, and continuous functions on $P_1(\mathcal{A})$ with the supremum norm, and define the left and right translation operators l_a and r_b on $C_b(P_1(\mathcal{A}))$ by $(l_a\phi)(b) = \phi(ab) = (r_b\phi)(a)$ for all $a, b \in P_1(\mathcal{A})$ and $\phi \in C_b(P_1(\mathcal{A}))$.

A function ϕ in $C_b(P_1(\mathcal{A}))$ is called *additively uniformly continuous* on $P_1(\mathcal{A})$ if given $\varepsilon > 0$, there is $\delta > 0$ such that $\mid \phi(a) - \phi(b) \mid < \varepsilon$ for all $a, b \in P_1(\mathcal{A})$ with $\|a - b\| < \delta$. Let $C_{au}(P_1(\mathcal{A}))$ denote the set of all additively uniformly continuous functions on $P_1(\mathcal{A})$. Then $C_{au}(P_1(\mathcal{A}))$ is norm closed, translation invariant subspace of $C_b(P_1(\mathcal{A}))$ containing constants and restrictions of elements in $\mathcal{A}^*$ to $P_1(\mathcal{A})$ [14]. An element m of $C_{au}(P_1(\mathcal{A}))^*$ is called a *mean* if $\langle m, 1\rangle = \|m\| = 1$; we call m *inner invariant* if, in addition, $\langle m, l_a\phi\rangle = \langle m, r_a\phi\rangle$ for all $\phi \in C_{au}(P_1(\mathcal{A}))$ and $a \in P_1(\mathcal{A})$.

Now, we can present the main result of this section.

Theorem 2.7. *The following statements are equivalent.*

(a) *$\mathcal{A}$ is inner amenable.*

(b) *There is a net (a_γ) in $P_1(\mathcal{A})$ such that $a_\gamma a - aa_\gamma \to 0$ in the weak topology of $\mathcal{A}$ for all $a \in P_1(\mathcal{A})$.*

(c) *There is a net (a_γ) in $P_1(\mathcal{A})$ such that $\|a_\gamma a - aa_\gamma\| \to 0$ for all $a \in P_1(\mathcal{A})$.*

(d) *There is an inner invariant mean on $C_{au}(P_1(\mathcal{A}))$.*

Proof. (a)$\Rightarrow$(b). Let $M \in \mathrm{TIIM}(\mathcal{A}^*)$. Then, since $P_1(\mathcal{A})$ is weak* dense in $P_1(\mathcal{A}^{**})$, there is a net (a_γ) in $P_1(\mathcal{A})$ such that $a_\gamma \to M$ in the weak* topology of $\mathcal{A}^*$. It follows that $a_\gamma a - aa_\gamma \to 0$ in the weak topology of $\mathcal{A}$ for all $a \in P_1(\mathcal{A})$.

(b)$\Rightarrow$(c). We follow Namioka's idea in [18]. Let Y be the vector space $\Pi\{\mathcal{A} : b \in P_1(\mathcal{A})\}$ and let $T : \mathcal{A} \to Y$ be the linear map defined by $T(a)(b) = ba - ab$ for all $a \in \mathcal{A}$ and $b \in P_1(\mathcal{A})$. By assumption, the weak closure of $T(P_1(\mathcal{A}))$ contains 0. Since Y with the product of the norm topology is a locally convex space and $P_1(\mathcal{A})$ is convex, the closure of $T(P_1(\mathcal{A}))$ in this topology contains 0. That is (c) holds.

(c)$\Rightarrow$(d). Let (a_γ) be as in (c). If we define $m_\gamma \in C_{au}(P_1(\mathcal{A}))^*$ by $\langle m_\gamma, \phi\rangle = \phi(a_\gamma)$ for all $\phi \in C_{au}(P_1(\mathcal{A}))$, then any weak* cluster point of (m_γ) in $C_{au}(P_1(\mathcal{A}))^*$ is an inner invariant mean.

(d)$\Rightarrow$(a). Let m be an inner invariant mean on $C_{au}(P_1(\mathcal{A}))^*$, and define $M \in \mathcal{A}^{**}$ by $\langle M, f\rangle = \langle m, f\mid_{P_1(\mathcal{A})}\rangle$ for $f \in \mathcal{A}^*$, where $f\mid_{P_1(\mathcal{A})}$ denotes the restriction of f to $P_1(\mathcal{A})$. Then M is a TIIM on $\mathcal{A}^*$. □

The equivalences (a)$\Leftrightarrow$(c) and (a)$\Leftrightarrow$(d) are similar analogues of Theorem 3.5 (a)$\Leftrightarrow$(b) in [10] and Lemma 2.1 in [14] for left amenability.

