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ON PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS IN UNBOUNDED DOMAINS

László Simon

Department of Applied Analysis, Faculty of Science, Eötvös Loránd University
1053 Budapest, Kecskeméti u. 10-12, Hungary
Email: simonl@ludens.elte.hu

Abstract. We shall consider weak solutions of initial–boundary value problems for nonlinear parabolic functional differential equations containing discontinuous terms in the unknown function. There will be proved the existence of solutions and formulated some properties of the solutions.

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1. Introduction

We shall consider initial-boundary value problems for the equation

$$
\begin{aligned}
&D_t u(t,x) - \sum_{j=1}^{n} D_j[f_j(t,x,u(t,x),\nabla u(t,x))] + f_0(t,x,u(t,x),\nabla u(t,x)) + \\
&\quad + g(t,x,u(t,x)) + h(t,x,[H(u)](t,x)) = F(t,x), \qquad (1)\\
&\qquad (t,x) \in Q_T = (0,T)\times\Omega
\end{aligned}
$$

where $\Omega \subset R^n$ is a (possibly unbounded) domain with sufficiently smooth boundary, H is a linear continuous operator in $L^p(Q_T)$, the functions f_j are measurable in (t,x), continuous with respect to $u(t,x)$, $\nabla u(t,x)$ but the functions g, h are assumed to be only measurable in all variables. Further, f_j, g, h have certain polynomial growth in $u(t,x)$, $\nabla u(t,x)$. The case when Ω is bounded, was considered, e.g., in [11] where certain terms were rapidly increasing in $u(t,x)$. In [13] there were considered equations of more general form where all the terms were continuous in $u(t,x)$ and $\nabla u(t,x)$.

The problem was motivated by the climate model considered by J.I. Díaz and G. Hetzer in [8] where a particular case of the equation (1) (also with discontinuous terms in u) was investigated on the unit sphere in R^3 (instead of Ω). Some qualitative properties of the solutions of the climate model (without delay terms) were proved in [1] and [7]. Functional partial differential equations arise also in population dynamics, plasticity, hysteresis (see, e.g., [2], [4], [10], [15]).

The aim of this work is to formulate and prove new results in the case of unbounded Ω. We shall formulate conditions which imply the existence of weak solutions of initial-boundary value problems for (1) and to show that in the case of unbounded Ω, the limit of solutions of problems in large bounded domains is a solution of the problem in Ω. There will also be proved the boundedness of the solutions under some conditions and a theorem on the stabilization of the solutions as $t \to \infty$. Our results can be easily extended to equations, containing higher order derivatives with respect to x.

2. Existence theorems

Let $\Omega \subset R^n$ be a (possibly unbounded) domain with sufficiently smooth boundary, $p \geq 2$. Denote by $W^{1,p}(\Omega)$ the usual Sobolev space with the norm

$$\| u \| = \left[\int_\Omega (\sum_{j=1}^n |D_j u|^p + |u|^p) \right]^{1/p}.$$

Let V be a closed linear subspace of $W^{1,p}(\Omega)$ and denote by $X_T = L^p(0,T;V)$ the Banach space of the set of measurable functions $u : (0,T) \to V$ such that $\| u \|^p$ is integrable. The dual space of $L^p(0,T;V)$ is $X_T^\star = L^q(0,T;V^\star)$ where $1/p + 1/q = 1$ and $V^\star$ is the dual space of V (see, e.g., [14]).

On functions f_j we assume that

A (i) $f_j : Q_T \times R^{n+1} \to R$ are measurable in $(t,x) \in Q_T$ and continuous in $\eta \in R, \zeta \in R^n$;

(ii) $|f_j(t,x,\eta,\zeta)| \leq c_1(|\eta|^{p-1} + |\zeta|^{p-1}) + k_1(x)$ with some constant c_1 and a function $k_1 \in L^q(\Omega)$ $(j = 0,1,...,n)$;

(iii) $\sum_{j=1}^n [f_j(t,x,\eta,\zeta) - f_j(t,x,\eta,\tilde{\zeta})](\zeta_j - \tilde{\zeta}_j) > 0$ if $\zeta \neq \tilde{\zeta}$;

(iv) $\sum_{j=1}^n f_j(t,x,\eta,\zeta)\zeta_j + f_0(t,x,\eta,\zeta)\eta \geq c_2[|\zeta|^p + |\eta|^p] - k_2(x)$ with some constant $c_2 > 0$ and $k_2 \in L^1(\Omega)$.

Remark 1. A simple example for f_j, satisfying A (i) - (iv) is

$$f_j(t,x,\eta,\zeta) = a_j(t,x)\zeta_j|\zeta_j|^{p-2}, \quad j = 1,...,n,$$

$$f_0(t,x,\eta,\zeta) = a_0(t,x)\eta|\eta|^{p-2},$$

where a_j are measurable functions, satisfying $0 < c_0 \leq a_j(t,x) \leq c_0'$ with some constants c_0, c_0'.

