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POSITIVE SOLUTIONS AND OSCILLATION OF HIGHER ORDER NEUTRAL DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions are established for the existence of positive solutions and oscillation of bounded solutions of p-th order neutral difference equations of the form

$$\Delta^p[x_n + a_n x_{\tau(n)}] + \delta\, q_n f(x_{\sigma(n)}) = h_n, \quad n \in \mathbb{N}(n_0),$$

where $\delta = \pm 1$, $\mathbb{N}(n_0) = \{n_0, n_0+1, \dots\}$, n_0 is fixed in $\mathbb{N} = \{1, 2, \dots\}$, $a, q, h : \mathbb{N}(n_0) \to \mathbb{R}$, $\tau, \sigma \in \mathbb{N}(n_0) \to \mathbb{N}$ with $\tau(n) < n$ and $\lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \sigma(n) = \infty$. Combining the sufficient conditions we are able to give necessary and sufficient conditions for every bounded solution of the above equation to be oscillatory or almost oscillatory. Our results improve and generalize several oscillation criteria obtained previously.

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1. Introduction

In this paper we consider p-th order neutral difference equations of the form

$$\Delta^p[x_n + a_n x_{\tau(n)}] + \delta\, q_n f(x_{\sigma(n)}) = h_n, \quad n \in \mathbb{N}(n_0), \tag{1}$$

where $\delta = \pm 1$, $\mathbb{N}(n_0) = \{n_0, n_0+1, \dots\}$, n_0 is fixed in $\mathbb{N} = \{1, 2, \dots\}$, $a, q, h : \mathbb{N}(n_0) \to \mathbb{R}$, $\tau, \sigma \in \mathbb{N}(n_0) \to \mathbb{N}$ with $\tau(n) < n$ and $\lim_{n\to\infty} \tau(n) = \lim_{n\to\infty} \sigma(n) = \infty$. Throughout this paper it is assumed that $f \in C(\mathbb{R}, \mathbb{R})$.

In what follows $n^{(s)}$ denotes the factorial function; that is, $n^{(0)} = 1$ and $n^{(s)} = n(n-1)\cdots(n-s+1)$ for any integer $s \geq 1$.

As usual a solution $\{x_n\}$ of equation (1) is called oscillatory if for a given $M \geq 0$, there exists $n \geq M$ such that $x_n x_{n+1} \leq 0$, and it is said to be almost oscillatory if $\{x_n\}$ is either oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

The oscillatory behavior of solutions of first and second order difference equations has been extensively studied by many authors However, much less has been done for higher order equations. For some results regarding the oscillation and asymptotic behavior of higher order difference equation, we refer in particular to [2]-[10] and the references cited therein. In [8], the first author of the present article considered a special case of (1), namely, the difference equation

$$\Delta^p[x_n + c\,x_{n-l}] + \delta\, q_n f(x_{n-k}) = h_n, \quad n \in \mathbb{N}(n_0), \tag{2}$$

where l and k are integers with $l > 0$, and proved that if

- (C_1) $c \neq \pm 1$,
- (C_2) f satisfies Lipschitz conditions on an interval $[a, b]$, where a and b depend upon the range of $c \neq 0$,
- (C_3) $\sum^{\infty} n^{(p-1)} |q_n| < \infty$,
- (C_4) $\sum^{\infty} n^{(p-1)} |h_n| < \infty$,

then (2) has a positive solution, and if

- (H_1) $x f(x) > 0$ for all $x \neq 0$,
- (H_2) $q_n \geq 0$ with infinitely many positive terms,
- (H_3) there exists an oscillatory function ρ on $\mathbb{N}$ such that $\Delta^p \rho_n = h_n$ and $\lim_{n\to\infty} \Delta^j \rho_n = 0$ for $j = 0, 1, \dots, p-1$,
- (H_4) $\sum^{\infty} n^{(p-1)} q_n = \infty$,

then every bounded solution $\{x_n\}$ of (2) is oscillatory when $(-1)^p \delta = 1$, and almost oscillatory when $(-1)^p \delta = -1$.

