ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 383–393, CDDE 2000 issue

# GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS AND DISCRETE SYSTEMS

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Abstract. This note is an attempt to show the possibility to deal with discrete equations in the frame of generalized ordinary differential equations defined by Jaroslav Kurzweil in 1957.
Generalized ordinary differential equations form a tool which makes it possible to use a unified approach to classical ordinary differential equations as well as discrete systems.

AMS Subject Classification. 34A99, 39A05, 26A42

Keywords. generalized differential equations, discrete systems.

## 1. Integration

The concept of a generalized ordinary differential equation is based on a special integration process wich is interesting by itself and plays a nice and important role in integration theory and in real analysis in general.

Assume that a bounded interval $[a, b] \subset \mathbb{R}$ is given, $-\infty < a < b < \infty$.

A finite set of points

$$D := a = \alpha_0 \le \tau_1 \le \alpha_1 \le \cdots \le \alpha_{k-1} \le \tau_k \le \alpha_k = b$$

with $\alpha_0 < \alpha_1 < \cdots < \alpha_k$ is called a *partition* of the interval $[a, b]$.

A positive function $\delta : [a, b] \to (0, \infty)$ will be called a *gauge* on the interval $[a, b]$.

The partition $D$ is called $\delta$- *fine* (with respect to the gauge $\delta$) if

$$[\alpha_{j-1}, \alpha_j] \subset [\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)].$$

Assume that a function $U(\tau, t) : [a, b] \times [a, b] \to \mathbb{R}^n$ is given.

For a partition $D$ we denote by

$$S(U, D) = \sum_{j=1}^{K} (U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1}))$$

the integral sum corresponding to the function $U$ and the partition $D$. The fundamental definition reads as follows.

**Definition 1.** The function $U : [a, b] \times [a, b] \to \mathbb{R}^n$ is called *Kurzweil integrable* over $[a, b]$ (shortly $U \in K([a, b])$) if there is a $J \in \mathbb{R}^n$ such that for every $\varepsilon > 0$ there is a gauge $\delta$ on $[a, b]$ and

$$\|S(U, D) - J\| < \varepsilon$$

if $D$ is a $\delta$-fine partition of $[a, b]$.

We use the formal notation $J = \int_a^b DU(\tau, t)$ for the *generalized Kurzweil integral* of $U$ over $[a, b]$.

*Remark 1.* Typical situations are for example $U(\tau, t) = f(\tau) \cdot t$ or $U(\tau, t) = f(\tau) \cdot g(t)$ where $f, g : [a, b] \to \mathbb{R}$ or $f : [a, b] \to \mathbb{R}^n$, $g : [a, b] \to \mathbb{R}$ or $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}^n$.

Looking for example at the sum $S(U, D)$ if $U(\tau, t) = f(\tau) \cdot t$, we can see easily that

$$S(U, D) = \sum_{j=1}^{K} f(\tau_j)(\alpha_j - \alpha_{j-1})$$

is the usual Riemann integral sum corresponding to the function $f : [a, b] \to \mathbb{R}$.

Reading the definition of the integral $\int_a^b DU(\tau, t) = \int_a^b f(s)d(s)$ we can see that it differs from the classical Darboux type definition of the Riemann integral in only one point, namely our $\delta$ is a gauge, i.e. a function which need not be a constant, instead of the positive constant gauge used for defining the Riemann integral.

Nevertheless, this slight change in the definition has dramatic consequences for the concept of the integral and integrability of functions.

It is well known that a function $f : [a, b] \to \mathbb{R}$ is integrable in the sense of our definition for $U(\tau, t) = f(\tau) \cdot t$ if and only if it is integrable in the sense of Perron (the narrow Denjoy integral) and this is a nonabsolutely convergent integral including the Lebesgue integral.

The definition of the integral is based strongly on the following statement which goes back to a paper of P. Cousin from 1895.

**Lemma 1.** *If $\delta$ is an arbitrary gauge on $[a,b]$, then there is a partition $D$ of $[a,b]$ which is $\delta$-fine.*

(See e.g. [2], [3].)