On functions g, h we assume that

B (i) $g = g_1 + g_2$, $g_j : Q_T \times R \to R$ and $h : Q_T \times R \to R$ are measurable functions;

(ii) $|g_1(t,x,\eta)| \leq k_3(x)|\eta|^{p-1}$ and $g_1(t,x,\eta)\eta \geq 0$ with some function $k_3 \in L^1(\Omega) \cup |L^\infty(\Omega)$;

(iii)

$$|g_2(t,x,\eta)| \leq k_3(x)k_4(|\eta|)|\eta|^{p-1} + k_5(x), \quad |h(t,x,\theta)| \leq k_3(x)k_4(|\theta|)|\theta|^{p-1} + k_5(x)$$

where $k_5 \in L^q(\Omega)$ and k_4 is a continuous function, satisfying $\lim_\infty k_4 = 0$.

Further,

C $H : L^p(Q_T) \to L^p(Q_T)$ is a linear and continuous operator such that for any compact $K \subset \Omega$ there is a compact $\tilde{K} \subset \Omega$ with the following property: the restriction of $H(u)$ to $(0,t) \times K$ depends only on the restriction of u to $(0,t) \times \tilde{K}$ for all $t \in (0,T]$.

Remark 2. The operator H may have e.g. one of the following forms:

$$[H(u)](t,x) = \int_0^t \beta_0(s,t,x)u(s,x)ds \text{ or } [H(u)](t,x) = u(\tau(t),x)$$

with some $\beta_0 \in L^\infty((0,T) \times Q_T)$ and a continuously differentiable function τ satisfying $\tau' > 0, 0 < \tau(t) \leq t$.

Since g_1 is locally bounded, for any $\epsilon > 0$ we may define (with fixed $(t,x) \in Q_T$)

$$\bar{g}_1^\varepsilon(t,x,\eta) = \text{ess sup}_{|\eta-\tilde{\eta}|<\varepsilon} g_1(t,x,\tilde{\eta}),$$

$$\underline{g}_1^\varepsilon(t,x,\eta) = \text{ess inf}_{|\eta-\tilde{\eta}|<\varepsilon} g_1(t,x,\tilde{\eta})$$

For fixed t,x,η $\bar{g}_1^\varepsilon(t,x,\eta)$ is nonincreasing and $\underline{g}_1^\varepsilon(t,x,\eta)$ is nondecreasing as ε is decreasing thus

$$\bar{g}_1(t,x,\eta) = \lim_{\varepsilon\to 0} \bar{g}_1^\varepsilon(t,x,\eta), \quad \underline{g}_1(t,x,\eta) = \lim_{\varepsilon\to 0} \underline{g}_1^\varepsilon(t,x,\eta)$$

exist. Similarly may be defined $\bar{g}_2, \underline{g}_2, \bar{h}, \underline{h}$ (by functions g_2, h, respectively).

Theorem 1. *Assume* **A** *(i) - (iv) and* **B** *(i) - (iii) and* **C**. *Then for each $F \in X_T^\star, u_0 \in V$ there exists $u \in X_T$ with $D_t u \in X_T^\star$ and $\varphi_1, \varphi_2, \psi \in L^q(Q_T)$ such that*

$$u(0,\cdot) = u_0, \tag{2}$$

for arbitrary $v \in V$ we have

(3)

$$\langle D_t u(t,\cdot), v\rangle + \sum_{j=1}^n \int_\Omega f_j(t,x,u(t,x),\nabla u(t,x))D_j v(x)dx +$$

$$\int_\Omega f_0(t,x,u(t,x),\nabla u(t,x))v(x)dx + \int_\Omega [\varphi_1(t,x) + \varphi_2(t,x) + \psi(t,x)]v(x)dx =$$

$$\langle F(t,\cdot), v\rangle \text{ for a.e. } t \in [0,T]$$

and for a.e. $(t,x) \in Q_T$

$$\underline{g}_l(t,x,u(t,x)) \le \varphi_l(t,x) \le \bar{g}_l(t,x,u(t,x)), \quad l = 1,2 \tag{4}$$

$$\underline{h}(t,x,[H(u)](t,x)) \le \psi(t,x) \le \bar{h}(t,x,[H(u)](t,x)).$$

Proof. Consider the function $j \in C_0^\infty(R)$ supported by $[-1,1]$ with the properties $j \ge 0$, $\int_R j = 1$ and for any positive integer k define the functions j_k by $j_k(\eta) = kj(k\eta)$. Then the convolutions (with fixed t,x) $g_{l,k} = g_l \star j_k \quad (l = 1,2), \quad h_k = h \star j_k$ are smooth functions (of η, θ, respectively). Further, define functions

$$\tilde{g}_{l,k}(t,x,\eta) = g_{l,k}(t,x,\eta) \text{ if } |x| \le k, \quad \tilde{g}_{l,k}(t,x,\eta) = 0 \text{ if } |x| > k$$

$$\tilde{h}_k(t,x,\theta) = h_k(t,x,\theta) \text{ if } |x| \le k, \quad \tilde{h}_k(t,x,\theta) = 0 \text{ if } |x| > k.$$

