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ALMOST CONTINUITY VS CLOSURE CONTINUITY

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Abstract. We provide an answer to a question: under what conditions almost continuity in the sense of Husain implies closure continuity?

1. Introduction

As a generalization of continuity, Levine [7] introduced weak continuity for a function from one topological space into another. The concept of closure continuity (also called θ-continuity), which implies weak continuity, was given independently by Fomin [4] in 1941 and Andrew and Whittlesy [1] in 1966. It appears that Andrew and Whittlesy were not aware of Fomin's paper. Other two different generalizations of continuity are the almost continuity of Husain [5] (in Euclidean spaces this notion was initially introduced and studied by Blumberg [2]) and the almost continuity of Singal and Singal [16]. Similarities and dissimilarities of these concepts were discussed in [8]. In 1968 Singal and Singal [16] remarked that closure continuous functions are not always almost continuous and asked whether every almost continuous function is closure continuous or not. Recently, Saleh [15] gave a positive answer to the above question. Indeed, in 1970s Papp [12] and Kim [6] independently showed that almost continuity implies closure continuity. It seems that the already published answers ([6], [12]) were not known to Saleh [15]. Naturally one may raise the following question:

Under what conditions almost continuity in the sense of Husain implies closure continuity?

In 1974 Noiri [11] considered the above question for weak continuous functions and showed that any almost continuous function $f : X \to Y$ (in the sense of Husain) is weakly continuous provided it satisfies the following condition:

$$\mathrm{cl}\left(f^{-1}(V)\right) \subset f^{-1}\left(\mathrm{cl}(V)\right) \tag{1}$$

for each open set $V \subset Y$, where X and Y are topological spaces. Rose [13] observed that condition (1) is equivalent to weak continuity and so almost continuity in Noiri's result is redundant. The aim of this paper is to provide an answer to the

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above question. In fact, we prove that any almost continuous function in the sense of Husain satisfying condition (1) is closure continuous. We observe that every closure continuous function satisfies condition (1) but the converse is not true.

2. Preliminaries

Unless otherwise specified, $f : X \to Y$ will denote an arbitrary function of a topological space X into a topological space Y. Let A be a subset of a topological space. Then $\mathrm{cl}(A)$ and $\mathrm{Int}\,(A)$ denote the closure and the interior of A respectively.

Definition 1. A function $f : X \to Y$ is weakly continuous (resp. closure continuous) if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subset \mathrm{cl}(V)$ (resp. $f\left(\mathrm{cl}(U)\right) \subset \mathrm{cl}(V)$. The weak continuity of $f : X \to Y$ is characterized by condition (1) [13, Theorem 7].

Remark 1. Every closure continuous function $f : X \to Y$ satisfies condition (1). But a function $f : X \to Y$ which satisfies condition (1) is not necessarily closure continuous, as the following example illustrates:

Example 1. Let $X = \{x, y, z, w\}$ have the topology

$$\tau = \{\phi, \{z\}, \{z, w\}, \{x, y, z\}, X\} .$$

Let $Y = \{a, b, c, d\}$ have the topology

$$\sigma = \{\phi, \{b\}, \{d\}, \{a, b\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, Y\} .$$

Let $f : X \to Y$ be defined by

$$f(x) = a\,, \quad f(y) = b\,, \quad f(z) = c\,, \quad f(w) = d\,.$$

Then

$$\mathrm{cl}\left(f^{-1}(V)\right) \subset f^{-1}\left(\mathrm{cl}(V)\right)$$

for all $V \in \sigma$. But f is not closure continuous (see, for instance, [3]).

Definition 2. A mapping $f : X \to Y$ is said to be almost continuous in the sense of Singal and Singal if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U in X containing x such that

$$f(U) \subset \mathrm{Int}\ \mathrm{cl}(V)\,.$$

Definition 3. A mapping $f : X \to Y$ is said to be almost continuous in the sense of Husain if for each point $x \in X$ and each open set V in Y containing $f(x), \mathrm{cl}\left(f^{-1}(V)\right)$ is a neighbourhood of x.

Remark 2.

(i) We observe that almost continuity in the sense of Singal and Singal implies closure continuity, which in turn implies weak continuity but neither implication is reversible. However, the three notions are equivalent if the range space is regular. For details, we refer the reader to [6, 7, 12, 16].

(ii) Examples in [9,13,16] show that Definitions 2 and 3 are completely independent to each other.

(iii) Example 1 of Rose [13] (or Example 1 of Long and McGehee [9]) shows that an almost continuous function in the sense of Husain need not be closure continuous. The following example shows that every closure continuous function is not necessarily almost continuous in the sense of Husain.

Example 2. Let $X = \mathbb{R}$ with the countable complement topology. Let $Y = \{a, b\}$ with the topology $\{\phi, \{a\}, Y\}$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} a & \text{if } x \text{ is rational,} \\ b & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is closure continuous but is not almost continuous in the sense of Husain ([11, 16]). This shows the independence of closure continuity and almost continuity in the sense of Husain.

Definition 4. A function $f : X \to Y$ is almost open if $f(U) \subset \operatorname{Int} \operatorname{cl}(f(U))$ for each open subset U of X.