The generalized Kurzweil integral given by Definition 1 has all the good properties usual in reasonable integration theories. Among others we have the following

**Theorem 1.** *If $U, V \in K([a,b])$ and $c_1, c_2 \in \mathbb{R}$, then $c_1U + c_2V \in K([a,b])$ and*

$$\int_a^b D[c_1U(\tau,t) + c_2V(\tau,t)] = c_1 \int_a^b DU(\tau,t) + c_2 \int_a^b DV(\tau,t).$$

*If $U \in K([a,b])$, then $U \in K([c,d])$ for every $[c,d] \subset [a,b]$.*

*If $c \in [a,b]$ and $U \in K([a,c])$ and $U \in K([c,b])$, then $U \in K([a,b])$ and*

$$\int_a^b DU(\tau,t) = \int_a^c DU(\tau,t) + \int_c^b DU(\tau,t).$$

(See Theorems 1.9, 1.10 and 1.11 in [3].)

Also a less usual result holds for the integral.

**Theorem 2.** *If $U \in K([a,c])$ for every $c \in [a,b)$ and*

$$\lim_{c \to b-} \Big( \int_a^c DU(\tau,t) - [U(b,c) - U(b,b)] \Big) = J \in \mathbb{R}, \tag{1}$$

*then $U \in K([a,b])$ and*

$$\int_a^b DU(\tau,t) = J.$$

*If $U \in K([c,b])$ for every $c \in (a,b]$ and*

$$\lim_{c \to a+} \Big( \int_c^b DU(\tau,t) + U(a,c) - U(a,a) \Big) = J \in \mathbb{R}, \tag{2}$$

*then $U \in K([a,b])$ and*

$$\int_a^b DU(\tau,t) = J.$$

(See Theorem 1.14 in [3].)

*Remark 2.* The property of the integral presented in the previous Theorem 2 is sometimes called Hake's Theorem and it is essential when considering generalized ordinary differential equations.

Let us mention that e.g. in the special case of $U(\tau, t) = f(\tau) \cdot t$ the relation (1) represents the existence of the limit

$$\lim_{c \to b-} \int_a^c f(s) ds = J \in \mathbb{R}$$

and by Theorem 2 we have the existence $\int_a^b f(s) ds$ as well as the equality

$$\lim_{c \to b-} \int_a^c f(s) ds = \int_a^b f(s) ds.$$

This property is not possessed by the Riemann or Lebesgue integral. This is a typical property of the Denjoy-Perron integral.

## 2. Generalized ordinary differential equations

Let us have a function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and assume that $[\alpha, \beta] \subset \mathbb{R}$ is a compact interval.

A function $x : [\alpha, \beta] \to \mathbb{R}^n$ is called a *solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DF(x, t) \tag{3}$$

on the interval $[\alpha, \beta]$ if

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t)$$

for every $s_1, s_2 \in [\alpha, \beta]$. (The integral on the right hand side of this relation is the integral presented in Definition 1 in the previous section.)

It can be shown easily that $x : [\alpha, \beta] \to \mathbb{R}^n$ is a solution of (3) if and only if

$$x(s) = x(\gamma) + \int_{\gamma}^{s} DF(x(\tau), t)$$

for every $s \in [\alpha, \beta]$ where $\gamma \in [\alpha, \beta]$ is fixed.

Theorem 2 yields the following

**Proposition 1.** *If $x : [\alpha, \beta] \to \mathbb{R}^n$ is a solution of (3) then*

$$\lim_{s \to \sigma} [x(s) - [F(x(\sigma), s) - F(x(\sigma), \sigma)]] = x(\sigma)$$

*for every $\sigma \in [\alpha, \beta]$.*

*Moreover, if the limit*

$$\lim_{s\to\sigma+}[F(x(\sigma),s)-F(x(\sigma),\sigma)]=J^{+}(\sigma)\in\mathbb{R}^{n}$$

*or*

$$\lim_{s\to\sigma-}[F(x(\sigma),s)-F(x(\sigma),\sigma)]=J^{-}(\sigma)\in\mathbb{R}^{n}$$

*exists, then*

$$\lim_{s\to\sigma+}x(s)=x(\sigma+)=x(\sigma)+J^{+}(\sigma)$$

*or*

$$\lim_{s\to\sigma-}x(s)=x(\sigma-)=x(\sigma)+J^{-}(\sigma),$$

*respectively.*

This proposition shows that in the solution of the generalized ordinary differential equation (3) discontinuities can occur if $J^{+}(\sigma)$ or $J^{-}(\sigma)$ is different from zero. Consequently, a solution of (3) can be a discontinuous function in general.

