ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 595–602, CDDE 2000 issue

METHOD OF LOWER AND UPPER SOLUTIONS FOR A GENERALIZED BOUNDARY VALUE PROBLEM

Boris Rudolf*

Dept. of Math., Faculty of Electrical Engineering and Information Technology SUT
Ilkovičova 3, 812 19 Bratislava, Slovakia
Email: rudolf@kmat.elf.stuba.sk

Abstract. A method of lower and upper solutions is used to prove the existence of a solution of a boundary value problem with generalized boundary conditions given by continuous linear functionals. The cases of Dirichlet, Neumann, multipoint and integral conditions are covered.

AMS Subject Classification. 34B10, 34B15

Keywords. Multiplicity result, lower and upper solutions, coincidence degree.

The method of lower and upper solutions is, in connection with the topological degree theory, widely used to prove the existence or multiplicity results for various types of boundary value problems. See [1] – [8].

The aim of this paper is to extend the method of lower and upper solutions to the case of boundary conditions given by the continuous linear functionals. Such conditions are given by Riemann-Stjeltjes integrals.

We consider the second order differential equation

$$x'' = f(t, x, x') \tag{1}$$

with the generalized boundary conditions

$$\begin{aligned} x(a) &= \int_a^b x(t)\, dg_1(t) + k_1 x'(a) \\ x(b) &= \int_a^b x(t)\, dg_2(t) - k_2 x'(b), \end{aligned} \tag{2}$$

* Research supported by grant Vega 1/7176/20

where $f : I = [a,b] \times R^2 \to R$ is a continuous function, $g_i(t)$ are nondecreasing functions with bounded variation, $1 \geq g_i(b) - g_i(a)$ and $k_i \geq 0$.

We assume that g_i, k_i, are such that the boundary conditions are linearly independent. Our purpose is to extend some existence results of [6] to the case of the problem (1), (2).

Definition 1. The function $\alpha(t)$ is called a lower solution for the problem (1), (2) if

$$\begin{aligned} \alpha''(t) &\geq f(t, \alpha(t), \alpha'(t)), \\ \alpha(a) &\leq \int_a^b \alpha(t)\, dg_1(t) + k_1\alpha'(a) \\ \alpha(b) &\leq \int_a^b \alpha(t)\, dg_2(t) - k_2\alpha'(b), \end{aligned} \tag{3}$$

Similarly the function $\beta(t)$ is called an upper solution for the problem (1), (2) if

$$\begin{aligned} \beta''(t) &\leq f(t, \beta(t), \beta'(t)), \\ \beta(a) &\geq \int_a^b \beta(t)\, dg_1(t) + k_1\beta'(a) \\ \beta(b) &\geq \int_a^b \beta(t)\, dg_2(t) - k_2\beta'(b), \end{aligned} \tag{4}$$

If the strict inequalities for α'', β'' hold α, β are called strict lower and upper solutions.

Remark 1. In the case of Dirichlet conditions $x(a) = x(b) = 0$, continuity of the function f implies that for $\varepsilon > 0$ sufficiently small $\alpha(t) - \varepsilon$, $\beta(t) + \varepsilon$ are strict lower and upper solutions satisfying the strict inequalities (3), (4).

Therefore below we assume that in the case of Dirichlet conditions the strict lower and upper solutions satisfy also the strict inequalities (3), (4).

Lemma 1. *[8, p. 214] Let $h(s)$ be a positive continuous function such that*

$$\int^{\infty} \frac{s}{h(s)} ds = \infty, \tag{5}$$

f be a continuous function satisfying

$$|f(t,x,y)| \leq h(|y|) \quad \text{for each } |x| \leq r,\ t \in I,$$

and let $x(t)$ be a solution of the problem (1), (2) such that $\|x\| \leq r$. Then there is a constant $\rho_0 > 0$ such that $\|x'\| < \rho_0$.

Lemma 2. *Let α, β be a strict lower and upper solutions and $u(t)$ be a solution of the problem (1), (2).*

Then $\alpha(t) \leq u(t)$ implies $\alpha(t) < u(t)$ and $\beta(t) \geq u(t)$ implies $\beta(t) > u(t)$.

Proof. Let $0 = u(t_0) - \beta(t_0)$ at $t_0 \in (a, b)$. Then

$$0 \geq u(t_0)'' - \beta(t_0)'' = f(t_0, u(t_0), u'(t_0)) - \beta(t_0)'' \geq f(t_0, \beta(t_0), \beta'(t_0)) - \beta(t_0)'' > 0,$$

a contradiction.