Rose [13] obtained that almost openness is equivalent to the following condition:

$$f^{-1}(\operatorname{cl}(V)) \subset \operatorname{cl}\left(f^{-1}(V)\right)$$

for each open subset V of Y.

Definition 5. A function $f : X \to Y$ is said to be almost open in the sense of Singal and Singal if the image $f(U)$ of every regularly open set U in X is open in Y.

Remark 3. Examples in [13] show the independence of almost openness and almost openness in the sense of Singal and Singal (see, for instance, [4]).

Definition 6. A space X is said to be nearly-compact (resp. closure compact) if every open cover of X has a finite sub-family, the interiors of the closures (resp. the closures) of whose members cover X.

3. Main Results

The following theorem gives a condition which causes almost continuity in the sense of Husain to imply closure continuity for functions.

Theorem 1. *If $f : X \to Y$ is an almost continuous function in the sense of Husain satisfying condition* (1), *then f is closure continuous.*

Proof. Let $x \in X$ and let $V \subset Y$ be any open set containing $f(x)$. Since f is almost continuous in the sense of Husain, there exists an open set $U \subset X$ containing x such that

$$U \subset \operatorname{cl}\left(f^{-1}(V)\right) .$$

Let $y \in \operatorname{cl}(U)$. Again, since f is almost continuous in the sense of Husain, for each open set $W \subset Y$ containing $f(y)$, there exists an open set $S \subset X$ containing y such that

$$S \subset \operatorname{cl}\left(f^{-1}(W)\right) .$$

Also, we have

$$S \cap U \neq \emptyset$$

because $y \in \mathrm{cl}(U)$. It further implies that

$$S \cap \mathrm{cl}\left(f^{-1}(V)\right) \neq \emptyset\,.$$

Since S is open, it follows that

$$S \cap f^{-1}(V) \neq \emptyset\,.$$

Furthermore,

$$\mathrm{cl}\left(f^{-1}(W)\right) \cap f^{-1}(V) \neq \emptyset$$

and so

$$f^{-1}\left(\mathrm{cl}(W)\right) \cap f^{-1}(V) \neq \emptyset\,,$$

which in turn implies that

$$\mathrm{cl}(W) \cap V \neq \emptyset\,.$$

Thus $W \cap V \neq \emptyset$. But W is any open set which contains $f(y)$. Therefore

$$f(y) \in \mathrm{cl}(V)\,.$$

Hence

$$f\left(\mathrm{cl}(U)\right) \subset \mathrm{cl}(V)$$

and so f is closure continuous. □

Remark 4.

(i) Theorem 1 shows that almost continuity in the sense of Husain and weak continuity together imply closure continuity. It is clear from Example 1 (above) and Example 1 of Rose [13] that almost continuity in the sense of Husain or weak continuity from Theorem 1 cannot be dropped.

(ii) If, in addition, Y is regular or Hausdorff and locally compact, then f is continuous.

Question. Does Theorem 1 hold if condition (1) is replaced by the following condition:

$$\mathrm{cl}\left(f^{-1}(V)\right) \subset f^{-1}\left(\mathrm{cl}(V)\right)$$

for each $V \in B$, where B is an open basis for the topology on Y.

Corollary 1. *An almost continuous function in the sense of Husain $f : X \to Y$ is closure continuous if and only if f satisfies condition* (1).

Corollary 2. *Let $f : X \to Y$ be an almost open function in the sense of Singal and Singal satisfying condition* (1). *If f is almost continuous in the sense of Husain, then f is almost continuous in the sense of Singal and Singal.*

Proof. If f is almost continuous in the sense of Husain, then by Theorem 1, f is closure continuous. By Theorem 4 of Noiri [10], f is almost continuous in the sense of Singal and Singal. □

Remark 5. For a function which is both almost open (in the sense of Singal and Singal) and almost continuous (in the sense of Husain), the following are equivalent:

(i) almost continuity in the sense of Singal and Singal,
(ii) closure continuity,
(iii) weak continuity,
(iv) condition (1).

Corollary 3. *Let $f : X \to Y$ be an almost continuous function in the sense of Husain satisfying condition* (1)*. If K is a closure compact subset of X, then $f(K)$ is a closure compact subset of Y.*

Proof. The proof follows from Theorem 1 (above) and Theorem 2 of Saleh [15].

Corollary 4. *Let $f : X \to Y$ be an almost open function in the sense of Singal and Singal satisfying condition* (1)*. If f is almost continuous in the sense of Husain and $K \subset X$ is nearly-compact, then $f(K)$ is nearly-compact.*

Proof. It follows immediately from Theorem 1 (above) and Corollary 3 of Noiri [10].

Corollary 5. *Let $f : X \to Y$ be a function of closure-compact space X onto a Urysohn space X satisfying condition* (1)*. If f is almost continuous in the sense of Husain, then f is almost continuous in the sense of Singal and Singal.*

Proof. If f is almost continuous in the sense of Husain, then, by Theorem 1, f is closure continuous. By Theorem 5 of Noiri [10], f is almost continuous in the sense of Singal and Singal.

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