Details on these concepts and properties of a solution of a generalized ordinary differential equation (3) can be found in [3].

Let us now turn our attention to a class of functions $F:\mathbb{R}^{n}\times\mathbb{R}\to\mathbb{R}^{n}$ which leads to a reasonable theory for equations of the form (3).

Assume that $h:\mathbb{R}\to\mathbb{R}$ is a nondecreasing function and that $\omega:[0,\infty)\to\mathbb{R}$ is continuous, increasing with $\omega(0)=0$ (a modulus of continuity).

Let us define the class $\mathcal{F}(h,\omega)$ of functions $F:\mathbb{R}^{n}\times\mathbb{R}\to\mathbb{R}^{n}$ satisfying

$$\|F(x,t_2)-F(x,t_1)\|\le|h(t_2)-h(t_1)| \tag{4}$$

and

$$\|F(x,t_2)-F(x,t_1)-[F(y,t_2)-F(y,t_1)]\|\le\omega(\|x-y\|)\cdot|h(t_2)-h(t_1)| \tag{5}$$

for $x,y\in\mathbb{R}^{n}$, $t_1,t_2\in\mathbb{R}$. (See 3.8 Definition in [3].)

The main statement concerning the class $\mathcal{F}(h,\omega)$ is a local existence result for a solution of (3) which has to satisfy a given initial condition.

**Theorem 3.** *If $\widetilde{x}\in\mathbb{R}^{n}$, $t_0\in\mathbb{R}$ and $F\in\mathcal{F}(h,\omega)$, then there exist $\Delta^{-},\Delta^{+}>0$ such that on $[t_0-\Delta^{-},t_0+\Delta^{+}]$ there exists a solution $x:[t_0-\Delta^{-},t_0+\Delta^{+}]\to\mathbb{R}^{n}$ of the generalized ordinary differential equation (3) for which $x(t_0)=\widetilde{x}$.*

(See 4.2 Theorem in [3].)

3.10 Lemma in [3] states the following:

*If $F\in\mathcal{F}(h,\omega)$ and $x:[\alpha,\beta]\to\mathbb{R}^{n}$ is a solution of (3), then for every $s_1,s_2\in[\alpha,\beta]$ we have*

$$\|x(s_2)-x(s_1)\|\le|h(s_2)-h(s_1)|. \tag{6}$$

This implies immediately that if $F \in \mathcal{F}(h, \omega)$ and $x : [\alpha, \beta] \to \mathbb{R}^n$ is a solution of (3) on $[\alpha, \beta]$ then $x \in BV([\alpha, \beta])$ ($x$ is a function of bounded variation on $[\alpha, \beta]$) and

$$\operatorname{var}_\alpha^\beta x \leq h(\beta) - h(\alpha) < \infty$$

if $-\infty < \alpha < \beta < \infty$.

Moreover, if $h$ is continuous from the left (i.e. $\lim_{s \to t-} h(s) = h(t-) = h(t)$) then $x(t-) = x(t)$ and the solution of (3) is continuous from the left. This is an easy consequence of the inequality (6).

Concerning the uniqueness of solutions of (3) we have the following general result.

**Theorem 4.** *If $F \in \mathcal{F}(h, \omega)$, $h(t-) = h(t)$ and*

$$\lim_{v \to 0+} \int_v^u \frac{1}{\omega(r)} dr = \infty$$

*for some $u > 0$, then every solution of (3) with $x(t_0) = \widetilde{x}$ is locally unique for $t > t_0$.*

(See 4.8 Theorem in [3].)

*Remark 3.* If $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $g(x, \cdot)$ is Lebesgue measurable for $x \in \mathbb{R}$ and

$$\|g(x, s)\| \leq m(s),$$

$$\|g(x, s) - g(y, s)\| \leq l(s)\omega(\|x - y\|),$$

where $m, l \in L^1_{\text{loc}}(\mathbb{R})$, then for

$$G(x, t) = \int_0^t g(x, s) ds : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

we have $G \in \mathcal{F}(h, \omega)$ with

$$h(t) = \int_0^t m(s) ds + \int_0^t l(s) ds.$$

The following result connects generalized ordinary differential equations with the classical ordinary differential equations in the Carathéodory sense.