Then we may define operators $A, B_k, C_k : X_T \to X_T^\star$ by

$$[A(u),v] = \int_0^T \langle A(u)(t), v(t)\rangle dt,$$

$$\langle A(u)(t), v(t)\rangle = \sum_{j=1}^n \int_\Omega f_j(t,x,u,\nabla u) D_j v dx + \int_\Omega f_0(t,x,u,\nabla u) v dx,$$

$$[B_k^l(u),v] = \int_0^T \langle B_k^l(u)(t), v(t)\rangle dt = \int_{Q_T} \tilde{g}_{l,k}(t,x,u) v dt dx, \quad l = 1,2,$$

$$[B_k(u),v] = [B_k^1(u),v] + [B_k^2(u),v],$$

$$[C_k(u),v] = \int_0^T \langle C_k(u)(t), v(t)\rangle dt = \int_{Q_T} \tilde{h}_k(t,x,H(u)) v dt dx, \quad u,v \in X_T.$$

By using the assumptions of our theorem, Hölder's inequality and Vitali's theorem it is not difficult to show that the operator $A + B_k + C_k : X_T \to X_T^\star$ is bounded (i.e. it maps bounded sets of X_T into bounded sets of $X_T^\star$) and demicontinuous, i.e.

$$(u_l) \to u \text{ in } X_T \text{ implies } (A + B_k + C_k)(u_l) \to (A + B_k + C_k)(u) \text{ weakly in } X_T^\star.$$

Further, by using compact imbedding theorems we obtain (as in [12]) that $A + B_k + C_k$ is pseudomonotone with respect to

$$D(L) = \{u \in X_T : D_t u \in X_T^\star, u(0) = 0\},$$

i.e. if $u_l, u \in D(L)$,

$$(u_l) \to u \text{ weakly in } X_T, \quad (D_t u_l) \to D_t u \text{ weakly in } X_T^\star \text{ and}$$

$$\limsup_{l\to\infty} [(A + B_k + C_k)(u_l), u_l - u] \le 0$$

then

$$(A + B_k + C_k)(u_l) \to (A + B_k + C_k)(u) \text{ weakly in } X_T^\star \text{ and}$$
$$\lim_{l\to\infty} [(A + B_k + C_k)(u_l), u_l - u] = 0.$$

Finally, we show that $A + B_k + C_k$ is coercive, i.e.

$$\lim_{\|u\|\to\infty} \frac{[(A + B_k + C_k)(u), u]}{\| u \|_{X_T}} = +\infty. \tag{5}$$

Assumption **A** (iv) implies

$$\int_0^t \langle A(u)(\tau), u(\tau)\rangle d\tau \geq c_2 \| u \|_{X_t}^p - t \int_\Omega k_2. \tag{6}$$

By **B** (ii)

$$\tilde{g}_{1,k}(t, x, \eta)\eta \geq 0 \text{ if } |\eta| > 1, \quad \tilde{g}_{1,k}(t, x, \eta)\eta \geq -k_3(x) \text{ if } |\eta| \leq 1$$

thus

$$\int_0^t \langle B_k^1(u)(\tau), u(\tau)\rangle d\tau \geq -t \int_\Omega k_3. \tag{7}$$

Let $a > 0$ be an arbitrary number. Since $\lim_\infty k_4 = 0$, there exists $b > 0$ such that $|\eta| \geq b$ implies $k_4(|\eta|) \leq a$. Hence, by using the notation $Q_t^b = \{(\tau, x) \in Q_t : |u(\tau, x)| \leq b\}$ we obtain from **B** (iii)

$$\begin{gathered} | \int_0^t \langle B_k^2(u)(\tau), u(\tau)\rangle d\tau| \leq \\ | \int_{Q_t^b} \tilde{g}_{2,k}(\tau, x, u) u d\tau dx| + | \int_{Q_t\setminus Q_t^b} \tilde{g}_{2,k}(\tau, x, u) u d\tau dx| \leq \\ C(a) + a \| k_3 \|_{L^\infty(\Omega)} \| u \|_{X_t}^p + \left[t \int_\Omega |k_5|^q\right]^{1/q} \| u \|_{X_t} \end{gathered} \tag{8}$$

with a constant $C(a)$ (not depending on u).

One gets similarly

$$\begin{gathered} | \int_0^t \langle C_k(u)(\tau), u(\tau)\rangle d\tau| \leq \\ C(a) + a \| k_3 \|_{L^\infty(\Omega)} \| u \|_{X_t}^p + \left[t \int_\Omega |k_5|^q\right]^{1/q} \| u \|_{X_t} . \end{gathered} \tag{9}$$

Choosing sufficiently small $a > 0$, from (6) - (9) we obtain for all $t \in [0, T]$

$$\int_0^t \langle (A + B_k + C_k)(u)(\tau), u(\tau)\rangle d\tau \geq c_2/2 \| u \|_{X_t}^p - c_2' \| u \|_{X_t} - c_3' \tag{10}$$

(with some constants c_2', c_3', not depending on u) which implies (5) since $p \geq 2$.