Let $0 = u(a) - \beta(a)$, $u(t) < \beta(t)$ for $t \in (a, b)$. If $u'(a) = \beta'(a)$ we obtain the same contradiction as above. Suppose $u'(a) < \beta'(a)$.

We consider several cases.

Let $k_1 > 0$. Then

$$u(a) - \beta(a) < \int_a^b u(t) - \beta(t)\, dg_1(t) \leq (g_1(b) - g_1(a)) \max_{t \in I} (u(t) - \beta(t)) \leq 0,$$

a contradiction.

Let $k_1 = 0$. If g_1 is nonconstant on a subinterval $[c, d] \subset (a, b)$ then

$$u(a) - \beta(a) \leq \int_a^b u(t) - \beta(t)\, dg_1(t) < (g_1(b) - g_1(a)) \max_{t \in I} (u(t) - \beta(t)) \leq 0,$$

a contradiction.

If g_1 is constant on $(a, b]$ then the first condition of (2) is reduced to Dirichlet condition. With respect to Remark 1 we assume $\beta(a) > 0$. Then $u(a) - \beta(a) < 0$, a contradiction.

If g_1 is constant on $[a, b)$ then $u(a) - \beta(a) \leq c(u(b) - \beta(b))$, $c \leq 1$. That means $u(a) = \beta(a)$ implies $u(b) = \beta(b)$. Using the boundary condition at point b and considering the same cases as above we obtain a contradiction with the equality $u(b) - \beta(b) = 0$. The last case g_2 is constant on $(a, b]$ leads either to the Dirichlet conditions case, or to the linear dependance of boundary conditions.

Let $X = C^1(I)$, $\operatorname{dom} L = \{x(t) \in C^2(I),\ x \text{ satisfies } (2)\}$, $Z = C(I)$. We denote

$$L : \operatorname{dom} L \subset X \to Z, \qquad Lx = x'',$$
$$N : X \to Z, \qquad Nx(t) = f(t, x(t), x'(t)).$$

The problem (1), (2) is equivalent to the operator equation

$$Lx = Nx,$$

where the operator N is L-compact [2].

We denote

$$\Omega_{r,\rho} = \{x(t) \in C^1(I), \quad \|x\| < r, \quad \|x'\| < \rho\}.$$

Lemma 3. *Let*

(i) there is a constant $r > 0$ such that $f(t, r, 0) > 0$ and $f(t, -r, 0) < 0$,

(ii) $|f(t, x, y)| \leq h(|y|)$, $h \geq \varepsilon > 0$ satisfies (5), for each $t \in I$, $|x| < r$.

Then there is $\rho_0 > 0$ such that the topological degree

$$D(L, N, \Omega_{r,\rho}) = 1 \qquad (\text{mod } 2)$$

for each $\rho > \rho_0$ i.e. there is a solution $x(t)$ of (1), (2) such that $|x(t)| < r$, $|x'(t)| < \rho$.

Proof. We consider the homotopy

$$Lx = \tilde{N}(x, \lambda)$$

defined by the parametric system of equations

$$x'' = \lambda f(t, x, y) + (1 - \lambda)x, \qquad (2). \tag{6}$$

Now $-r$, r are a strict lower and upper solutions of the problem (6).

As $|\lambda f(t, x, y) + (1 - \lambda)x| \le h(|y|) + r$, the assumptions of Lemma 1 are satisfied for the function $\lambda f(t, x, y) + (1 - \lambda)x$. Then the a priori bound of derivative and Lemma 2 imply that no solution of (6) lies on the boundary of $\partial\Omega_{r,\rho}$, $\rho \ge \rho_0$.

By the generalized Borsuk theorem [3]

$$D(L, \tilde{N}(.,1), \Omega_{r,\rho}) = D(L, \tilde{N}(.,0), \Omega_{r,\rho}) = 1 \qquad (\mathrm{mod}\ 2)$$

and Lemma 3 is proved.

Theorem 1. *Let*

(i) $\alpha(t) < \beta(t)$ *be a lower and upper solutions of the problem* (1), (2).

(ii) $|f(t, x, y)| \le h(|y|)$, *for each* (t, x, y), $t \in I$, $\alpha(t) \le x \le \beta(t)$, $y \in R$, *where* $h \ge \varepsilon > 0$ *satisfies* (5),

Then there is a constant ρ_0 *such that for each* $\Omega = \{x(t) \in C^1(I), \quad \alpha(t) < x(t) < \beta(t), \quad \|x'\| < \rho\}$, $\rho > \rho_0$ *there is a solution* $x \in \bar{\Omega}$ *of* (1), (2).