Remark 2.8. (a) Let us remark that if $\mathcal{A}$ has a bounded approximate identity (e_γ) in $P_1(\mathcal{A})$, then (e_γ) satisfies the conditions (b) and (c) of the preceding theorem.

(b) One can see easily that the following equivalent statements characterize commutative Lau algebras in terms of topological inner invariant means.

(1) $P_1(\mathcal{A}^{**}) = \mathrm{TIIM}(\mathcal{A}^*)$.
(2) $P_1(\mathcal{A}) \subseteq \mathrm{TIIM}(\mathcal{A}^*)$.
(3) $P_1(\mathcal{A})$ is a commutative semigroup.
(4) Any mean on $C_{au}(P_1(\mathcal{A}))$ is inner invariant.

In particular, any commutative Lau algebra is inner amenable. For example, the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G are always inner amenable.

(c) Let G be a locally compact group. Then any mixed identity E with norm one in $L^\infty(G)^*$ is an extension of δ_e from $C_b(G)$ to $L^\infty(G)$, where δ_e denotes the Dirac measure at the identity e of G; indeed an argument similar to the first part of the proof of [16, Theorem 2] shows that $\langle E, g\rangle = g(e)$ for each continuous function g with compact support, and then [16, Lemma 3] implies that $\langle E, g\rangle = g(e)$ for all $g \in C_b(G)$. Conversely, it is readily verified that any extension of δ_e to an element in $P_1(L^\infty(G)^*)$ is a mixed identity. Consequently, the set of mixed identities with norm one in $L^\infty(G)^*$ are exactly the extensions of δ_e to elements of $P_1(L^\infty(G)^*)$.

In the case where G is discrete, δ_e is the only mixed identity with norm one in $\ell^\infty(G)^*$, and of course δ_e is an inner invariant mean. However, a mixed identity with norm one in $L^\infty(G)^*$ is not in general an *inner invariant mean* (i.e. an element M of $P_1(L^\infty(G)^*)$ such that $\langle M, xf\rangle = \langle M, fx\rangle$ for all $x \in G$ and $f \in L^\infty(G)$), although it is a TIIM on $L^\infty(G)$. Losert and Rindler [16] and Yuan [28] studied the possibility of the extension of δ_e to an inner invariant mean on $L^\infty(G)$. See also Bekka [2], Lau and Paterson [12, 13], Losert and Rindler [17], and Paterson [21].

(d) There are many interesting and important relations between left invariant means and topological left invariant means on $L^\infty(G)$ of a locally compact group G. For example, topological left invariant means on $L^\infty(G)$ are special types of left invariant means (with additional invariance properties), and left amenability of $L^1(G)$ is equivalent to (left) amenability of G; see Pier [23] for details.

However, (in the non-discrete case) topological inner invariant means on $L^\infty(G)$ do not seem to be related to inner invariant means. Furthermore, G need not be inner amenable whereas $L^1(G)$ is always inner amenable. For example, non-amenable connected locally compact groups are not inner amenable; this is because that in this case, as shown in Losert and Rindler [17], amenability of G is equivalent to *inner amenability*; i.e. existence of an inner invariant mean on $L^\infty(G)$.

On the other hand, in the case where G is non-trivial and connected, $L^1(G)$ is strictly inner amenable if G is *strictly inner amenable*; i.e. there is an inner invariant mean on $L^\infty(G)$ that is not a mixed identity in $L^\infty(G)^*$. Moreover, in the case where G is amenable or discrete, $L^1(G)$ is strictly inner amenable if and only if so is G. These observations lead us to ask:

Question. Is strict inner amenability of $L^1(G)$ equivalent to strict inner amenability of G?

3. Fixed Point Characterization of Inner Amenability

Let $\mathcal{A}$ be a Lau algebra and X be a left Banach $\mathcal{A}$-module, i.e. a Banach space X equipped with a bounded bilinear map from $\mathcal{A} \times X$ into X, denoted by $(a, x) \mapsto a \cdot x$ $(a \in \mathcal{A}, x \in X)$ such that $a \cdot (b \cdot x) = (ab) \cdot x$ for all $a, b \in \mathcal{A}$, and $x \in X$. Define

$$\langle x^* \cdot a, x \rangle = \langle x^*, a \cdot x \rangle, \quad \text{and} \quad \langle a \cdot x^{**}, x^* \rangle = \langle x^{**}, x^* \cdot a \rangle$$