Later the same author [9] gave a necessary and sufficient condition for the oscillation of bounded solutions of (1) when $\tau(n) = n - l$, $\sigma(n) = n - k$, and $-b_0 \leq c_n \leq -b_1 < -1$, where b_0 and b_1 are fixed real numbers. The dependence mentioned in (C_2) was obtained as $a/b < (b_1 - 1)/b_0$.

A similar result was also established in [7] for equation (1) when p is even, $\tau(n) = n - l$, $\sigma(n) = n - k$, $h_n \equiv 0$, and $0 \leq c_n < b_2 < 1$. Instead of (H_4), they had imposed the condition that

$$\sum^{\infty} q_n f\Big(\big(\frac{n-k}{2^{p-1}}\big)^{p-1}\Big) = \infty.$$

Our purpose here in this paper is to find sufficient conditions for the existence of positive solutions and oscillation of bounded solutions of equation (1), and thereby establish necessary and sufficient conditions for oscillation or almost oscillation of bounded solutions of equation (1).

For simplicity we first consider the difference equation

$$\Delta^p[x_n + c\, x_{\tau(n)}] + \delta\, q_n f(x_{\sigma(n)}) = 0, \quad n \in \mathbb{N}(n_0) \tag{3}$$

in sections 2 and 3, and next extend the results obtained to equation (1) in section 4.

2. Existence of positive solutions

In this section we are concerned with the existence of positive solutions of neutral type difference equations of the form (3). It will be proved that (3) has a positive solution when $|c| \neq 1$ provided that the function f satisfies a Lipschitz condition on an interval $[a, b]$, where a and b are arbitrary positive real numbers.

Theorem 1. *If (C_1) and (C_3) hold and*

($\bar{C}_2$) *for some positive numbers a and b, the function f satisfies the Lipschitz condition with a constant L on the interval $[a, b]$,*

then equation (3) has a positive solution.

Proof. Let $K = \max\{|f(x)|/|x| : a \le x \le b\}$ and $M = \max\{K, L\}$.

We first consider the case $|c| < 1$. Because of (C_3), there exists a sufficiently large integer $n_1 \ge n_0$ such that

$$\sum_{s=n_1}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!}{Mb}\beta, \quad \beta = \frac{(b-a)(1-|c|)}{2}, \tag{4}$$

and such that $\tau(n) \ge n_0$ and $\sigma(n) \ge n_0$ for all $n \in \mathbb{N}(n_1)$.

We introduce the Banach Space

$$Y = \Big\{ x : \sup_{n \ge N_0} |x_n| < \infty \Big\}$$

with the norm

$$||x|| = \sup_{n \ge N_0} |x_n|,$$

where $N_0 = \inf_{n\geq n_1}\{\tau(n),\sigma(n)\}$.

Set $X = \{x \in Y : a \leq x \leq b\}$. It is clear that X is a bounded, convex and closed subset of Y. Define an operator $S : X \to Y$ by

$$\begin{aligned} Sx_n &= \alpha - c\,x_{\tau(n)} + \frac{(-1)^p}{(p-1)!}\sum_{s=n}^{\infty}(s+p-1-n)^{(p-1)}q_s f(x_{\sigma(s)}), \quad n \geq n_1 \\ &= Sx_{n_1}, \quad N_0 \leq n \leq n_1, \end{aligned}$$

where

$$\alpha = \frac{(b+a)(1+c)}{2}.$$

We shall show that S is a contraction mapping on X. We prove this when $0 \leq c < 1$, the case $-1 < c < 0$ is similar. It is easy to see that S maps X into itself. In fact, for $x \in X$, $n \geq n_1$, using (4) it follows that