Thus, by Theorem 4 of [3], for any $F \in X_T^\star, u_0 \in V$ there exists $u_k \in X_T$ such that $D_t u_k \in X_T^\star$ and

$$D_t u_k + (A + B_k + C_k)(u_k) = F, \tag{11}$$

$$u_k(0) = u_0. \tag{12}$$

Since

$$\langle D_t u_k(t), u_k(t)\rangle = \frac{1}{2}\frac{d}{dt}\langle u_k(t), u_k(t)\rangle = \frac{1}{2}\frac{d}{dt}(u_k(t), u_k(t))_{L^2(\Omega)}$$

(see, e.g., [14]), applying both sides of (11) to u_k , we find by (10), (12)

$$\begin{gathered} 1/2 \parallel u_k(t) \parallel^2_{L^2(\Omega)} -1/2 \parallel u_0 \parallel^2_{L^2(\Omega)} +c_2/2 \parallel u_k \parallel^p_{X_t} \leq \\ [\parallel F \parallel_{X_T^\star} +c_2'] \parallel u_k \parallel_{X_t} +c_3', \quad t \in [0,T]. \end{gathered} \tag{13}$$

This inequality implies that

$$\parallel u_k \parallel_{X_T}, \parallel u_k \parallel_{L^\infty(0,T;L^2(\Omega))} \text{ are bounded.} \tag{14}$$

Hence the sequence $(A+B_k+C_k)(u_k)$ is bounded in $X_T^\star$ and so $(D_t u_k)$ is bounded in $X_T^\star$, too.

Consequently, there exist $u \in X_T$, $w \in X_T^\star$, $\varphi_l, \psi \in L^q(Q_T)$ and a subsequence of (u_k), again denoted by (u_k) such that

$$(u_k) \to u \text{ weakly in } X_T, \tag{15}$$

(16)
$$(u_k) \to u \text{ in } L^p((0,T) \times \Omega_0) \text{ for each fixed bounded } \Omega_0 \subset \Omega \text{ and a.e. in } Q_T;$$

thus by **C**

$$(H(u_k)) \to H(u) \text{ a.e. in } Q_T; \tag{17}$$

$$(A + B_k + C_k)(u_k) \to w \text{ weakly in } X_T^\star, \tag{18}$$

$$\tilde{g}_{l,k}(t,x,u_k) \to \varphi_l \text{ and } \tilde{h}_k(t,x,u_k) \to \psi \text{ weakly in } L^q(Q_T). \tag{19}$$

From (11), (12), (14), (15), (18), (19) it follows (see, e.g., [14]) $u(0) = u_0$,

$$D_t u + w + \varphi_1 + \varphi_2 + \psi = F. \tag{20}$$

Now we prove $w = A(u)$. Apply (11) to $(u_k - u)\zeta$ with arbitrary fixed $\zeta \in C_0^\infty(\Omega)$ having the properties : $\zeta \geq 0$, $\zeta(x) = 1$ in a compact subset K of Ω. So we obtain

$$\begin{aligned} &[D_t u_k - D_t u, (u_k - u)\zeta] + [D_t u, (u_k - u)\zeta] + [A(u_k), (u_k - u)\zeta] + \\ &\qquad [(B_k + C_k)(u_k), (u_k - u)\zeta] = [F, (u_k - u)\zeta]. \end{aligned} \tag{21}$$

For the first term we have

$$\begin{aligned}[D_t u_k - D_t u, (u_k - u)\zeta] =& 1/2 \int_0^T \left[\frac{d}{dt} \int_\Omega (u_k(t) - u(t))^2 \zeta dx \right] dt = \\ & 1/2 \int_\Omega (u_k(T) - u(T))^2 \zeta dx \geq 0,\end{aligned} \tag{22}$$

further, by (15), (16), (19)

$$\begin{aligned}\lim_{k\to\infty} [D_t u, (u_k - u)\zeta] = 0, \quad \lim_{k\to\infty} [(B_k + C_k)(u_k), (u_k - u)\zeta] = 0, \\ \lim_{k\to\infty} [F, (u_k - u)\zeta] = 0.\end{aligned} \tag{23}$$

Thus (21) - (23) imply

$$\limsup_{k\to\infty} [A_k(u_k), (u_k - u)\zeta] \leq 0. \tag{24}$$

Since by **A** (ii) and (16)

$$\lim_{k\to\infty} \int_{Q_T} f_0(t, x, u_k, \nabla u_k)(u_k - u)\zeta dt dx = 0,$$

from (24) we obtain

$$\limsup_{k\to\infty} \sum_{j=1}^n \int_{Q_T} f_j(t, x, u_k, \nabla u_k)(u_k - u)\zeta dt dx \leq 0. \tag{25}$$

By using arguments of [5], we obtain from (25)

$$\nabla u_k \to \nabla u \text{ a.e. in } (0, T) \times K$$

(see [13]). Since K can be chosen as any compact subset of Ω, we find

$$\nabla u_k \to \nabla u \text{ a.e. in } Q_T. \tag{26}$$

Thus Vitali's theorem and Hölder's inequality imply

$$A(u_k) \to A(u) \text{ weakly in } X_T^\star$$

(see, e.g., [5]), i.e. $w = A(u)$.