A function $x : [\alpha, \beta] \to \mathbb{R}^n$ is a solution of

$$\dot{x} = g(x, t)$$

if and only if $x$ is a solution of the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DG(x, t)$$

on $[\alpha, \beta]$.

If $J_i : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\|J_i(x) - J_i(y)\| \leq \omega(\|x - y\|)$$

for $i \in \mathbb{N}$, $x, y \in \mathbb{R}^n$ and if $H_d : \mathbb{R} \to \mathbb{R}$ is given for $d \in \mathbb{R}$ by the relations

$$H_d(t) = 0, \quad t \leq d, \quad H_d(t) = 1, \quad t > d,$$

then define

$$F(x, t) = G(x, t) + \sum_{j=1}^{\infty} J_j(x) H_j(t).$$

The function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is well defined, it belongs to a certain class $\mathcal{F}(h, \omega)$ and the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DF(x, t)$$

is equivalent to the so called system with impulses given by the ordinary differential equation

$$\dot{x} = g(x, t)$$

and the conditions

$$x(i+) = x(i) + J_i(x(i)), \quad i \in \mathbb{N}$$

describing the jumps of a solution at the instants $i \in \mathbb{N}$.

Let us now consider the function

$$F(x, t) = \sum_{i=1}^{\infty} J_i(x) H_i(t).$$

with $J_i, H_i$, $i \in \mathbb{N}$ given above and assume for simplicity that

$$\|J_i(x)\| < K = \text{const.}, \quad x \in \mathbb{R}^n. \tag{7}$$

Then $F \in \mathcal{F}(h, \omega)$ with $h(t) = K \sum_{i=1}^{\infty} H_i(t)$.

Note that the assumption (7) of the uniform boundedness of the functions $J_i$ is very strong and restrictive. We use it for simplicity only, in fact for a reasonable theory it is sufficient to require (7) on compact subsets of $\mathbb{R}^n$ only and moreover the constant $K$ need not be the same for all $i \in \mathbb{N}$.

Consider the generalized ordinary differential equation

$$\frac{dx}{d\tau} = DF(x, t) = D[\sum_{i=1}^{\infty} J_i(x) H_i(t)],$$

i.e. the integral equation

$$x(s) = x(\gamma) + \int_\gamma^s D[\sum_{j=1}^{\infty} J_j(x(\tau))H_j(t)],\ s \in [0,\infty),$$

or more conveniently

$$x(s) = x(\gamma) + \int_\gamma^s \sum_{j=1}^{\infty} J_j(x(t))dH_j(t),\ s \in [0,\infty), \tag{8}$$

where $\gamma \in \mathbb{R}$ is fixed.

Since $F(x,s_2) - F(x,s_1) = 0$ for $s_1, s_2 \in (j, j+1]$, $j \in \mathbb{N}$ and for $s_1, s_2 \in [0,1]$, we get for a solution $x$ of (8) on $[0,\infty)$ the relation

$$x(s_2) = x(s_1)$$

if $s_1, s_2 \in (j, j+1]$, $j \in \mathbb{N}$ or $s_1, s_2 \in [0,1]$, i.e the solution $x$ is constant on $[0,1]$ and on intervals $(j, j+1]$, $j \in \mathbb{N}$.

Moreover, we have

$$x(j+) = x(j) + J_j(x(j)), \quad j \in \mathbb{N}.$$

If we assume that $\gamma = 0$ and $x(\gamma) = x(0) = \widetilde{x} \in \mathbb{R}^n$, then for a solution $x$ of (8) on $[0,\infty)$ we have

$$x(s) = \widetilde{x}, \quad s \in [0,1],$$

$$x(s) = x(1) + J_1(x(1)), \quad s \in (1,2],$$

$$x(s) = x(k) + J_k(x(k)), \quad s \in (k,k+1], \quad k \in \mathbb{N}.$$

The solution of (8) is evidently a piecewise constant function defined on $[0,\infty)$ which is constant on the intervals $[0,1]$, $(k,k+1]$, $k \in \mathbb{N}$

## 3. Discrete equations

Let us consider equations of the form

$$x_{k+1} = S_k(x_k), \quad k \in \mathbb{N} \tag{9}$$

where $S_k : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\|S_k(x) - S_k(y)\| \le \omega_1(\|x - y\|) \tag{10}$$

and $\omega_1 : [0,\infty) \to [0,\infty)$ has the character of a modulus of continuity.