Moreover if $\alpha(t)$, $\beta(t)$ *are strict lower and upper solutions then*

$$D(L, N, \Omega) = 1 \qquad (\mathrm{mod}\ 2).$$

Proof. Let $r = \max\{\|\alpha\|, \|\beta\|\}$, $M > \max|f(t, x, 0)|$ for $t \in I$, $|x| \le r$.

We define a perturbation

$$f^*(t, x, y) = \begin{cases} f(t, \beta(t), y) + M(r - \beta(t)) + M & x > r + 1, \\ f(t, \beta(t), y) + M(x - \beta(t)) & \beta(t) < x \le r + 1, \\ f(t, x, y) & \alpha(t) \le x \le \beta(t), \\ f(t, \alpha(t), y) - M(\alpha(t) - x) & -r - 1 \le x < \alpha(t), \\ f(t, \alpha(t), y) - M - M(\alpha(t) + r) & x < -r - 1. \end{cases}$$

The function f^* satisfies the Nagumo condition as well as the assumptions of Lemma 3 for $\Omega_{r+1,\rho}$, $\rho > \rho_0$ where ρ_0 is a constant from Lemma 1 for the function f^*.

Suppose $u(t) \in \Omega_{r+1,\rho}$ is a solution of the problem

$$x'' = f^*(t, x, x'), \quad (2). \tag{7}$$

We show that $\alpha \le u \le \beta$.

Let $v(t) = u(t) - \beta(t)$ attains its maximum $v_{max} > 0$. Then $\beta(t) + v_{max}$ is a strict upper solution of (7). Lemma 2 implies $u(t) < \beta(t) + v_{max}$ a contradiction.

That means $u(t)$ is a solution of (1), (2).

If $\alpha(t)$, $\beta(t)$ are a strict lower and upper solutions then moreover

$$D(L, N^*, \Omega_{r,\rho}) = D(L, N^*, \Omega) = D(L, N, \Omega) = 1 \qquad (\mathrm{mod}\ 2).$$

Theorem 2. *Let*

(i) $|f(t, x, y)| < M$,

(ii) α, β, $\beta(t) < \alpha(t)$, *be a strict lower and upper solutions for the problem* (1), (2).

Then there are constants $r, \rho > 0$ *such that*

$$D(L, N, \Omega) = 1 \qquad (\mathrm{mod}\ 2)$$

where $\Omega = \{x(t) \in C^1(I),\ \exists t_x \in I,\ \beta(t_x) < x(t_x) < \alpha(t_x),\ \|x\| < r,\ \|x'\| < \rho\}$.

Proof. Let $\rho = (b - a)2M$ and $r = \max(\|\alpha\|, \|\beta\|) + (b - a)\rho$.

We define a perturbation f^* by

$$f^*(t, x, y) = \begin{cases} f(t, x, y) + M & x > r + 1, \\ f(t, x, y) + M(x - r) & r < x \le r + 1, \\ f(t, x, y) & -r \le x \le r, \\ f(t, x, y) + M(x + r) & -r - 1 \le x < -r, \\ f(t, x, y) - M & x < -r - 1. \end{cases}$$

Clearly $r + 1$, $-r - 1$ are a strict lower and upper solutions of the problem

$$x'' = f^*(t, x, x'), \quad (2). \tag{8}$$

As $|f^*| < 2M$ then for each solution of (8) the boundary conditions (2) imply that there is a constant ρ such that $|x'(t)| < \rho$.

Therefore

$$D(L, N^*, \Omega_{r+1,\rho}) = 1 \qquad (\mathrm{mod}\ 2)$$

Let now

$$\Omega_l = \{x(t) \in \Omega_{r+1,\rho}, \quad -r - 1 < x < \beta\},$$
$$\Omega_u = \{x(t) \in \Omega_{r+1,\rho}, \quad \alpha < x < r + 1\}.$$

Then

$$D(L, N^*, \Omega_l) = D(L, N^*, \Omega_u) = 1 \qquad (\mathrm{mod}\ 2)$$

Set $\Omega_m = \Omega_{r+1,\rho} \setminus \overline{(\Omega_l \cup \Omega_u)}$.

As $-r - 1$, α, $r + 1$, β are strict lower and upper solutions, Lemma 2 implies there is no solution $u \in \partial\Omega_m$.

