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ON NEUMANN ELLIPTIC PROBLEMS WITH DISCONTINUOUS NONLINEARITIES

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Abstract. In this paper we study a class of nonlinear Neumann elliptic problems with discontinuous nonlinearities. We examine elliptic problems with multivalued boundary conditions involving the subdifferential of a locally Lipschitz function in the sense of Clarke.

1. Introduction

In this paper, using the critical point theory of Chang [3] for locally Lipschitz functionals, we study nonlinear boundary problems with discontinuous nonlinearities and nonlinear Neumann boundary conditions. Let $Z \subseteq R^N$ be a bounded domain with a C^1-boundary Γ. The problem under consideration is:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = f(z,x(z)) \text{ a.e. on } Z \\ -\frac{\partial x}{\partial n_p}(z) \in \partial j(z,\tau(x)(z)) \text{ a.e. on } \Gamma, 2 \le p < \infty\,. \end{cases} \tag{1}$$

Here, ∂ is the subdifferential for locally Lipschitz functionals in the sense of Clarke [4].

First, we convert the single-valued problem to a multivalued problem by filling in the gaps. The multivalued one is:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in [f_1(z,x(z)), f_2(z,x(z))] \text{ a.e. on } Z \\ -\frac{\partial x}{\partial n_p} \in \partial j(z,\tau(x)(z)) \text{ a.e. on } \Gamma, 2 \le p < \infty\,. \end{cases} \tag{2}$$

with $f_1(z,x) = \liminf\limits_{x\to x'} f(z,x')$ and $f_2(z,x) = \limsup\limits_{x\to x'} f(z,x')$. First we prove an existence result for problem (2) using the critical point theory of Chang [3]. Finally we prove an existence result for the single-valued one under more restrictive hypothesis on the right hand side. Stuart-Tolland [7] proved an analogous theorem for a Dirichlet boundary value problem involving the laplacian operator. Also Ambrosetti-Badiale [1] studied a Dirichlet problem by using Clarke's dual action

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principle and they used a very interesting technic so that the energy functional remains a C^1 functional. Finally, Heikkila-Lakshmikantham [5] had studied these problems using the method of upper and lower solutions.

Here we study Neumann problems with the p-Laplacian operator and nonlinear boundary conditions. It seems that is the first such result on Neumann problems. In the following section we state some facts from the critical point theory and the subdifferential for locally lipschitz functionals.

2. Preliminaries

Let Y be a subset of X. A function $f : Y \to R$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative scalar K, one has

$$|f(y) - f(x)| \leq K\|y - x\|$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^o(x; v)$ is defined as follows:

$$f^o(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t} ,$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \to f^o(x; v)$ is finite, positively homogeneous, subadditive and satisfies $|f^o(x; v)| \leq K\|v\|$. In addition f^o satisfies $f^o(x; -v) = -f^o(x; v)$. Now we are ready to introduce the generalized gradient which denoted by $\partial f(x)$ as follows:

$$\partial f(x) = \{w \in X^* : f^o(x; v) \geq \langle w, v\rangle \quad \text{for all} \quad v \in X\} .$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of X^* and $\|w\|_* \leq K$ for every w in $\partial f(x)$.

(b) For every v in X, one has

$$f^o(x; v) = \max\{\langle w, v\rangle : w \in \partial f(x)\} .$$

If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2 .$$

Let us recall the (P.S)-condition introduced by Chang.

Definition. We say that Lipschitz function f satisfies the Palais-Smale condition if any sequence $\{x_n\}$ along which $|f(x_n)|$ is bounded and

$$\lambda(x_n) = \mathrm{Min}_{w \in \partial f(x_n)} \|w\|_{X^*} \to 0$$

possesses a convergent subsequence.

Let us now recall a theorem of Chang for minimization of locally Lipschitz functionals.

Theorem 1. *Suppose a locally Lipschitz function f defined on a reflexive Banach space, satisfies the (P.S)-condition and it is bounded from below. Then $c = \inf_X f(x)$ is a critical value of f.*

Recall that we say that c is a critical value of f if there exists a critical point x_o in $f^{-1}(c)$.

In the next section we will use the following inequality which appears in Tolksdorff [8].

$$\sum_{j=1}^{N}(a_j(\eta) - a_j(\eta'))(\eta - \eta') \geq C|\eta - \eta'|^p \,. \tag{3}$$

for $\eta, \eta' \in R^N$, with $a_j(\eta) = |\eta|^{p-2}\eta_j$.

3. Existence theory

Let us state the hypotheses for the function f and j of problem (1).

H(f): $f : Z \times R \to R$ is a function such that

(i) $f(z, \cdot)$ is N-measurable (i.e. if $x(\cdot) \in W^{1,p}(Z)$ is measurable so is $f(z, x(z))$).