Given $x_1 = \widetilde{x} \in \mathbb{R}^n$, by (9) a sequence $(x_k)$, $k \in \mathbb{N}$ in $\mathbb{R}^n$ is uniquely determined.

Also, if $x_{k^*} \in \mathbb{R}^n$ is given for some $k^* \in \mathbb{N}$, the values $x_k$ for $k \geq k^*$, $k \in \mathbb{N}$ can be computed according to (9).

In this situation it is sometimes useful to know the "ancestors" of $x_{k^*}$, i.e. the values $x_k$ for $k \in \mathbb{N}$, $k < k^*$ for which (9) is satisfied and of course it is nice to have these values determined uniquely. For this reason we require that

$$\text{the inverse } S_k^{-1} : \mathbb{R}^n \to \mathbb{R}^n \text{ to } S_k \text{ exists for } k \in \mathbb{N}$$

and $S_k^{-1}$ is defined on the whole $\mathbb{R}^n$, i.e. the range $\mathcal{R}(S_k)$ of $S_k$ equals $\mathbb{R}^n$ for every $k \in \mathbb{N}$.

Let us set

$$J_k(x) = S_k(x) - x \tag{11}$$

for $x \in \mathbb{R}^n$, $k \in \mathbb{N}$. Then by (10) we have

$$\|J_k(x) - J_k(y)\| \leq \omega_1(\|x-y\|) + \|x-y\| = \omega(\|x-y\|)$$

and $\omega : [0,\infty) \to [0,\infty)$ has again the shape of a modulus of continuity.

Let us now consider the generalized ordinary differential equation of the form (8) with $J_k$ given by (11).

It can be seen immediately that given $\widetilde{x} \in \mathbb{R}^n$ the sequence $(x_k)$, $k \in \mathbb{N}$ defined by the discrete system (9) with $x_1 = \widetilde{x}$ is such that the piecewise constant function defined by $x(s) = x_1 = \widetilde{x}$ for $s \in [0,1]$, $x(s) = x_k$ for $s \in (k, k+1]$, $k \in \mathbb{N}$ is a solution of the generalized ordinary differential equation (8) and vice versa: if $x$ is a solution of the generalized ordinary differential equation (8) on $[0,\infty)$ with $x(0) = \widetilde{x}$ then $x_{k+1} = x(s)$, $s \in (k, k+1]$, $k \in \mathbb{N}$ gives the sequence in $\mathbb{R}^n$ defined by (9) with $x_1 = \widetilde{x}$.

We conclude this section by stating that

*there is a one-to-one correspondence between sequences* $(x_k)$, $k \in \mathbb{N}$ *given by* (9) *and solutions of the generalized ordinary differential equation in the special form* (8), *where* $J_k(x) = S_k(x) - x$ *for* $x \in \mathbb{R}^n$, $k \in \mathbb{N}$.

## 4. Some possible applications

Results known for generalized ordinary differential equations can be used for the investigation of discrete systems of the form (9).

For example, there are many stability concepts known for discrete systems (9) (see e.g. the book [1]). They are mostly motivated by analogous concepts for classical ordinary differential equations.

Let us define a new stability concept for discrete equations (9)

$$x_{k+1} = S_k(x_k), \quad k \in \mathbb{N},$$

where we assume that $S_k(0) = 0$ for every $k \in \mathbb{N}$.

The sequence $x_k \equiv 0$ evidently satisfies (9) and we will consider stability of this sequence.

**Definition 2.** $x_k \equiv 0$ is called *variationally stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $y_{k_0}, \dots, y_{k_0+l}$, $l \in \mathbb{N}$ satisfies

$$\|y_{k_0}\| < \delta \text{ and } \sum_{j=k_0}^{k_0+l} \|S_j(y_j)\| < \delta,$$

then $\|y_j\| < \varepsilon$ for $j = k_0, \dots, k_0 + l$.