$$Sx_n \geq \alpha - c\,b - \beta = a$$

and

$$Sx_n \leq \alpha - c\,a + \beta = b,$$

and hence $Sx \in X$. To show that S is a contraction, let $x, y \in X$. It is easy to see that

$$\begin{aligned} |Sx_n - Sy_n| &\leq c\,|x_{\tau(n)} - y_{\tau(n)}| \\ &\quad + \frac{M}{(p-1)!}\sum_{s=n}^{\infty}(s+p-1-n)^{(p-1)}|q_s||x_{\sigma(s)} - y_{\sigma(s)}| \\ &\leq c||x-y|| + \frac{\beta}{b}||x-y||, \end{aligned}$$

and so

$$||Sx - Sy|| \leq (c + \frac{\beta}{b})||x-y||.$$

Since $c + \beta/b < 1$, S is a contraction on X. It follows that S has a fixed point $x \in X$, that is, $Sx = x$. It is easy to check that x is a positive solution of equation (3).

Suppose that $|c| > 1$. In this case we fix

$$\beta = \frac{(b-a)(|c|-1)}{2|c|}$$

and let n_1 be so large that

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!}{Mb}|c|\beta. \tag{5}$$

Define an operator $S : X \to Y$ as follows:

$$Sx_n = \frac{1}{c}[\alpha - x_{\tau^{-1}(n)} + \frac{(-1)^p}{(p-1)!}\sum_{s=\tau^{-1}(n)}^{\infty}(s+p-1-\tau^{-1}(n))^{(p-1)}q_s f(x_{\sigma(s)})], \quad n \geq n_1$$

$$= Sx_{n_1}, \quad N_0 \leq n \leq n_1$$

where

$$\alpha = \frac{(b+a)(1+c)}{2}.$$

We may claim that S is contraction on X. We shall prove our claim when $c > 1$, the case $c < -1$ is similar. In view of (5) we see that

$$Sx_n \geq \frac{\alpha}{c} - \frac{b}{c} - \beta = a$$

and

$$Sx_n \leq \frac{\alpha}{c} - \frac{a}{c} + \beta = b.$$

Thus we have $Sx \in X$. It is not also difficult to see that if $x, y \in X$ then

$$\begin{aligned}|Sx_n - Sy_n| &\leq \frac{1}{c}|x_{\tau^{-1}(n)} - y_{\tau^{-1}(n)}| \\ &+ \frac{1}{c}\frac{M}{(p-1)!}\sum_{s=\tau^{-1}(n)}^{\infty}(s+p-1-\tau^{-1}(n))^{(p-1)}|q_s||x_{\sigma(s)} - y_{\sigma(s)}| \\ &\leq (\frac{1}{c} + \frac{\beta}{b})||x-y||.\end{aligned}$$

Since $1/c + \beta/b < 1$, S is a contraction on X. This completes the proof.

3. Oscillation of bounded solutions

In this section we investigate the oscillation behavior of bounded solutions of (3) and establish necessary and sufficient conditions under which every solution $\{x_n\}$ of (3) is either oscillatory or almost oscillatory.

The following lemmas will be needed in the proof of our theorems. The first three of them can be found in [1]. The last one is essentially new and may be of interest for other studies as well.

Lemma 1. *Let $\{y_n\}$ and $\{\Delta^p y_n\}$ be sequences defined on $\mathbb{N}(n_0)$ with $y_n\Delta^p y_n < 0$ on $\mathbb{N}(n_0)$. Then there exists an integer l, $0 \leq l \leq p-1$, with $p-l$ odd such that for $n \in \mathbb{N}(n_0)$,*

$$y_n\Delta^j y_n > 0, \quad j = 0, 1, \ldots, l,$$
$$(-1)^{j-l}y_n\Delta^j y_n > 0, \quad j = l+1, \ldots, p-1.$$