In order to show the inequalities (4), one applies arguments of [9], by using (16), (17). (16) implies that for each positive a there exists a subset $\omega \subset Q_T$ with Lebesgue measure $\lambda(\omega) < a$ such that

$$(u_k) \to u \text{ uniformly on } Q_T \setminus \omega \text{ and } u \in L^\infty(Q_T \setminus \omega).$$

Thus for any $\varepsilon > 0$ there is k_0 such that $k_0 > 2/\varepsilon$ and $k > k_0$ implies

$$|u_k(t, x) - u(t, x)| < \varepsilon/2 \text{ if } (t, x) \in Q_T \setminus \omega. \tag{27}$$

Let $k > k_0$, $(t,x) \in Q_T \setminus \omega$. From $1/k < \varepsilon/2$, (27) and the definition of $g_{1,k}$, $\underline{g}_1^\varepsilon$, $\bar{g}_1^\varepsilon$ it easily follows

$$\underline{g}_1^\varepsilon(t,x,u(t,x)) \le g_{1,k}(t,x,u_k(t,x)) \le \bar{g}_1^\varepsilon(t,x,u(t,x)),$$

hence for sufficiently large k

$$\underline{g}_1^\varepsilon(t,x,u(t,x)) \le \tilde{g}_{1,k}(t,x,u_k(t,x)) \le \bar{g}_1^\varepsilon(t,x,u(t,x)).$$

Consequently, for any $\varphi \in C_0^\infty(Q_T)$ with $\varphi \ge 0$ we have

$$\int_{Q_T\setminus\omega} \underline{g}_1^\varepsilon(t,x,u)\varphi \le \int_{Q_T\setminus\omega} \tilde{g}_{1,k}(t,x,u_k)\varphi \le \int_{Q_T\setminus\omega} \bar{g}_1^\varepsilon(t,x,u)\varphi$$

which implies by (19)

$$\int_{Q_T\setminus\omega} \underline{g}_1^\varepsilon(t,x,u)\varphi \le \int_{Q_T\setminus\omega} \varphi_1\varphi \le \int_{Q_T\setminus\omega} \bar{g}_1^\varepsilon(t,x,u)\varphi.$$

Since $u \in L^\infty(Q_T \setminus \omega)$, Lebesgue's dominated convergence theorem implies as $\varepsilon \to 0$

$$\int_{Q_T\setminus\omega} \underline{g}_1(t,x,u)\varphi \le \int_{Q_T\setminus\omega} \varphi_1\varphi \le \int_{Q_T\setminus\omega} \bar{g}_1(t,x,u)\varphi. \tag{28}$$

(28) holds for arbitrary nonnegative $\varphi \in C_0^\infty(Q_T)$, thus we find

$$\underline{g}_1(t,x,u(t,x)) \le \varphi_1(t,x) \le \bar{g}_1(t,x,u(t,x)) \tag{29}$$

for a.e. $(t,x) \in Q_T \setminus \omega$. Inequality (29) holds true for any $a > 0$ and $\omega \subset Q_T$ with $\lambda(\omega) < a$, thus we obtain that (29) is valid a.e. in Q_T.

Remark 3. In certain particular cases (if some Lipschitz conditions are satisfied) one can prove uniqueness of the solution (see also [11]).

It is not difficult to prove an existence theorem for the interval $(0,\infty)$. Denote by X_∞ and $X_\infty^\star$ the set of functions $u : [0,\infty) \to V$, $\quad w : [0,\infty) \to V^\star$, respectively, such that for any finite T $u \in L^p(0,T;V)$, $\quad w \in L^q(0,T;V^\star)$, respectively. Further, define $Q_\infty = (0,\infty) \times \Omega$ and let $L^p_{loc}(Q_\infty)$ be the set of functions $v : Q_\infty \to R$ such that $v \in L^p(Q_T)$ for arbitrary finite T.

Theorem 2. *Assume that functions*

$$f_j : Q_\infty \times R^{n+1} \to R, \quad g,h : Q_\infty \times R \to R$$

satisfy **A** *(i) - (iv),* **B** *(i) - (iii) and* **C** *for any finite* $T > 0$.

Then for arbitrary $F \in X_\infty^\star$ *there exists* $u \in X_\infty$ *such that for any finite* T *the assertion of Theorem 1 holds with some functions* $\varphi_l, \psi \in L^q_{loc}(Q_\infty)$.

Theorem 2 is a consequence of Theorem 1, the proof is based on simple and standard arguments. (Similar arguments can be found e.g. in [12].)

By using arguments of the proof of Theorem 1 we obtain that in the case when Ω is unbounded, the limit (as $k \to \infty$) of certain problems in "large" bounded $\Omega_k \subset \Omega$ is a solution in Ω. Now we give the exact formulation of this statement.