(ii) there exists $h : R \to R$ such that $h(x) \to \infty$ as $n \to \infty$ and there exists $M > 0$ such that $-F(z, x) \geq h(|x|)$ for $|x| \geq M$ with $F(z, x) = \int_Z f(z, r)dr$.

(iii) $|f(z, x)| \leq a(z) + c|x|^{\mu-1}$, $\mu < p$ for almost all $z \in Z$ and all $x \in R$.

H(j): $j : Z \times R \to R$ such that $z \to j(z, x)$ is measurable and $x \to j(z, x)$ locally Lipschitz. Also $j(z, \cdot) \geq 0$ for almost all $z \in Z$ and finally $|w(z)| \leq a_1(z) + c|x|^{p^*-1}$ with $p^* = \frac{Np}{N-p}$ for every $w(z) \in \partial j(z, x)$.

Remark. If hypothesis $H(j)$ holds, then theorem 2.7.5 of Clarke [4] is satisfied.

Proposition 1. *If hypotheses* H(f), H(j) *holds, then problem* (2) *has a solution $x \in W^{1,p}(Z)$.*

Proof. Let $\Phi(x) = -\int_Z F(z, x(z))dz$ and $\psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_\Gamma j(z, \tau(x(z))d\sigma$. Then the energy functional is $R(x) = \Phi(x) + \psi(x)$.

Claim 1: $R(\cdot)$ satisfies the (P.S)-condition of Chang [3].

Indeed, let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}$ is such that $R(x_n) \to c$ as $n \to \infty$. We shall prove that this sequence is bounded in $W^{1,p}(Z)$. Suppose not. Then $\|x_n\| \to \infty$. Let $y_n(z) = \frac{x_n(z)}{\|x_n\|}$. Then clearly we have $y_n \xrightarrow{w} y$ in $W^{1,p}(Z)$. From the choice of the sequence we have

$$\Phi(x_n) + \frac{1}{p}\|Dx_n\|_p^p \leq M \tag{4}$$

(recall that $j(z, \cdot) \geq 0$). Dividing with $\|x_n\|^p$ the last inequality, we have

$$-\int_Z \frac{F(z, x(z))}{\|x_n\|^p}dz + \frac{1}{p}\|Dy_n\|_p^p \leq \frac{M}{\|x_n\|^p} \,.$$

By virtue of hypothesis $H(f)$(iii) we have that $\frac{F(z,x(z))}{\|x_n\|_p^p} \to 0$. Hence $\limsup \|Dy_n\|_p^p \to 0$. Thus, $\|Dy\| = 0$ and it arises that $y = c \in R$. But $\|y_n\| = 1$, so $c \neq 0$ and we have that $|x_n(z)| \to \infty$. Going back to (3) and using hypothesis $H(f)$(ii) we have a contradiction. So $\|x_n\|$ is bounded, i.e $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$. It remains to show that $x_n \to x$ in $W^{1,p}(Z)$. From the properties of the subdifferential of Clarke, we have

$$\begin{aligned}\partial R(x_n) &\subseteq \partial\Phi(x_n) + \partial\psi(x_n)\\ &\subseteq \partial\Phi(x_n) + \partial(\frac{1}{p}\|Dx_n\|_p^p) + \int_\Gamma \partial j(z, \tau(x_n(z)))\, d\sigma\end{aligned}$$

(see Clarke [4], p. 83).

So we have

$$\langle w_n, y\rangle = \langle Ax_n, y\rangle + \langle r_n, y\rangle - \int_Z v_n(z)y(z)\, dz\ ,$$

where $r_n(z) \in \partial j(z, x_n(z))$, $v_n(z) \in [f_1(z, x_n(z)), f_2(z, x_n(z))]$, w_n the element with minimal norm of the subdifferential of R and finally $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ is such that $\langle Ax, y\rangle = \int_Z(\|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{R^N}\, dz$. But $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$, so $x_n \to x$ in $L^p(Z)$ and $x_n(z) \to x(z)$ a.e. on Z by virtue of the compact embedding $W^{1,p}(Z) \subseteq L^p(Z)$. Thus, r_n is bounded in $L^q(Z)$ (see Chang [3], p. 104 Proposition 2), i.e $r_n \xrightarrow{w} r$ in $L^q(Z)$. Choose $y = x_n - x$. Then in the limit we have that $\limsup\langle Ax_n, x_n - x\rangle = 0$ (note that v_n is bounded). By virtue of the inequality (3) of Tolksdorff we have that $Dx_n \to Dx$ in $L^p(Z)$. So we have $x_n \to x$ in $W^{1,p}(Z)$. The Claim is proved.