for all $a \in \mathcal{A}, x \in X, x^* \in X^*$, and $x^{**} \in X^{**}$. Let $\mathcal{B}(X^{**})$ denote the Banach space of all bounded operators on X^{**}. By *weak* operator topology* on $\mathcal{B}(X^{**})$, we shall mean the locally convex topology of $\mathcal{B}(X^{**})$ determined by the family

$$\{T \mapsto | \langle Tx^{**}, x^* \rangle | : \ x^{**} \in X^{**}, x^* \in X^*\}$$

of seminorms on $\mathcal{B}(X^{**})$. We also denote by $\mathcal{P}(\mathcal{A}, X^{**})$ the closure of the set $\{\Lambda_a : a \in P_1(\mathcal{A}) \}$ in the weak* operator topology, where $\Lambda_a \in \mathcal{B}(X^{**})$ is defined by $\Lambda_a(x^{**}) = a \cdot x^{**}$ for all $x^{**} \in X^{**}$.

Proposition 3.1. *If $\mathcal{A}$ is inner amenable, then for each left Banach $\mathcal{A}$-module X, there is $\Lambda \in \mathcal{P}(\mathcal{A}, X^{**})$ such that $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in P_1(\mathcal{A})$.*

Proof. First note that if we identify $\mathcal{B}(X^{**})$ with $(X^{**} \otimes X^*)^*$, then the weak* operator topology of $\mathcal{B}(X^{**})$ coincides with the weak* topology of $(X^{**} \otimes X^*)^*$. So $\mathcal{P}(\mathcal{A}, X^{**})$ is compact in the weak* operator topology. Using Theorem 2.7, there exists a net (a_γ) in $P_1(\mathcal{A})$ such that $||a_\gamma a - a a_\gamma|| \to 0$ for all $a \in P_1(\mathcal{A})$. Hence we may find $\Lambda \in \mathcal{P}(\mathcal{A}, X^{**})$ with $||\Lambda|| \leq 1$ and a subnet (a_δ) of (a_γ) such that $\Lambda_{a_\delta} \to \Lambda$ in the weak* operator topology. For each $a \in P_1(\mathcal{A})$, we therefore have $\Lambda_{a_\delta}\Lambda_a \to \Lambda\Lambda_a$ and $\Lambda_a\Lambda_{a_\delta} \to \Lambda_a\Lambda$ in the weak* operator topology. Also

$$||\Lambda_{a_\delta}\Lambda_a - \Lambda_a\Lambda_{a_\delta}|| \leq K||a_\delta a - a a_\delta|| \to 0,$$

where K is a constant satisfying $||b \cdot x|| \leq K||b|| \, ||x||$ for all $b \in \mathcal{A}$ and $x \in X$. Consequently, $\Lambda\Lambda_a = \Lambda_a\Lambda$. □

We are now in a position to give a fixed point property characterizing inner amenability of certain Lau algebras.

Theorem 3.2. *Suppose that $\mathcal{A}$ has a bounded right approximate identity. Then the following are equivalent.*

(a) *$\mathcal{A}$ is inner amenable.*
(b) *There exists $\Lambda \in \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ such that $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in P_1(\mathcal{A})$.*
(c) *For each left Banach $\mathcal{A}$-module X, there exists $\Lambda \in \mathcal{P}(\mathcal{A}, X^{**})$ such that $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in P_1(\mathcal{A})$.*

Proof. By Proposition 3.1, (a) implies (b). That (b) implies (c) is trivial. Now, suppose that (c) holds, and choose an element Λ of $\mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ such that $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in P_1(\mathcal{A})$. To prove (a), we follow the proof of [13, Theorem 5.1]. Choose a net (a_γ) in $P_1(\mathcal{A})$ such that $\Lambda_{a_\gamma} \to \Lambda$ in the weak* operator topology of $\mathcal{B}(\mathcal{A}^{**})$, and let M be a weak* cluster point of (a_γ) in $P_1(\mathcal{A}^{**})$. Then $\Lambda F = M \odot F$ for $F \in \mathcal{A}^{**}$. Indeed, for each $f \in \mathcal{A}^*$,

$$\begin{aligned}\langle \Lambda F, f\rangle &= \lim_\delta \langle \Lambda_{a_\delta} F, f\rangle = \lim_\delta \langle a_\delta \cdot F, f\rangle \\ &= \lim_\delta \langle a_\delta \odot F, f\rangle = \lim_\delta \langle a_\delta, Ff\rangle \\ &= \langle M, Ff\rangle = \langle M \odot F, f\rangle \,,\end{aligned}$$

where (a_δ) is a subnet of (a_γ) converging to M in the weak* topology.

