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SOME REMARKS ABOUT THE NONOSCILLATORY SOLUTIONS

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Abstract. The paper investigates the relation between a linear homogeneous differential equation and its nonhomogeneous variant concerning the nonoscillatory property.

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The aim of the paper is to investigate the relation between a linear homogeneous differential equation and its nonhomogeneous variant concerning the nonoscillatory property. More precisely, we formulate the problem as follows.

Problem. If the homogeneous linear differential equation is nonoscillatory and $f(x)$ is a continuous one-signed function (i. e. $f(x) \geq 0$ or $f(x) \leq 0$) which is not identically zero for large x, we ask which other properties has the homogeneous differential equation to have so that also the nonhomogeneous differential equation will have the nonoscillatory property.

For the simplicity we will consider the selfadjoint differential equation

$$y^{(4)} + p(x)y = 0 \tag{1}$$

and

$$z^{(4)} + p(x)y = f(x) \tag{2}$$

We assume that $p(x) \in C([a,\infty))$ is nonnegative function defined on $J = [a,\infty)$ and $f(x) \in C([a,\infty))$ is a one-signed function on J not identically zero for large x.

It follows from the assumptions about $p(x)$ that either all solutions of (1) are oscillatory or all are nonoscillatory [1].

Definition 1. A solution of (1) or (2) is oscillatory if it has an upper unbounded set of zeros. A solution is nonoscillatory if it is not oscillatory.

Definition 2. Equation (1) or (2) is oscillatory if it has at least one oscillatory solution. Otherwise the equation is nonoscillatory.

Definition 3. Equation (1) is said to be disconjugate (on an interval I) if no nontrivial solution of (1) has more than 3 zeros (on I).

The above problem was solved for the linear differential equations of the second order in paper [2].

Theorem 1. ([2]). *Let the equation*

$$y'' + p(x)y = 0$$

be a nonoscillatory equation and let $f(x)$ be a one-signed function not identically zero for large x. Then the equation

$$z'' + p(x)z = f(x)$$

is also nonoscillatory.

For the equation of higher order our problem was solved in the paper [3], where the condition for the nonoscillatory behaviour of the homogeneous differential equation was substituted by the condition of disconjugacy of the homogeneous differential equation. It has to be mentioned that the disconjugacy doesn't follow from the nonoscillatory property.

Our problem was discussed in the paper [4] for the linear differential equations of the n-th order, where the condition of disconjugacy is assumed for the so-called reduced operator $\hat{L}_{n-1}$ associated to the operator L_n.

Definition 4. Equation

$$L_n y = y^{(n)} + a_1 y^{n-1} + \dots + a_n y = 0, \tag{3}$$

where $a_i \in C([a,\infty))$, $i = 1, 2, \dots, n$, is said to be disconjugate (on an interval I), if no nontrivial solution of (3) has more than $n-1$ zeros (on I).

Assume that the equation (3) is nonoscillatory and that $\Phi(x)$ is a nonoscillatory solution of (3). If we set $y = \Phi z$, then for sufficiently large x we get

$$L_n y = z L_n \Phi + \Phi \left[z^{(n)} + \sum_{i=1}^{n-1} \hat{a}_i(x) z^{(n-i)} \right] = z L_n \Phi + \Phi \hat{L}_{n-1} z',$$

where $\hat{a}_i(x)$ depend on $\Phi(x)$. Operator $\hat{L}_{n-1}$ is called the reduced operator for L_n associated with Φ.

Our problem is partially solved in the paper [4].

Lemma 1. ([4]). *Let the equation* (3) *be nonoscillatory and let for solution Φ of* (3) *be $\hat{L}_{n-1}z = 0$ disconjugate for large x. Let $f(x)$ be a one-signed continuous function on $[a, \infty)$ not identically zero for large x. Then the equation*

$$L_n y = f(x) \tag{4}$$

is also nonscillatory.

In the following we will consider our problem for the equations (1) and (2). Instead of the disconjugacy we will use the condition of selfadjointness of (1) and the property that each solution $y(x)$ of (1) can have at most one double zero.

We know that all solutions of (1) are of the same oscillatory character. We will assume that all solutions of (1) are nonoscillatory.