$x_k \equiv 0$ is called *variationally attractive* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$ there exist $K(\varepsilon) \in \mathbb{N}$, $\gamma(\varepsilon) > 0$ such that if $y_{k_0}, \dots, y_{k_0+l}$, $l \in \mathbb{N}$ satisfy

$$\|y_{k_0}\| < \delta_0 \text{ and } \sum_{j=k_0}^{k_0+l} \|S_j(y_j)\| < \gamma(\varepsilon),$$

then $\|y_j\| < \varepsilon$ provided $j = k_0 + K(\varepsilon), \dots, k_0 + l$.

$x_k \equiv 0$ is called *asymptotically variationally stable* if it is variationally stable and variationally attractive.

Another concept is given by the following definition.

**Definition 3.** $x_k \equiv 0$ is called *stable with respect to perturbations* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $p_{k_0}, \dots, p_{k_0+l}$, $l \in \mathbb{N}$ satisfies

$$\sum_{j=k_0}^{k_0+l} \|p_j\| < \delta,\ y_{k_0} \in \mathbb{R}^n,\ \|y_{k_0}\| < \delta$$

and

$$y_{k+1} = S_k(y_k) + p_k, \quad k = k_0, \dots, k_0 + l,$$

then $\|y_j\| < \varepsilon$ for $j = k_0, \dots, k_0 + l$.

$x_k \equiv 0$ is called *attractive with respect to perturbations* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$ there exist $K(\varepsilon) \in \mathbb{N}$, $\gamma(\varepsilon) > 0$ such that if

$$\|y_{k_0}\| < \delta_0 \text{ and } \sum_{j=k_0}^{k_0+l} \|p_j\| < \gamma,$$

then for

$$y_{k+1} = S_k(y_k) + p_k, \quad k = k_0, \dots, k_0 + l,$$

we have $\|y_j\| < \varepsilon$ if $j = k_0 + K(\varepsilon), \dots, k_0 + l$.

$x_k \equiv 0$ is called *asymptotically stable with respect to perturbations* if it is stable and attractive with respect to perturbations.

Similar concepts have been presented for generalized differential equations in Chapter 10 of [3]. Presenting the results from [3] in terms of discrete systems we can state e.g. the following

**Theorem 5.** *$x_k \equiv 0$ is variationally stable if and only if it is stable with respect to perturbations.*

*$x_k \equiv 0$ is variationally attractive if and only if it is attractive with respect to perturbations.*

For characterizing e.g. the concept of variational stability of $x_k \equiv 0$ for (9) the following Ljapunov-type result can be derived using the theory of generalized ordinary differential equations (see Theorems 10.13 and 10.23 in [3]).

**Theorem 6.** *$x_k \equiv 0$ is variationally stable if and only if there is a sequence of functions $V_k : \overline{B_d} \subset \mathbb{R}^n \to \mathbb{R}$, $d > 0$ ($\overline{B_d} = \{x \in \mathbb{R}^n; \|x\| \le d\}$ is the closed ball in $\mathbb{R}^n$ centered at 0 with radius $d$) such that*

$$a(\|x\|) \le V_k(x), \quad V_k(0) = 0,$$

$$|V_k(x) - V_k(y)| \le K\|x - y\|$$

*for $x, y \in \overline{B_d}$, $K$ is a constant and $a : [0, \infty) \to \mathbb{R}$ is a continuous increasing function such that $a(r) = 0$ if and only if $r = 0$.*

There is a fairly complete theory for linear generalized ordinary differential equations (see Chapter VI in [3]) which can be used in the above described way for investigating linear discrete systems of the form

$$x_{k+1} = S_k x_k + b_k, \quad k \in \mathbb{N},$$

where $S_k \in L(\mathbb{R}^n)$ are $n \times n$-matrices, $b_k \in \mathbb{R}^n$, $k \in \mathbb{N}$. With the assumption of existence of the inverse $S_k^{-1}$, $k \in \mathbb{N}$ we get a theory of linear systems with nice properties and the results known for linear generalized ordinary differential equations presented in [3] lead to results for linear discrete systems like variation-of-constant formula, periodic systems, Floquet theory, multipliers, etc.

## References

1. Halanay A., Wexler D., *Qualitative Theory of Systems with Impulses*, Editura Acad. Rep. Soc. Romania, Bucuresti, 1968 (Romanian).
2. Kurzweil J.,*Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. J., **7** (82), 1957, 418–449.
3. Schwabik Š.,*Generalized Ordinary Differential Equations*, World Scientific, Singapore 1992.