Claim 2: $R(\cdot)$ is bounded from below.

Indeed, suppose not. Then there exists some sequence $\{x_n\}_{n\geq 1}$ such that $R(x_n) \leq -n$. Then we have

$$\Phi(x_n) + \psi(x_n) \leq -n$$

(recall that $j(z, \cdot) \geq 0$.) By virtue of the continuity of $\Phi+\psi$ we have that $\|x_n\| \to \infty$ (because if $\|x_n\|$ is bounded then $\Phi(x_n) + \psi(x_n)$ is also bounded). Dividing with $\|x_n\|^p$ and letting $n \to \infty$ we have as before a contradiction (by virtue of hypothesis $H(f)$(ii)). Therefore $R(\cdot)$ is bounded from below.

So by Theorem 1 we have that there exists $x \in W^{1,p}(Z)$ such that $0 \in \partial R(x)$. That is $0 \in \partial\Phi(x) + \partial\psi(x)$. Let $\psi_1(x) = \frac{\|Dx\|^p}{p}$ and $\psi_2(x) = \int_\Gamma j(z, \tau(x)(z) d\sigma$. Then let $\widehat{\psi}_1 : L^p(Z) \to R$ the extension of ψ_1 in $L^p(Z)$. Then $\partial\psi_1(x) \subseteq \partial\widehat{\psi}_1(x)$ (see Chang [3]). It is easy to prove that the nonlinear operator $\widehat{A} : D(A) \subseteq L^p(Z) \to L^q(Z)$ such that

$$\langle\widehat{A}x, y\rangle = \int_Z \|Dx(Z)\|^{p-2}(Dx(z), Dy(z))dz \quad \text{for all} \quad y \in W^{1,p}(Z)$$

with $D(A)=\{x\in W^{1,p}(Z):\widehat{A}x\in L^q(Z)\}$, satisfies $\widehat{A}=\partial\widehat{\psi}_1$. Indeed, first we show that $\widehat{A}\subseteq\partial\widehat{\psi}$ and then it suffices to show that $\widehat{A}$ is maximal monotone.

$$\begin{aligned}
\langle\widehat{A}x,y-x\rangle &= \int_Z \|Dx(z)\|^{p-2}(Dx(z),Dy(z)-Dx(z))_{R^N}\,dz\\
&= \int_Z \|Dx(z)\|^{p-2}(Dx(z),Dy(z))_{R^N}\,dz-\int_Z\|Dx(z)\|^p dz\\
&\le \int_Z\Big(\frac{\|Dx(z)\|^{q(p-2)}\|Dx(z)\|^q}{q}+\frac{\|Dy(z)\|^p}{p}\Big)dz-\|Dx\|_p^p\\
&= \frac{\|Dx\|_p^p}{q}-\|Dx\|^p+\frac{\|Dy\|_p^p}{p}\\
&= \widehat{\psi}_1(y)-\widehat{\psi}_1(x)\,.
\end{aligned}$$

The monotonicity part is obvious using inequality (3). The maximality needs more work. Let $J:L^p(Z)\to L^q(Z)$ be defined as $J(x)=|x(\cdot)|^{p-2}x(\cdot)$. We will show that $R(\widehat{A}+J)=L^q(Z)$. Assume for the moment that this holds. Then let $v\in L^p(Z), v^*\in L^q(Z)$ such that

$$(\widehat{A}(x)-v^*,x-v)_{pq}\ge 0$$

for all $x\in D(\widehat{A})$. Therefore there exists $x\in D(\widehat{A})$ such that $\widehat{A}(x)+J(x)=v^*+J(v)$ (recall that we assumed that $R(\widehat{A}+J)=L^q(Z)$). Using this in the above inequality we have that

$$(J(v)-J(x),x-v)_{pq}\ge 0\,.$$

But J is strongly monotone. Thus we have that $v=x$ and $\widehat{A}(x)=v^*$. Therefore $\widehat{A}$ is maximal monotone. It remains to show that $R(\widehat{A}+J)=L^q(Z)$. But $\widehat{J}=J\mid_{W^{1,p}(Z)}: W^{1,p}(Z)\to W^{1,p}(Z)^*$ is maximal monotone, because is demicontinuous and monotone. So $A+\widehat{J}$ is maximal monotone. But it is easy to see that the sum is coercive. So is surjective. Therefore, $R(A+\widehat{J})=W^{1,p}(Z)^*$. Then for every $g\in L^q(Z)$, we can find $x\in W^{1,p}(Z)$ such that $A+\widehat{J}(x)=g\Rightarrow A(x)=g-\widehat{J}(x)\in L^q(Z)\Rightarrow A(x)=\widehat{A}(x)$. Thus, $R(\widehat{A}+J)=L^q(Z)$.