Let be $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ nonoscillatory solutions of (1) on J given by the initial conditions in $x_0 \in [a, \infty)$

$$y_i^{(j)}(x_0) = \begin{cases} 1 & , \quad for \;\; j = i-1 \\ 0 & , \quad for \;\; j \neq i-1 \end{cases} , i = 1,2,3,4; \;\; j = 0,1,2,3. \tag{5}$$

These solutions form a fundamental system. Their wronskian is

$$W(y_1, y_2, y_3, y_4)(x) = 1. \tag{6}$$

From the fact that (1) is selfadjoint it follows ([5], Chap. II,5) that the wronskians

$$\begin{array}{l} W_1 = W(y_2, y_3, y_4)(x), \;\; W_2 = W(y_1, y_3, y_4)(x) \\ W_3 = W(y_1, y_2, y_4)(x), \;\; W_4 = W(y_1, y_2, y_3)(x) \end{array} \tag{7}$$

are solutions of (1) on J. It is easy to see that

$$\left. \begin{array}{rcl} W_k^{(j)}(x_0) & = & 0, \\ W_k^{k-1}(x_0) & = & 1. \end{array} \right\} k = 1,2,3,4, \;\; j \neq k-1, \tag{8}$$

Thus

$$W_1 = y_4(x), W_2 = y_3(x), W_3 = y_2(x), W_4 = y_1(x). \tag{9}$$

Using the method of variation of constants we get for the general solution $z(x)$ of (2) the expression

$$z(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + c_4 y(x) + \int_{x_0}^{x} A(t,x) f(t) dt, \tag{10}$$

where

$$A(t,x)=\begin{vmatrix} y_1(t), & y_2(t), & y_3(t), & y_4(t)\\ y_1'(t), & y_2'(t), & y_3'(t), & y_4'(t)\\ y_1''(t), & y_2''(t), & y_3''(t), & y_4''(t)\\ y_1(x), & y_2(x), & y_3(x), & y_4(x)\end{vmatrix}, x_0\le t\le x. \tag{11}$$

Respecting (7) and (8) we get

$$A(t,x)=-y_1(x)y_4(t)+y_2(x)y_3(t)-y_3(x)y_2(t)+y_4(x)y_1(t),\quad x_0\le t\le x. \tag{12}$$

It is evident that $A(t,x)$ as the function of t is a solution of (1). It is easy to see that $t=x$ is a triple zero of the solution $A(t,x)$. Using the expression (12) we get from (10)

$$z(x)=\sum_{i=1}^{4} y_i(x)\left[c_i+(-1)^i\int_{x_0}^{x} y_{5-i}(t)f(t)dt\right]. \tag{13}$$

We remark that evidently $\int_{x_0}^{x} y_{5-i}(t)f(t)dt,\ i=1,2,3,4$, is a monotone function in a neighbourhood of $+\infty$.

Lemma 2. *Let $p(x)$ be continuous and nonnegative on $[a,\infty)$. Let all solutions of the equation* (1) *be nonoscillatory. Then not all solutions of the equation* (1) *are bounded.*

Proof. Let be all solutions of the equation (1) nonoscillatory and bounded. Thus, the solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ satisfying (5) are nonoscillatory and bounded on $[x_0,\infty)$. From this it folows that $\lim_{x\to\infty} y_i^{(j)}=0$, $i=1,2,3,4$, $j=1,2,3$ and $\lim_{x\to\infty} y_i(x)$ is finite. Therefore, $\lim_{x\to\infty} W(y_1,y_2,y_3,y_4)(x)=0$, which contradicts the fact that $W(y_1,y_2,y_3,y_4)(x)=1$ for all $x\in[a,\infty)$.

Lemma 3. *Let be $p(x)\in C([a,\infty))$ nonnegative and not identically zero on some subinterval of $[a,\infty)$. Then every nontrivial solution $y(x)$ of* (1) *has at most one double (triple) zero point on $[a,\infty)$.*

Proof. Multiplying (1) by $y(x)$ we get $y^{(4)}y+p(x)y^2=0$ or after modification $(y'''y-y'y'')'=-y''^2-p(x)y^2\le 0$. It means that the function $F(y(x))=y'''(x)y(x)-y'(x)y''(x)$ is a nonincreasing one. From this the assertion of Lemma 3 follows.