Lemma 2. *Let $\{y_n\}$ and $\{\Delta^p y_n\}$ be sequences defined on $\mathbb{N}(n_0)$ with $y_n \Delta^p y_n > 0$ on $\mathbb{N}(n_0)$. Then for $n \in \mathbb{N}(n_0)$, either*

$$y_n \Delta^j y_n > 0, \quad j = 1, \dots, p$$

or there exists an integer l, $0 \le l \le p-2$, with $p-l$ even such that for $n \in \mathbb{N}(n_0)$,

$$\begin{aligned} y_n \Delta^j y_n &> 0, \quad j = 0, 1, \dots, l, \\ (-1)^{j-l} y_n \Delta^j y_n &> 0, \quad j = l+1, \dots, p-1. \end{aligned}$$

Lemma 3. *If $\{y_n\}$ is a sequence defined on $\mathbb{N}(n_0)$, then*

$$\sum_{s=n_1}^{n-1} s^{(p-1)} \Delta^p y_s = \sum_{k=1}^{p} (-1)^{k+1} \Delta^{k-1} s^{(p-1)} \Delta^{p-k} y_{s+k-1} \Big|_{s=n_1}^{n}.$$

Lemma 4. *Let g be a continuous monotone function such that $\lim_{n\to\infty} g(n) = \infty$. Set*

$$z_n = x_n + a_n x_{g(n)}. \tag{6}$$

If x_n is eventually positive, $\liminf_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} z_n = \ell \in \mathbb{R}$ exists, then $\ell = 0$ provided that for some real numbers b_1, b_2, b_3 and b_4 the sequence $\{a_n\}$ satisfies one of the following:

$$(a) \quad b_1 \le a_n \le 0, \qquad (b) \quad 0 \le a_n \le b_2 < 1, \qquad (c) \quad 1 < b_3 \le a_n \le b_4.$$

Proof. We see from (6) that

$$z_{g^{-1}(n)} - z_n = x_{g^{-1}(n)} + a_{g^{-1}(n)} x_n - x_n - a_n x_{g(n)}$$

and so

$$\lim_{n\to\infty} \left\{ x_{g^{-1}(n)} + a_{g^{-1}(n)} x_n - x_n - a_n x_{g(n)} \right\} = 0. \tag{7}$$

Let $\{n_k\}$ be a sequence of real numbers such that

$$\lim_{k\to\infty} x_{n_k} = 0. \tag{8}$$

Assume that (a) holds. It follows from (7) and (8) that

$$\lim_{k\to\infty} \left\{ x_{g^{-1}(n_k)} - a_{n_k} x_{g(n_k)} \right\} = 0.$$

As $x_{g^{-1}(n_k)} > 0$ and $-a_{n_k} x_{g(n_k)} \ge 0$, we see that

$$\lim_{k\to\infty} x_{g^{-1}(n_k)} = 0$$

and so from (6) we get

$$\ell = \lim_{k\to\infty} z_{g^{-1}(n_k)} = \lim_{k\to\infty} \left\{x_{g^{-1}(n_k)} + a_{g^{-1}(n_k)} x_{n_k}\right\} = 0.$$

Assume that (b) holds. By replacing n by $g(n)$ in (7) and using (8) we have

$$\lim_{n\to\infty} \left\{x_n + a_n x_{g(n)} - x_{g(n)} - a_{g(n)} x_{g(g(n))}\right\} = 0. \tag{9}$$

It is clear from (8) and (9) that

$$\lim_{k\to\infty} \left\{[a_{n_k} - 1] x_{g(n_k)} - a_{g(n_k)} x_{g(g(n_k))}\right\} = 0$$

and so

$$\lim_{k\to\infty} x_{g(n_k)} = 0.$$

Thus,

$$\ell = \lim_{k\to\infty} z_{g(n_k)} = \lim_{k\to\infty} \left\{x_{g(n_k)} + a_{g(n_k)} x_{g(g(n_k))}\right\} = 0.$$

Finally, let (c) be satisfied. Replacing n by $g^{-1}(n)$ in (7) and using (8) leads to

$$\lim_{k\to\infty} \left\{x_{g^{-1}(g^{-1}(n_k))} + [a_{g^{-1}(g^{-1}(n_k))} - 1] x_{g^{-1}(n_k)}\right\} = 0$$

and hence

$$\lim_{k\to\infty} x_{g^{-1}(n_k)} = 0. \tag{10}$$

In view of (6) and (10), it follows that

$$\ell \;=\; \lim_{k\to\infty} z_{g^{-1}(n_k)} = 0.$$

This completes the proof.