Let $\Omega_k \subset \Omega$ be bounded domains with sufficiently smooth boundary such that $B_k \cap \Omega \subset \Omega_k$ $(B_k = \{x \in R^n : |x| < k\})$ and introduce the notations

$$V_k = W_0^{1,p}(\Omega_k), \quad X_T^k = L^p(0,T;V_k), \quad (X_T^k)^\star = L^q(0,T;V_k^\star)$$

where $W_0^{1,p}(\Omega_k)$ is the completion of $C_0^\infty(\Omega_k)$ with respect to the norm of $W^{1,p}(\Omega_k)$. Further, let $M_k : X_T^k \to X_T$ be the following (extension) operator:

$$M_k v_k(t,x) = v_k(t,x) \text{ for } x \in \Omega_k, \qquad M_k v_k(t,x) = 0 \text{ for } x \in \Omega \setminus \Omega_k$$

Define the restriction F_k of $F \in X_T^\star$ (to Ω_k) by

$$\int_0^T \langle F_k(t), v_k(t)\rangle dt = \int_0^T \langle F(t), (M_k v_k)(t)\rangle dt, \quad v_k \in X_T^k.$$

Finally, let $\varphi \in C_0^\infty(R^n)$ be a function with the properties

$$\varphi(x) = 1 \text{ if } |x| \le 1/2, \quad \varphi(x) = 0 \text{ if } |x| \ge 1$$

and define φ_k by $\varphi_k(x) = \varphi(x/k)$.

Theorem 3. *Assume that the conditions of Theorem 1 are satisfied and the functions $u_k \in X_T^k$ are solutions of the following problems in Ω_k:*

$$u_k(0,\cdot) = \varphi_k u_0 \quad (\in V_k);$$

$D_t u_k \in (X_T^k)^\star$ *and for any* $v_k \in V_k$

$$\begin{aligned}\langle D_t u_k(t,\cdot), v_k\rangle + \sum_{j=1}^n \int_{\Omega_k} & f_j(t,x,u_k(t,x),\nabla u_k(t,x)) D_j v_k(x)dx + \\ \int_{\Omega_k} & f_0(t,x,u_k(t,x),\nabla u_k(t,x)) v_k(x)dx + \\ \int_{\Omega_k} & [\varphi_{1,k}(t,x) + \varphi_{2,k}(t,x) + \psi_k(t,x)] v_k(x)dx = \\ & \langle F_k(t,\cdot), v_k\rangle \text{ for a.e. } t \in [0,T]\end{aligned}$$

with some functions $\varphi_{1,k}, \varphi_{2,k}, \psi_k \in L^q((0,T)\times\Omega_k)$ such that for a.e. $(t,x) \in (0,T)\times\Omega_k$

$$\underline{g}_l(t,x,u_k(t,x)) \le \varphi_{l,k}(t,x) \le \bar{g}_l(t,x,u_k(t,x)), \quad l = 1,2$$

$$\underline{h}(t,x,[H(M_k u_k)](t,x)) \le \psi_k(t,x) \le \bar{h}(t,x,[H(M_k u_k)](t,x)).$$

Then the sequence $(M_k u_k)$ is bounded in X_T and it has a subsequence which is weakly convergent in X_T to a function $u \in X_T$ satisfying (2) – (4).

3. Boundedness and stabilization

Theorem 4. *Assume that the conditions of Theorem 2 are satisfied such that* c_2 *and* k_2 *in* **A** *(iv) are independent of* T, $p > 2$, $\| F(t) \|_{V^\star}$ *is bounded,*

$$|g(t,x,\eta)|^q \le c_4^\star |\eta|^2 + k_4^\star(x), \quad |h(t,x,\theta)|^q \le c_4^\star |\theta|^2 + k_4^\star(x) \tag{30}$$

with some constant $c_4^\star$ *and a function* $k_4^\star \in L^1(\Omega)$. *Further, for any* $u \in L^p_{loc}(Q_\infty)$

$$\int_\Omega |H(u)|^2(t,x)dx \le const \sup_{\tau\in[0,t]} \int_\Omega |u(\tau,x)|^2 dx. \tag{31}$$

Then for the solution u *the function*

$$y(t) = \int_\Omega |u(t,x)|^2 dx$$

is bounded in $(0,\infty)$ *and there exist constants* $c', c"$ *such that for sufficiently large* T_1, T_2

$$\int_{T_1}^{T_2} \| u(t) \|_V^p \, dt \le c'(T_2 - T_1) + c".$$

Idea of the proof. Apply (3) to $v = u(t,\cdot)$ and integrate over (T_1, T_2). Then one obtains the inequality

$$y(T_2) - y(T_1) + c^\star \int_{T_1}^{T_2} [y(t)]^{p/2} dt \le \text{const} \int_{T_1}^{T_2} [\sup_{[0,t]} |y| + 1] dt$$

with some constant $c^\star > 0$ which implies the assertion of Theorem 4. (See, e.g., the proof of Theorem 2 in [12].)

Now we formulate a theorem on the stabilization of the solution as $t \to \infty$. Assume that the conditions of Theorem 4 are satisfied. Consider a sequence $(t_l) \to +\infty$ and define for a solution u

$$U_l(s,x) = u(t_l + s, x), \quad s \in (-a,b), \quad x \in \Omega$$

with some fixed numbers $a, b > 0$. By Theorem 4 (U_l) is bounded in $L^p(-a,b;V)$.