The addition property of the degree means

$$D(L, N^*, \Omega_m) = 1 \qquad (\mathrm{mod}\ 2)$$

on the set $\Omega_m = \Omega_{r+1} \setminus \overline{(\Omega_l \cup \Omega_u)}$, and finally the excision property implies

$$D(L, N^*, \Omega_m) = D(L, N^*, \Omega) = D(L, N, \Omega) = 1 \qquad (\mathrm{mod}\ 2).$$

The Nagumo condition in Theorem 1 and the a priori bound of f in Theorem 2 are in the following theorems replaced by the one sided growth condition.

Theorem 3. *Let*

(i) $k_1 > 0, k_2 > 0$,

(ii) there is $M > 0$ *such that* $f(t, x, y) \leq M$ *for each* $t \in I$, *and each* $x, y \in R$.

(iii) α, β, $\alpha(t) < \beta(t)$, *be a strict lower and upper solutions of the problem* (1), (2).

Then there is $\rho_0 > 0$ *such that for each* $\rho > \rho_0$ *and* $\Omega = \{x(t) \in C^1(I),\ \alpha(t) < x(t) < \beta(t),\ \|x'\| < \rho\}$ *there is*

$$D(L, N, \Omega) = 1 \qquad \pmod 2.$$

Proof. Let $r = \max\{\|\alpha\|, \|\beta\|\}$.

Let $x(t)$ be a solution of (1), (2) such that $\|x\| < r$. Then the boundary conditions (2) imply $x'(a) \leq \dfrac{2r}{k_1}$ and $x'(b) \geq -\dfrac{2r}{k_2}$. Therefore $\|x'\| \leq \dfrac{2r}{k} + (b-a)M$, where $k = \min\{k_1, k_2\}$.

Let $\rho_1 = \dfrac{2r}{k} + (b-a)M + \max\{\|\alpha'\|, \|\beta'\|\}$.

We define

$$\chi(s, t) = \begin{cases} 1 & s \leq t \\ \frac{2t-s}{t} & t < s \leq 2t \\ 0 & s > 2t \end{cases}$$

and

$$f^* = \chi(\|x\|, r)\chi(\|y\|, \rho_1) f(t, x, y). \tag{9}$$

Now f^* is a bounded function and α, β, are strict lower and upper solutions of the problem

$$x'' = f^*(t, x, x'), \quad (2). \tag{10}$$

Theorem 1 implies that there is ρ_2 such that for each $\rho > \rho_2$

$$D(L, N^*, \Omega) = 1 \qquad \pmod 2.$$

We choose $\rho > \max\{\rho_1, \rho_2\} = \rho_0$. For each solution x of (10) such that $\|x\| < r$ there is $\|x'\| < \rho_1$. Then $f(t, x(t), x'(t)) = f^*(t, x(t), x'(t))$ and

$$D(L, N, \Omega) = D(L, N^*, \Omega) = 1 \qquad \pmod 2.$$

Theorem 4. *Let*

(i) $k_1, k_2 > 0$,

(ii) there is $M > 0$ *such that* $f(t, x, y) \leq M$ *for each* $t \in I$, *and each* $x, y \in R$.

(iii) α, β, $\beta(t) < \alpha(t)$, *be a strict lower and upper solutions of the problem* (1), (2).

Then there is $r, \rho > 0$ such that

$$D(L, N, \Omega) = 1 \qquad (\mathrm{mod}\ 2)$$

where

$$\Omega = \{x(t) \in C^1(I),\ \exists t_x \in I\ \beta(t_x) < x(t_x) < \alpha(t_x),\ \|x\| < r,\ \|x'\| < \rho\}.$$

Proof. Let $m = \max\{\|\alpha\|, \|\beta\|\}$, $x(t)$ be a solution and let $\exists t_x \in I\ \beta(t_x) < x(t_x) < \alpha(t_x)$. Then $|x(t_x)| \le m$.

Let t_1 be such that $\min x(t) = x(t_1)$, and suppose that $x(t_1) < -m$.

Let $t_1 < t_x$. Then either $x'(t_1) = 0$ or $t_1 = a$.

In the case $x'(t_1) = 0$ there is $x'(t) = \int_{t_1}^{t} x''(s)\, ds \le (b - t_1)M$, for $t \ge t_1$. Then

$$x(t_1) = x(t_x) - \int_{t_1}^{t_x} x'(s)\, ds \ge -m - (t_x - t_1)(b - t_1)M.$$

If $t_1 = a$ then the boundary condition implies $x(a) > x(a)\,(g_1(b) - g_1(a)) + k_1 x'(a)$. Hence $k_1 x'(a) < (1 - (g_1(b) - g_1(a)))\, x(a)$ which implies $x'(a) < 0$, a contradiction.