Theorem 2. *Suppose that (H_1), (H_2) and (H_4) hold.*

(i) *If $c \geq 0$ and $c \neq 1$, then every bounded solution $\{x_n\}$ of (3) is oscillatory when $(-1)^p\delta = 1$, and is almost oscillatory when $(-1)^p\delta = -1$.*

and

(ii) *If $c < -1$ and $\inf_{n\geq 0}[n - \tau(n)] > 0$, then every bounded solution $\{x_n\}$ of (3) is oscillatory when $(-1)^p\delta = -1$, and is almost oscillatory when $(-1)^p\delta = 1$.*

Proof. Suppose on the contrary that $\{x_n\}$ is a nonoscillatory bounded solution of (3). Without loss of generality we may assume that $\{x_n\}$ is eventually positive. Set

$$z_n = x_n + c\, x_{\tau(n)}.$$

Clearly, $\{z_n\}$ is bounded and

$$\delta \Delta^p z_n = -q_n f(x_{\sigma(n)}) < 0. \tag{11}$$

Let $c \geq 0$ and $c \neq 1$. It is obvious that $\{z_n\}$ is eventually positive and $\delta z_n \Delta^p z_n < 0$. Applying Lemma 1 and Lemma 2 we see that there exist n_1 and integer $l \in \{0,1\}$ with $(-1)^{p-l}\delta = -1$ such that

$$\begin{aligned} \Delta^k z_n &> 0, \quad k = 0,1,\ldots,l \\ (-1)^{k-l}\Delta^k z_n &> 0, \quad k = l, l+1, \ldots, p-1 \end{aligned} \tag{12}$$

for all $n \geq n_1$. Multiplying (3) by $s^{(p-1)}$ and summing from n_1 to $n-1$ we obtain

$$\sum_{s=n_1}^{n-1} s^{(p-1)} \delta \Delta^p z_s + \sum_{s=n_1}^{n-1} s^{(p-1)} q_s f(x_{\sigma(s)}) = 0. \tag{13}$$

Applying Lemma 3 to the first term in the left side of (13) we have

$$\begin{aligned} \sum_{s=n_1}^{n-1} s^{(p-1)} \delta \Delta^p z_s &= \sum_{k=1}^{p-1} (-1)^{k+1} \delta \Delta^{k-1} s^{(p-1)} \Delta^{p-k} z_{s+k-1}\Big|_{s=n_1}^{n} \\ &\quad + (-1)^{p+1} \delta \Delta^{p-1} s^{(p-1)} \delta \Delta^{p-p} z_{s+p-1}\Big|_{s=n_1}^{n} \\ &= \sum_{k=1}^{p-1} (-1)^{k+1} \delta \Delta^{k-1} n^{(p-1)} \Delta^{p-k} z_{n+k-1} \\ &\quad + (-1)^{p+1} \delta (p-1)! [z_{n+p-1} - z_{n_1+p-1}] - K \end{aligned} \tag{14}$$

where in view of (12)

$$K = \sum_{k=1}^{p-1} (-1)^{k+1} \delta \Delta^{k-1} n_1^{(p-1)} \Delta^{p-k} z_{n_1+k-1} \geq 0.$$

Using (14) in (13) leads to

$$\sum_{s=n_1}^{n-1} s^{(p-1)} q_s f(x_{\sigma(s)}) \leq K + (-1)^p \delta (p-1)! [z_{n+p-1} - z_{n_1+p-1}]. \tag{15}$$