Lemma 4. *Let $y_i(x),\ i=1,2,3,4$ be the nonoscillatory solutions of* (1) *satisfying* (5)*. Then there exists $\bar{x}\in[a,\infty)$ such that for $x\ge\bar{x}\quad y_i(x)\ne 0,\ i=1,2,3,4$,*

$$\begin{gathered} W(y_4,y_3,y_2,y_1)(x)\ne 0,\quad W(y_4,y_3,y_2)(x)\ne 0,\\ W(y_4,y_3)(x)\ne 0,\quad W(y_4)(x)=y_4\ne 0. \end{gathered} \tag{14}$$

Proof. It follows from the assumption of nonoscillatority of $y_i(x),\ i = 1,2,3,4$ that there exists $\bar{x} > x_0$ such that $y_i(x) \neq 0$ for $x \geq \bar{x}$ and $i = 1,2,3,4$. Moreover, we know that $W(y_4,y_3,y_2,y_1)(x) = const \neq 0$ for all $x \in [a,\infty)$ and $W(y_4,y_3,y_2)(x) = -y_4(x) \neq 0$ for $x \geq \bar{x}$. Consider the solution $u(x) = c_1y_4(x) + c_2y_3(x)$. Evidently, $u(x_0) = u'(x_0) = 0$, $u''(x_0) = c_2$. Thus $u(x)$ has no double zero for $x > x_0$ and therefore there doesn't exist $t > x_0$ such that

$$u(t) = c_1y_4(t) + c_2y_3(t) = 0$$

$$u'(t) = c_1y_4'(t) + c_2y_3'(t) = 0.$$

From this we have that $W(y_4,y_3)(t) \neq 0$ for all $t > x_0$ and therefore also for $t = x \geq \bar{x}$. Evidently, $W(y_4)(x) = y_4(x) \neq 0$ for $x \geq \bar{x}$. This ends the proof of Lemma 4.

Lemma 5. *Let be $p(x),\ f(x) \in C([a,\infty))$, $p(x)$ nonnegative and not identically zero on some subinterval of $[a,\infty)$ and $f(x)$ a one-signed function not identically zero for large x. Then the equation* (2) *allows the Frobenius factorization ([6], Chap. IV, §8, IX.)*

$$a_4(a_3(a_2(a_1(a_0z)')')')' = f(x),\ x \geq \bar{x}, \tag{15}$$

where

$$a_j(x) = \frac{W_j^2(x)}{W_{j-1}(x)W_{j+1}(x)},\ j = 0,1,2,3,4, \tag{16}$$

$$W_0(x) = W_{-1}(x) = W_5(x) = 1,$$

$$W_j(x) = W(y_4,...,y_{5-j})(x),\ j = 1,2,3,4. \tag{17}$$

Proof. From Lemma 4 we have that for $x \geq \bar{x}$

$$W_1(x) = W(y_4)(x) = y_4(x) \neq 0,\quad W_2(x) = W(y_4,y_3)(x) \neq 0,$$

$$W_3(x) = W(y_4,y_3,y_2)(x) = -y_4(x) \neq 0,\quad W(y_4,y_3,y_2,y_1)(x) = 1.$$

Thus

$$a_0(x) = \frac{1}{y_4(x)} \neq 0,\quad a_1(x) = \frac{y_4^2(x)}{W(y_4,y_3)(x)} \neq 0,\quad a_2(x) = \frac{W^2(y_4,y_3)(x)}{y_4^2(x)} \neq 0,$$

$$a_3(x) = \frac{y_4^2(x)}{W(y_4,y_3)(x)} \neq 0,\quad a_4(x) = \frac{1}{y_4(x)} \neq 0,$$

and (2) or (15) will have the form

(18)

$$\frac{1}{y_4(x)}\left[\frac{y_4^2(x)}{W(y_4,y_3)(x)}\left[\frac{W^2(y_4,y_3)(x)}{y_4^2(x)}\left[\frac{y_4^2(x)}{W(y_4,y_3)(x)}\left[\frac{z(x)}{y_4(x)}\right]'\right]'\right]'\right]' = f(x).$$

Theorem 2. *Let $p(x)$, $f(x) \in C([a,\infty))$, $p(x)$ nonnegative and not identically zero on some subinterval of $[a,\infty)$ and $f(x)$ a one-signed function in a neighbourhood of $+\infty$ not identically zero for large x. Let be equation* (1) *nonoscillatory. Then the equation* (2) *is also nonoscilatory.*

Proof. Under the given conditions on $p(x)$ and $f(x)$ the equation (2) can be transformed to the equivalent equation (15) and also (18), where the functions $a_i(x) \neq 0$, $i = 0,1,2,3,4$ on some neighbourhood of $+\infty$. The nonoscillatory character of solutions of (15) and (18) is evident.

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