Theorem 5. *Let the assumptions of Theorem 4 be satisfied; assume that* f_j, g, h *are not depending on* t, *there exists a (finite)* ρ *such that for sufficiently large* $t > 0$, $[H(u)](t,x)$ *depends only on the restriction of* u *to* $(t-\rho,t)\times\Omega$ *and it is not depending on* t *if* u *is not depending on* t. *Further, there exists* $F_\infty \in V^\star$ *such that*

$$\lim_{T\to\infty} \int_{T-1}^{T+1} \| F(t) - F_\infty \|_{V^\star} \, dt = 0.$$

Finally,

$$\exists u_\infty \in L^p(\Omega) \text{ and } (t_l) \to +\infty \text{ such that } (U_l) \to u_\infty \text{ weakly in } L^p((-1-\rho,1)\times\Omega). \tag{32}$$

(u_∞ is not depending on t!)

Then there is a subsequence of (t_l) (again denoted by (t_l)) such that for the sequence (U_l) (defined by the subsequence (t_l))

$$(U_l) \to u_\infty \text{ weakly in } L^p(-1,1;V), \tag{33}$$

$$(U_l) \to u_\infty \text{ in } L^p((-1,1)\times\Omega_0) \tag{34}$$

for each bounded $\Omega_0 \subset \Omega$ and a.e. in $(-1,1)\times\Omega$.

Moreover, u_∞ is a solution of the stationary problem

(35)

$$\sum_{j=1}^{n}\int_\Omega f_j(x,u_\infty(x),\nabla u_\infty(x))D_jw(x)dx + \int_\Omega f_0(x,u_\infty(x),\nabla u_\infty(x))w(x)dx +$$

$$\int_\Omega [\tilde\varphi_1(x)+\tilde\varphi_2(x)+\tilde\psi(x)]w(x)dx = \langle F_\infty, w\rangle, \quad w\in V$$

with some functions $\tilde\varphi_l, \tilde\psi \in L^q(\Omega)$ satisfying for a.e. $x\in\Omega$

$$\underline{g}_l(x,u_\infty(x)) \le \tilde\varphi_l(x) \le \bar g_l(x,u_\infty(x)), \quad l=1,2 \tag{36}$$

$$\underline{h}(x,[H(u_\infty)](x)) \le \tilde\psi(x) \le \bar h(x,[H(u_\infty)](x)).$$

Remark 4. In (36) u_∞ means the constant function in t, defined in an interval $(t-\rho,t)$. By the assumption of our theorem, $H(u_\infty)$ does not depend on t.

Remark 5. The operators H, defined in Remark 2 satisfy the assumptions of Theorem 5 if

$$\beta_0(s,t,x) = \beta(s-t,x) \text{ for } \max\{t-\rho,0\} \le s \le t,$$

$$\beta_0(s,t,x) = 0 \text{ for } 0 \le s \le \max\{t-\rho,0\}$$

with a function $\beta \in L^\infty((-\rho,0)\times\Omega)$; $t-\rho \le \tau(t)$, respectively.

Remark 6. By Theorem 4 (U_l) is bounded in $L^p((-1-\rho)\times\Omega)$ for any sequence $(t_l)\to+\infty$, hence a subsequence of (U_l) is weakly convergent to a function $U \in L^p((-1-\rho)\times\Omega)$. In (32) we assume that there exists U, not depending on t.

A sufficient condition for (32) is

$$D_tu \in L^2(0,\infty;L^2(\Omega)). \tag{37}$$

For the proof see [11]. In [11] there are given simple sufficient conditions for (37) which imply a stabilization result in the case when g,h are depending on t and Ω is bounded. The formulation and proof of this result for unbounded Ω is similar to the case of bounded Ω.

The sketch of the proof of Theorem 5. By Theorem 4 (U_l) is bounded in $L^p(-2\rho-1,1;V)$ thus $D_t U_l$ is bounded in $L^q(-\rho-1,1;V^\star)$ which implies by (32) that there is a subsequence of (U_l) (again denoted by (U_l)) such that

(38)
$$(U_l) \to u_\infty \text{ weakly in } L^p(-\rho-1,1;V) \text{ and strongly in } L^p((-\rho-1,1)\times\Omega_0)$$

for any bounded $\Omega_0 \subset \Omega$;

$$(U_l) \to u_\infty \text{ a.e. in } (-1,1)\times\Omega. \tag{39}$$

Define the functions $\varphi_{1,l}, \varphi_{2,l}, \psi_l$ by

$$\varphi_{1,l}(s,x) = \varphi_1(t_l+s,x), \quad \varphi_{2,l}(s,x) = \varphi_2(t_l+s,x), \quad \psi_l(s,x) = \psi(t_l+s,x).$$

Since $(\varphi_{1,l}), (\varphi_{2,l}), (\psi_l)$ are bounded in $L^q((-1,1)\times\Omega)$, we may assume that

$$(\varphi_{1,l}) \to \varphi_1^\star, \quad (\varphi_{2,l}) \to \varphi_2^\star, \quad (\psi_l) \to \psi^\star \text{ weakly in } L^q((-1,1)\times\Omega). \tag{40}$$

Finally, we may assume that

$$\hat{A}(U_l(t)) \to Y \text{ weakly in } L^q(-1,1;V^\star) \tag{41}$$

with some $Y \in L^q(-1,1;V^\star)$ where the operator $\hat{A}: V \to V^\star$ is defined by

$$\langle \hat{A}(v), w\rangle = \sum_{j=1}^n \int_\Omega f_j(x,v,\nabla v)D_j w + \int_\Omega f_0(x,v,\nabla v)w, \quad v,w \in V.$$