Since $\{z_n\}$ is bounded and (H_4) holds, we obtain from (15) that

$$\liminf_{n\to\infty} f(x_n) = 0$$

or

$$\liminf_{n\to\infty} x_n = 0.$$

It follows from Lemma 4 that $\ell = \lim_{n\to\infty} z_n = 0$. But $\ell = 0$ is possible only when $l = 0$, since in the case $l = 1$, $\{z_n\}$ being positive and increasing cannot approach zero. This means that bounded solutions of (3) must be oscillatory when $(-1)^p\delta = 1$. It is clear that if $\ell = 0$, then in view of $0 < x_n \le z_n$ we have

$$\lim_{n\to\infty} x_n = 0.$$

Suppose that $c < -1$. We claim that $\{z_n\}$ is eventually negative. Otherwise, for sufficiently large values of n, $x_n > -c\,x_{\tau(n)}$. Replacing n by $\tau^{-1}(n)$, using mathematical induction one can see that

$$x_{r_m(n)} > (-c)^m x_n, \tag{16}$$

where

$$r_1(n) = \tau^{-1}(n) \ \text{ and } r_m(n) = \tau^{-1}(r_{m-1}(n)) \text{ for } m \ge 2.$$

We shall show that $\lim_{m\to\infty} r_m = \infty$. In that case since $\{x_n\}$ is bounded we get a contradiction. We first notice that $\tau(n) < n$ and so $r_1(n) > n$. In view of $\inf_{n\ge 0}[n - \tau(n)] > 0$ there exists $\varepsilon > 0$ such that $r_1(n) > n + \varepsilon$. By mathematical induction we obtain

$$r_m(n) > n + m\varepsilon$$

and hence $\lim_{m\to\infty} r_m = \infty$. Therefore $\{z_n\}$ is eventually negative. Since

$$\delta\Delta^p z_n = -q_n f(x_{\sigma(s)}) < 0$$

we have $\delta z_n \Delta^p z_n > 0$. Applying Lemma 1 and Lemma 2 it follows that there are n_1 and $l \in \{0,1\}$ with $(-1)^{p-l}\delta = 1$ such that

$$\begin{gathered} \Delta^j z_n < 0, \quad j = 0,1,....l, \\ (-1)^{j-l}\Delta^j z_n < 0, \quad j = l+1,\dots,p-1. \end{gathered}$$

Using the arguments of the previous case we see that

$$\liminf_{n\to\infty} x_n = 0$$

and hence by Lemma 4, $\ell = \lim_{n\to\infty} z_n = 0$. Moreover, we observe as in the previous case that $\ell = 0$ is possible only when $l = 0$. In this case since $z_n < 0$ it follows that for a given $\epsilon > 0$ there exists an n_2 so large that

$$z_n > -\epsilon \quad \text{for } n \ge n_2.$$

This means that

$$x_n > -\epsilon - c\,x_{\tau(n)} \quad \text{for } n \ge n_2. \tag{17}$$

If we define $\tilde{c} = -1/c$, then we see from (17) that

$$x_n < \tilde{c}\,\epsilon + x_{r_1(n)}.$$

It follows that

$$x_n < (\tilde{c} + \tilde{c}^2 + \cdots \tilde{c}^m)\epsilon + \tilde{c}^m \, x_{r_m(n)}$$

and therefore

$$x_n < \frac{\tilde{c}}{1-\tilde{c}}\,\epsilon + \tilde{c}^m \, x_{r_m(n)}. \tag{18}$$

In view of $0 < \tilde{c} < 1$ we easily deduce from (18) that $\lim_{n\to\infty} x_n = 0$. This completes the proof.

In view of Theorem 1 and Theorem 2, we obtain a necessary and sufficient condition for oscillation of bounded solutions of (3), which gives an improvement of the theorem given in Section 1.