So, we can say that

$$\int_Z w(z)y(z)=\int_Z\|Dx(z)\|^{p-2}(Dx(z),Dy(z))\,dz+\int_\Gamma v(z)y(z)\,d\sigma \tag{5}$$

with $w(z)\in[f_1(z,x(z)),f_2(z,x(z))]$ and $v(z)\in\partial j(z,\tau(x(z)))$, for every $y\in W^{1,p}(Z)$. Let $y=\phi\in C_o^\infty(Z)$. Then we have

$$\int_Z w(z)\phi(z)\,dz=\int_Z\|Dx(z)\|^{p-2}(Dx(z),D\phi(z))\,dz\,.$$

But $\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z))\in W^{-1,q}(Z)$ then we have that

$$\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)\in L^q(Z)\quad\text{because}\quad w(Z)\in L^q(Z)\,.$$

Then we have that $-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z))\in[f_1(z,x(z)),f_2(z,x(z))]$ a.e. on Z. Going back to (5) and letting $y\in C^\infty(Z)$ and finally using the Green formula

1.6 of Kenmochi [6], we have that $-\frac{\partial x}{\partial n_p} \in \partial j(z, \tau(x)(z))$. So $x \in W^{1,p}(Z)$ solves (2). □

Let now state the following condition on f.

$\mathbf{H(f)_1}$: f satisfies $H(f)$ and in addition there exists $g : Z \times R \to R$ Carathéodory such that $x \to g(z,x) - f(z,x)$ is increasing.

Remark. If f satisfies the hypothesis $\mathrm{H(f)}_1$ then has countable number of discontinuities.

Theorem 2. *If the hypotheses* $\mathrm{H(f)}_1$, $\mathrm{H(j)}$ *holds, then problem* (1) *has a solution* $x \in W^{1,p}(Z)$.

Proof. If

$$\Phi(x) = -\int_Z F(z, x(z))\, dz + \int_Z \int_o^{x(z)} g(z,r)\, dr dz\,,$$

$$\psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_\Gamma j(z, x(z))\, dz - \int_Z \int_o^{x(z)} g(z,r)\, dr dz\,,$$

then the energy functional is $R = \Phi + \psi$.

From Proposition (1) we know that there exists $x \in W^{1,p}(Z)$ such that $0 \in \partial R(x)$. Then from definition of the subdifferential of Clarke we have $0 \le R^o(x; v)$ for all $v \in W^{1,p}(Z)$. So, we have $0 \le \Phi^o(x;v) + \psi^o(x;v) \Rightarrow -\Phi^o(x;v) \le \psi^o(x;v)$, that is $-\partial\Phi(x) \subseteq \partial\psi(x)$.

We will show that $\lambda\{z \in Z : x(z) \in D(f)\} = 0$ with $D(f) = \{x \in R : f(x^+) < f(x^-)\}$, that is the set of downward-jumps.

Let $w \in \partial(-\Phi(x))$ and for any $t \in D(f)$, set

$$\rho^{\pm}(z) = [1 - \chi_t(x(z))]w(z) + \chi_t(x(z))[f(z, x(z)^{\pm})]\,, \tag{6}$$

where

$$\chi_t(s) = \begin{cases} 1 & \text{if} \quad s = t \\ 0 & \text{if} \quad \text{otherwise.} \end{cases} \tag{7}$$

Then $\rho^{\pm} \in L^p(Z)$ and $\rho^{\pm} \in \partial\psi(x)$. Hence

$$\int_Z \rho^{\pm}(z) y(z)\, dz = \int_Z (\|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{R^N}\, dz + \int_\Gamma v(z) y(z)\, d\sigma$$

for all $y \in W^{1,p}(Z)$.

So for $y = \phi \in C_o^\infty(Z)$ we have

$$\int_Z \rho^{\pm}(z)\phi(z)\, dz = \int_Z (\|Dx(z)\|^{p-2}(Dx(z), D\phi(z))_{R^N}\, dz\,.$$

Thus, $\rho^+ = \rho^-$ for almost all $z \in Z$. From this it follows that $\chi_t(x(z)) = 0$ for almost all $z \in Z$. Since $D(f)$ is countable, and

$$\chi(x(z)) = \sum_{t \in D(f)} \chi_t(x(z))\,,$$

it follows that $\chi(x(z)) = 0$ almost everywhere, (with $\chi(t) = 1$ if $t \in D(f)$ and $\chi(t) = 0$ otherwise).

Now it is clear that $x \in W^{1,p}(Z)$ solves problem (1). $\square$

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