Now we apply arguments of [7]. Let

$$\varphi \in C_0^\infty(-1,1), \quad 1 \geq \varphi \geq 0, \quad \int_{-1}^1 \varphi = 1, \quad w \in V. \tag{42}$$

Since u is a solution of (3), we have (for sufficiently large l)

$$\int_{-1}^1 \int_\Omega U_l w \varphi' dt dx + \int_{-1}^1 \langle \hat{A}(U_l(t)), w\rangle \varphi dt + \tag{43}$$
$$\int_{-1}^1 \int_\Omega (\varphi_{1,l} + \varphi_{2,l} + \psi_l) w\varphi dt dx = \int_{-1}^1 \langle F(t_l+t), w\rangle \varphi dt.$$

By (38), (40) - (42) we obtain from (43) as $l \to \infty$

$$\int_{-1}^1 \langle Y(t), w\rangle \varphi dt + \int_{-1}^1 \int_\Omega (\varphi_1^\star + \varphi_2^\star + \psi^\star) w\varphi dt dx = \langle F_\infty, w\rangle. \tag{44}$$

It is not difficult to costruct fuctions $\varphi = \varphi_j$ satisfying (42) such that

$$\lim_{j\to\infty} (\varphi_j) = 1/2 \text{ in } (-1,1).$$

Applying (44) to $\varphi = \varphi_j$, we obtain as $j \to \infty$

$$\frac{1}{2}\int_{-1}^{1} \langle Y(t), w\rangle dt + \int_{\Omega} (\tilde{\varphi}_1 + \tilde{\varphi}_2 + \tilde{\psi}) w dx = \langle F_\infty, w\rangle \tag{45}$$

where

$$\tilde{\varphi}_k = \frac{1}{2}\int_{-1}^{1} \varphi_k^\star dt, \quad \tilde{\psi} = \frac{1}{2}\int_{-1}^{1} \psi^\star dt. \tag{46}$$

Now we show $Y = \hat{A}(u_\infty)$. Let $\Omega_0 \subset \Omega$ be any bounded domain and $\zeta \in C_0^\infty(\Omega)$ with the properties: $\zeta \geq 0$, $\zeta(x) = 1$ for $x \in \Omega_0$ and denote by K the support of ζ. By (38) (for a suitable subsequence)

$$(U_l(t)) \to u_\infty \text{ in } L^2(K) \text{ for a.e. } t \in (-1, 1),$$

hence there exist $\delta_l, \varepsilon_l > 0$ such that (for a suitable subsequence of (U_l))

$$\begin{gathered} \lim_{l\to\infty} (\delta_l) = 0, \quad \lim_{l\to\infty} (\varepsilon_l) = 0, \text{ and } U_l(-1+\delta_l) \to u_\infty, \\ U_l(1-\varepsilon_l) \to u_\infty \text{ in } L^2(K). \end{gathered} \tag{47}$$

By (3) we find

(48)

$$\frac{1}{2}\int_{\Omega} |U_l(1-\varepsilon_l)|^2 \zeta dx - \frac{1}{2}\int_{\Omega} |U_l(-1+\delta_l)|^2 \zeta dx + \int_{-1+\delta_l}^{1-\varepsilon_l} \langle \hat{A}(U_l(t)), U_l(t)\zeta\rangle dt +$$

$$\int_{-1+\delta_l}^{1-\varepsilon_l} \int_{\Omega} (\varphi_{1,l} + \varphi_{2,l} + \psi_l) U_l \zeta dt dx = \int_{-1+\delta_l}^{1-\varepsilon_l} \langle F(t_l + t), U_l(t)\zeta\rangle dt,$$

hence by (38), (40), (45) - (47)

$$\lim_{l\to\infty} \int_{-1+\delta_l}^{1-\varepsilon_l} \langle \hat{A}(U_l(t)), U_l(t)\zeta\rangle dt = \tag{49}$$

$$2\langle F_\infty, u_\infty\zeta\rangle - \int_{-1}^{1}\int_{\Omega} (\varphi_1^\star + \varphi_2^\star + \psi^\star) u_\infty \zeta dt dx = \int_{-1}^{1} \langle Y(t), u_\infty\zeta\rangle dt.$$

By using arguments of [5] we obtain from (49)

$$\nabla U_l \to u_\infty \text{ a.e. in } (-1, 1) \times \Omega_0$$

which implies by (39)

$$(\hat{A}(U_l)) \to \hat{A}(u_\infty) \text{ weakly in } L^q(-1, 1; V^\star),$$

i.e. $Y = \hat{A}(u_\infty)$.

Finally, by (39), (40) we get (similarly to the proof of (4))

$$\underline{g}_l(x, u_\infty(x)) \leq \varphi_l^\star(t, x) \leq \bar{g}_l(x, u_\infty(x)), \quad l = 1, 2$$

$$\underline{h}(x, [H(u_\infty)](x)) \leq \psi^\star(t, x) \leq \bar{h}(x, [H(u_\infty)](x))$$

Integrating these inequalities over $(-1, 1)$, we obtain (36).

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