We show that M is a TIIM on $\mathcal{A}^*$. To observe this, let E be a weak* cluster point of a bounded right approximate identity of $\mathcal{A}$ in $\mathcal{A}^{**}$. Then E is a right identity for $\mathcal{A}^{**}$, and for each $a \in \mathcal{A}$, we conclude that

$$\begin{aligned} M \odot a &= M \odot (a \odot E) = \Lambda \Lambda_a E \\ &= \Lambda_a \Lambda E = a \odot (M \odot E) = a \odot M \,. \end{aligned}$$ □

Remark 3.3. Since $P_1(\mathcal{A})$ is weak* dense in $P_1(\mathcal{A}^{**})$ [11, Lemma 2.1], the mapping Φ from $P_1(\mathcal{A}^{**})$ into $\mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$, defined by $\Phi(N)(F) = N \odot F$ for all $N \in P_1(\mathcal{A}^{**})$ and $F \in \mathcal{A}^{**}$, is a semigroup homomorphism. The proof of Theorem 3.2 shows that Φ is onto. Also, if $\mathcal{A}^{**}$ has a right identity E, then $\Phi(M)(E) = M$ for all $M \in P_1(\mathcal{A}^{**})$, and so Φ is a semigroup isomorphism. Furthermore, if $\mathcal{A}^{**}$ has a left identity E, then $\Phi(E) = I_{\mathcal{A}^{**}}$, the identity operator on $\mathcal{A}^{**}$, and hence $I_{\mathcal{A}^{**}} \in \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$.

An application of the above remark gives the following characterization of strict inner amenability.

Corollary 3.4. *Suppose that $\mathcal{A}^{**}$ has an identity with norm one. Then the following assertions are equivalent.*

(a) *$\mathcal{A}$ is strictly inner amenable.*
(b) *There is $\Lambda \in \mathcal{P}(\mathcal{A}, \mathcal{A}^{**})$ such that $\Lambda \neq I_{\mathcal{A}^{**}}$ and $\Lambda\Lambda_a = \Lambda_a\Lambda$ for all $a \in P_1(\mathcal{A})$.*

4. Inner Amenability of Direct Sums

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two Lau algebras. Define the *direct sum* of $\mathcal{A}_1$ and $\mathcal{A}_2$ denoted by $\mathcal{A}_1 \oplus \mathcal{A}_2$ to be the complex algebra consisting of all ordered pairs (a_1, a_2), $a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2$, with coordinatewise addition, scalar multiplication, and product

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 + \langle u_2, a_2\rangle b_1 + \langle u_2, b_2\rangle a_1, a_2 b_2),$$

where u_i is the identity element of $(\mathcal{A}_i)^*$ $(i = 1, 2)$. Then $\mathcal{A}_1 \oplus \mathcal{A}_2$ with norm $\|(a_1, a_2)\| = \|a_1\| + \|a_2\|$ is a Lau algebra which the linear functional $(a_1, a_2) \mapsto \langle u_1, a_1\rangle + \langle u_2, a_2\rangle$ is the identity of $(\mathcal{A}_1 \oplus \mathcal{A}_2)^*$ [10, Prop. 3.6].

Lemma 4.1. *The semigroup $P_1(\mathcal{A}_1 \oplus \mathcal{A}_2)$ is equal to all $(t_1 a_1, t_2 a_2)$ such that $a_i \in \{0\} \cup P_1(\mathcal{A}_i)$, $t_i \geq 0$ $(i = 1, 2)$ and $t_1\|a_1\| + t_2\|a_2\| = 1$.*

Proof. Let $(b_1, b_2) \in P_1(\mathcal{A}_1 \oplus \mathcal{A}_2)$. Then we have

$$\|b_1\| + \|b_2\| = \langle u_1, b_1\rangle + \langle u_2, b_2\rangle = 1,$$

whence $|\langle u_1, b_1\rangle| + |\langle u_2, b_2\rangle| = 1$. It follows that $\langle u_i, b_i\rangle$ is a positive real number for $i = 1, 2$. This implies that $\langle u_i, b_i\rangle = \|b_i\|$ $(i = 1, 2)$.

Putting $a_i = \|b_i\|^{-1}b_i$ and $t_i = \|b_i\|$ if $b_i \neq 0$, and choosing a_i in $P_1(\mathcal{A}_i)$ arbitrarily and $t_i = 0$ if $b_i = 0$, we see that $b_i = t_i a_i$, $a_i \in P_1(\mathcal{A}_i)$ $(i = 1, 2)$, and $t_1\|a_1\| + t_2\|a_2\| = 1$. This proves the forward inclusion. The reverse inclusion is trivial. □

We now give the main result of this section which is a similar analogue of Lau [10, Prop. 4.5] concerning left amenability.