Theorem 3. *Let (H_1), (H_2) and $(\bar{C}_2)$ be satisfied. Then the conclusion of Theorem 2 holds if and only if (H_4) is satisfied.*

4. Some generalizations

In this section we extend the results obtained for equation (3) to equation (1). Since the proofs are similar, we will omit the details.

Theorem 4. *Suppose that (C_3) and (C_4) are satisfied, and (C_2) holds with positive real numbers a and b satisfying the following:*

(A) $a/b < (b_2+1)/(b_1+1)$, when $b_1 \le a_n \le b_2 < -1$,
(B) $a/b < (b_1+1)/(b_2+1)$, when $-1 < b_1 \le a_n \le b_2 \le 0$,
(C) $a/b < (1-b_2)/(1-b_1)$, when $0 \le b_1 \le a_n \le b_2 < 1$,
(D) $a/b < (b_1-1)/(b_2-1)$, when $1 < b_1 \le a_n \le b_2$,

where b_1 and b_2 are real numbers.

Then equation (1) has a positive solution.

Proof. Let $K = \max\{|f(x)|/|x| : a \le x \le b\}$ and $M = \max\{K, L\}$.

We first consider case (A). Let

$$\beta = \frac{b(b_2+1) - a(b_1+1)}{2b_2}.$$

In view of (C_3) and (C_4) we can find sufficiently large $n_1 \ge n_0$ such that if $n \ge n_1$ then

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!\beta}{2Mb}(-b_2) \tag{19}$$

and

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|h_s| < \frac{(p-1)!\beta}{2}(-b_2). \tag{20}$$

We may assume that $\tau(n) \geq n_0$ and $\sigma(n) \geq n_0$ for all $n \geq n_1$.

We introduce the Banach Space

$$Y = \left\{x : \sup_{n \geq N_0} |x_n| < \infty\right\}$$

with the supremum norm

$$||x|| = \sup_{n \geq N_0} |x_n|,$$

where $N_0 = \inf_{n \geq n_1} \{\tau(n), \sigma(n)\}$. Let

$$X = \{x \in Y : a \leq x \leq b\}.$$

It is clear that X is a bounded, convex and closed subset of Y.

Define an operator $S : X \to Y$ by

$$\begin{aligned} Sx_n = & \frac{1}{a_{\tau^{-1}(n)}}[\alpha - x_{\tau^{-1}(n)} + \frac{(-1)^p}{(p-1)!} \sum_{s=\tau^{-1}(n)}^{\infty} (s+p-1-\tau^{-1}(n))^{(p-1)} q_s f(x_{\sigma(s)}) \\ & + \frac{(-1)^{p-1}}{(p-1)!} \sum_{s=\tau^{-1}(n)}^{\infty} (s+p-1-\tau^{-1}(n))^{(p-1)} h_s], \qquad n \geq n_1 \\ = & Sx_{n_1}, \quad N_0 \leq n \leq n_1, \end{aligned}$$

where

$$\alpha = \frac{b(b_2+1) + a(b_1+1)}{2}.$$

We shall show that S is a contraction mapping on X. It is easy to show that S maps X into itself. In fact if $x \in X$ then, because of (19) and (20), it follows that

$$Sx_n \leq \frac{-1}{b_2}[-\alpha + b - b_2\beta] = b$$

and

$$Sx_n \geq \frac{-1}{b_1}[-\alpha + a + b_2\beta] = a.$$

Therefore $SX \subseteq X$.

To show that S is a contraction, we take $x, y \in X$. Obviously,

$$\begin{aligned} |Sx_n - Sy_n| &\leq \frac{-1}{b_2}|x_{\tau^{-1}(n)} - y_{\tau^{-1}(n)}| \\ &\quad + \frac{M}{(-b_2)(p-1)!}\sum_{s=\tau^{-1}(n)}^{\infty} s^{(p-1)}|q_s||x_{\sigma(s)} - y_{\sigma(s)}| \\ &\leq (\frac{-1}{b_2} + \frac{\beta}{2b})||x-y||. \end{aligned} \tag{21}$$

Since $\frac{-1}{b_2} + \frac{\beta}{2b} < 1$, S is a contraction on X, and therefore there exists a fixed point $x \in X$ such that $Sx = x$. It can easily be verified that x is a positive solution of equation (1). This completes the proof in the case when (A) is satisfied.