Let $t_1 > t_x$. Then either $x'(t_1) = 0$ or $t_1 = b$.

Again $x'(t_1) = 0$ implies that $x'(t) = -\int_{t}^{t_1} x''(s)\, ds \ge -(t_1 - a)M$, for $t \le t_1$. Then

$$x(t_1) = x(t_x) + \int_{t_x}^{t_1} x'(s)\, ds \ge -m - (t_1 - a)(b - t_x)M.$$

If $t_1 = b$ then $x(b) > x(b)\,(g_2(b) - g_2(a)) - k_2 x'(b)$ i.e. $k_2 x'(b) > -(1 - (g_2(b) - g_2(a)))x(b)$ which implies $x'(b) > 0$, a contradiction.

That means $x(t) > -m - (b - a)^2 M$.

Suppose that there is t_2 such that $\max x(t) = x(t_2) > m$.

Case $t_2 > t_1$.

There is $x'(t) = x'(t_1) + \int_{t_1}^{t} x''(s)\, ds \le (t_2 - t_1)M$, for $t \in [t_1, t_2]$, and

$$x(t_2) = x(t_1) + \int_{t_1}^{t_2} x'(s)\, ds \le m + (t_2 - t_1)^2 M.$$

Case $t_2 < t_1$.

There is $x'(t) = x'(t_1) - \int_{t}^{t_1} x''(s)\, ds \ge -(t_1 - t_2)M$, for $t \in [t_2, t_1]$, and

$$x(t_2) = x(t_1) - \int_{t_2}^{t_1} x'(s)\, ds \le m + (t_2 - t_1)^2 M.$$

The above estimations give a priori bound of a solution

$$|x(t)| < r = m + (b - a)^2 M.$$

Arguing as in the proof of the preceeding theorem we obtain that

$$|x'(t)| \le \frac{2r}{k} + (b-a)M,$$

where again $k = \min\{k_1, k_2\}$ and we put $\rho_1 = \frac{2r}{k} + (b-a)M + \max\{\|\alpha'\|, \|\beta'\|\}$.

Using again the perturbation (9) and Theorem 2 we obtain that there is ρ_2 such that for each $\rho > \rho_2$

$$D(L, N^*, \Omega) = 1 \qquad (\text{mod } 2),$$

where

$$\Omega = \{x(t) \in C^1(I),\ \exists t_x \in I\ \beta(t_x) < x(t_x) < \alpha(t_x),\ \|x\| < r,\ \|x'\| < \rho\}.$$

We choose $\rho > \max(\rho_1, \rho_2) = \rho_0$. A priori bounds of solutions imply

$$D(L, N, \Omega) = D(L, N^*, \Omega) = 1 \qquad (\text{mod } 2).$$

Remark 2. It is possible to replace the inequality in the condition (ii) of Theorem 3 and 4 by $f(t, x, y) \ge -M$ for each $t \in I,\ x, y \in R$.

References

1. Fabry C., Mawhin J. and Nkashama M.N., *A multiplicity result for periodic solutions of forced nonlinear boundary value problems*, Bull. London Math. Soc. **18** (1986), 173–186.
2. Mawhin J. *Points fixes, points critiques et problèmes aux limites*, Sémin. Math. Sup. no.92, Presses Univ. Montréal, Montréal 1985.
3. Mawhin J. *Equivalence theorems for Nonlinear operator equations and Coincidence degree theory for some mappings in locally convex topological vector spaces*, JDE **12** (1972), 610–636.
4. Rachůnková I. *Multiplicity results for four-point boundary value problems*, Nonlinear Analysis TMA **18** (1992), 497–505.
5. Rachůnková I. *On the existence of two solutions of the periodic problem for the ordinary second order differential equation*, Nonlinear Analysis TMA **22** (1994), 1315–1322.
6. Rachůnková I. *Upper and lower solutions and topological degree*, JMAA **234** (1999), 311–327.
7. Rudolf B. *A multiplicity result for a periodic boundary value problem*, Nonlinear Analysis TMA, **22** (1997), 137–144.
8. Šeda V. *On some nonlinear boundary value problems for ordinary differential equations*, Archivum Mathematicum (Brno) **25** (1989), 207–222.