Proposition 4.2. *The set* TIIM$((\mathcal{A}_1 \oplus \mathcal{A}_2)^*)$ *is equal to all* (t_1M_1, t_2M_2) *in* $P_1((\mathcal{A}_1 \oplus \mathcal{A}_2)^{**})$ *such that* $M_i \in \{0\}\cup$ TIIM$((\mathcal{A}_i)^*)$ *and* $t_i \geq 0$ $(i = 1, 2)$. *In particular,* $\mathcal{A}_1 \oplus \mathcal{A}_2$ *is inner amenable if and only if* $\mathcal{A}_1$ *or* $\mathcal{A}_2$ *is inner amenable.*

Proof. First note that the map J from the Lau algebra $(\mathcal{A}_1)^{**} \oplus (\mathcal{A}_2)^{**}$ onto the Lau algebra $(\mathcal{A}_1 \oplus \mathcal{A}_2)^{**}$ defined by

$$\langle J(F_1, F_2), (f_1, f_2)\rangle = \langle F_1, f_1\rangle + \langle F_2, f_2\rangle \quad (F_i \in (\mathcal{A}_i)^{**},\ f_i \in (\mathcal{A}_i)^*,\ i = 1, 2)$$

is a linear isometry and algebra isomorphism [10, Prop. 3.5]. Now, let $(a_1, a_2) \in P_1(\mathcal{A}_1 \oplus \mathcal{A}_2)$. If $M_i \in \{0\}\cup$ TIIM$((\mathcal{A}_i)^*)$ and $t_i \geq 0$ $(i = 1, 2)$ with $(t_1M_1, t_2M_2) \in P_1((\mathcal{A}_1 \oplus \mathcal{A}_2)^{**})$, then

$$\begin{aligned}
&(a_1, a_2) \odot (t_1M_1, t_2M_2)\\
&\qquad= \big(t_1(a_1 \odot M_1) + \langle u_2, a_2\rangle t_1M_1 + \langle t_2M_2, u_2\rangle a_1, t_2(a_2 \odot M_2)\big)\\
&\qquad= \big(t_1(M_1 \odot a_1) + \langle t_2M_2, u_2\rangle a_1 + \langle u_2, a_2\rangle t_1M_1, t_2(M_2 \odot a_2)\big)\\
(*)\quad &\qquad= (t_1M_1, t_2M_2) \odot (a_1, a_2)
\end{aligned}$$

That is $(t_1M_1, t_2M_2) \in$ TIIM$((\mathcal{A}_1 \oplus \mathcal{A}_2)^*)$.

Conversely, if $(N_1, N_2) \in$ TIIM$((\mathcal{A}_1 \oplus \mathcal{A}_2)^*)$, then by Lemma 4.1, we have $N_i = t_iM_i$ for some $M_i \in \{0\} \cup P_1((\mathcal{A}_i)^{**})$ and $t_i \geq 0$ $(i = 1, 2)$. Thus equalities similar to $(*)$ hold for all $(a_1, a_2) \in P_1(\mathcal{A}_1) \times P_1(\mathcal{A}_2)$; we therefore have $a_i \odot M_i = M_i \odot a_i$ whence $M_i \in$ TIIM$((\mathcal{A}_i)^*)$ $(i = 1, 2)$. □

Note that $(\mathcal{A}_1 \oplus \mathcal{A}_2)^{**}$ has a mixed identity with norm one if and only if $\mathcal{A}_2$ has a mixed identity with norm one. Indeed an element (F, E) of $(\mathcal{A}_1 \oplus \mathcal{A}_2)^{**}$ is a mixed identity for $(\mathcal{A}_1 \oplus \mathcal{A}_2)^{**}$ if and only if $F = 0$ and E is a mixed identity for $\mathcal{A}_2$; this follows from the fact that a net (a_γ, e_γ) is a bounded approximate identity for $\mathcal{A}_1 \oplus \mathcal{A}_2$ if and only if $a_\gamma \to 0$ and (e_γ) is a bounded approximate identity for $\mathcal{A}_2$ [10, Prop. 3.4]. So, using the above proposition, we have

Corollary 4.3. *Suppose that* $(\mathcal{A}_2)^{**}$ *has a mixed identity with norm one. Then* $\mathcal{A}_1 \oplus \mathcal{A}_2$ *is strictly inner amenable if and only if* $\mathcal{A}_1$ *is inner amenable or* $\mathcal{A}_2$ *is strictly inner amenable.*

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