To prove the theorem for the cases (B), (C), and (D) we need only to make the following modifications on β, α and S in each case:

Case (B) :

$$\beta = \frac{b(b_1+1) - a(b_2+1)}{2}, \quad \alpha = \frac{b(b_1+1) + a(b_2+1)}{2},$$

$$\begin{aligned} Sx_n &= \alpha - a_n x_{\tau(n)} + \frac{(-1)^p}{(p-1)!}\sum_{s=n}^{\infty}(s+p-1-n)^{(p-1)}q_s f(x_{\sigma(s)}) \\ &+ \frac{(-1)^{p-1}}{(p-1)!}\sum_{s=n}^{\infty}(s+p-1-n)^{(p-1)}h_s, \qquad n \geq n_1 \\ &= Sx_{n_1}, \quad N_0 \leq n \leq n_1, \end{aligned}$$

where n_1 is chosen so large that

$$\sum_{s=n_1}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!}{2Mb}\beta, \tag{22}$$

$$\sum_{s=n_1}^{\infty} s^{(p-1)}|h_s| < \frac{(p-1)!}{2}\beta \tag{23}$$

for all $n \geq n_1$.

Case(C) :

$$\beta = \frac{b(1-b_2) - a(1-b_1)}{2}, \quad \alpha = \frac{b(b_2+1) + a(b_1+1)}{2},$$

S is defined as in the case (B), and (22) and (23) are satisfied for all $n \geq n_1$.

Case(D):

$$\beta = \frac{b(b_1-1) - a(b_2-1)}{2b_1}, \quad \alpha = \frac{b(b_1+1) + a(b_2+1)}{2},$$

S is defined as in the case (i), and

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|q_s| < \frac{(p-1)!}{2Mb}\beta b_1$$

$$\sum_{s=\tau^{-1}(n_1)}^{\infty} s^{(p-1)}|h_s| < \frac{(p-1)!}{2}\beta b_1$$

for all $n \geq n_1$.

The next theorem is a generalization of the results given in Theorem 2 to equation (1). For a similar result and especially the technique about handling the difficulty of having a forcing term, we refer the reader to [8,9].

Theorem 5. *Suppose that (H_1) - (H_4) hold.*

(i) *If $0 \leq a_n \leq b_2 < 1$ or $1 < b_1 \leq a_n \leq b_2$, then every bounded solution $\{x_n\}$ of (1) is oscillatory when $(-1)^p\delta = 1$, and is almost oscillatory when $(-1)^p\delta = -1$.*

(ii) *If $b_1 \leq a_n \leq b_2 < -1$ and $\inf_{n\geq 0}[n - \tau(n)] > 0$, then every bounded solution $\{x_n\}$ of (1) is oscillatory when $(-1)^p\delta = -1$, and is almost oscillatory when $(-1)^p\delta = 1$.*

Finally, by combining Theorem 4 and Theorem 5 we obtain the following necessary and sufficient condition for oscillation of bounded solutions of (1).

Theorem 6. *Suppose that (C_4), (H_1) - (H_3) hold, and that (C_2) is fulfilled on $[a, b]$, where a and b are as in (A), (C), and (D). Then the conclusion of Theorem 5 holds if and only if (H_4) is satisfied.*

Remark 1. In this paper we have assumed that $\{a_n\}$ is bounded away from ± 1. It is not difficult to provide specific examples showing that this assumption cannot be dropped. Therefore, finding similar results concerning (1) when $\{a_n\}$ is not bounded away from ± 1 seems to be very interesting.

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