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ON NON-LINEAR BOUNDARY VALUE PROBLEMS CONTAINING PARAMETERS

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Abstract. We consider a boundary value problem containing two parameters both in the non-linear ordinary differential equation and in the non-linear boundary conditions. By using a suitable change of variables, we bring the given problem to a family of those with linear boundary conditions (plus some non-linear determining equations), and apply an iterative method to approximately find its solution.

Keywords. Parametrised boundary value problems, non-linear boundary conditions, numerical-analytic methods, successive approximations, determining equations.

1. Introduction

An analysis of the publications concerning the iterative methods in the theory of boundary value problems shows that various numerical-analytic methods, in particular, those based upon successive approximations, are now widely used and developed (see, e. g., [5] for a review).

According to the basic idea of the latter group of methods, the given boundary value problem is replaced by a problem for a "perturbed" differential equation containing some artificially introduced parameter, whose value should be determined later. The solution of the "perturbed" problem is sought for in the analytic form by iteration with all the iterations depending upon the parameter mentioned.

As to the way how the auxiliary problem is constructed, it is essential that the form of the "perturbation term" yields a certain system of (algebraic or transcendental) "determining equations," which give the numerical values of the parameter corresponding to the solutions sought-for. By studying these determining equations, it is possible to establish existence results for the original problem.

It is worth mentioning that, earlier, the parametrised boundary value problems were studied mostly in the case of the linear boundary conditions [4], or even in the case when the parameters are contained only in the differential equation [1,2].

It has been an open problem to find out how one can construct a numerical-analytic scheme suitable for problems with parameters both in the equation and in non-linear boundary conditions. Here, we give a possible approach to this question following the method from [3].

2. Problem setting

We consider the non-linear two-point parameterized boundary value problem

$$y'(t) = f(t, y(t), \lambda_1, \lambda_2), \qquad t \in [0, T], \tag{1}$$

$$g(y(0), y(T), \lambda_1, \lambda_2) = 0, \tag{2}$$

$$y_1(0) = y_{10}, \quad y_2(0) = y_{20}, \tag{3}$$

containing the parameters λ_1 and λ_2 both in Eq. (1) and in condition (2).

Here, we suppose that the functions $f : [0, T] \times G \times [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^n$ ($n \geq 3$) and $g : G \times G \times I_1 \times I_2 \to \mathbb{R}^n$ are continuous, $G \subset \mathbb{R}^n$ is a closed, connected, and bounded domain, and $\lambda_k \in I_k := [a_k, b_k]$ $(k = 1, 2)$ are unknown scalar parameters.

Assume that, for $t \in [0, T]$, $\lambda_1 \in I_1$, and $\lambda_2 \in I_2$ fixed, the function f satisfies the Lipschitz condition

$$|f(t, u, \lambda_1, \lambda_2) - f(t, v, \lambda_1, \lambda_2)| \leq K|u - v| \tag{4}$$

for all $\{u, v\} \subset G$ and some non-negative matrix $K = (K_{kl})_{k,l=1}^n$. In (4), as well as in similar relations below, the signs $|\cdot|$ and $\leq$ are understood component-wise.

The problem is to find the values of the parameters λ_1 and λ_2 such that problem (1), (2) has a classical solution satisfying the additional conditions (3). Thus, a solution is the triple $\{y, \lambda_1, \lambda_2\}$ and, therefore, (1)–(3) is similar, in a sense, to an eigen-value problem.

3. A reduction to the parametrised boundary value problem with linear conditions

Let us introduce the substitution

$$y(t) = x(t) + w, \tag{5}$$

where $w = \mathrm{col}(w_1, w_2, \dots, w_n) \in \Omega \subset \mathbb{R}^n$ is an unknown parameter. The domain Ω is chosen so that

$$D + \Omega \subset G,$$

whereas the new variable, x, is supposed to have range in D, the closure of a bounded subdomain of G.

Substitution (5) allows one to rewrite problem (1)–(3) as

$$x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2), \qquad t \in [0, T], \tag{6}$$

$$g(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = 0, \tag{7}$$

$$x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2. \tag{8}$$

Let us bring the boundary condition (7) to the form

$$Ax(0) + Bx(T) = \Phi(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = [A + B]w,$$

where $\Phi(u, v, \lambda_1, \lambda_2) := Au + Bv + g(u, v, \lambda_1, \lambda_2)$ and A, B are fixed square n-dimensional matrices such that $\det B \neq 0$.

The parameter w is natural to be determined from the determining equation

$$\Phi(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = [A + B]w$$

or, equivalently,

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = 0.$$

Thus, the essentially non-linear problem (1)–(3) turns out to be equivalent to

$$x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2), \qquad t \in [0, T], \tag{9}$$

$$Ax(0) + Bx(T) + g(x(0) + w, x(T) + w, \lambda_1, \lambda_2) = 0, \tag{10}$$

$$x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2. \tag{11}$$

On the other hand, system (9), (10), (11) can be regarded as a collection of problems

$$x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2), \qquad t \in [0, T], \tag{12}$$

$$Ax(0) + Bx(T) = 0, \tag{13}$$

$$x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2. \tag{14}$$

parametrised by the unknown vector w and considered together with the determining equation (10).

The essential advantage obtained thereby is that the boundary condition (13) is linear.

It follows from the consideration above that family (12)–(14) can be studied by using the numerical-analytic method developed in [5].

Assume that

$$D_\beta := \{x \in \mathbb{R}^n : B(x, \beta(x)) \subset D\} \neq \emptyset, \tag{15}$$

where

$$\beta(x) := \frac{T}{2}\delta_G(f) + |(B^{-1}A + E_n)x|$$

and

$$\delta_G(f) := \frac{1}{2}\Big[\max_{(t,x,\lambda_1,\lambda_2)\in[0,T]\times\Omega\times I_1\times I_2} f(t,x,\lambda_1,\lambda_2) - \min_{(t,x,\lambda_1,\lambda_2)\in[0,T]\times\Omega\times I_1\times I_2} f(t,x,\lambda_1,\lambda_2)\Big]. \tag{16}$$

Moreover, we suppose that K in (4) satisfies

$$r(K) < \frac{10}{3T}. \tag{17}$$

Set

$$D_1 := \big\{u \in \mathbb{R}^{n-2} : z \equiv \operatorname{col}(y_{10} - w_1, y_{20} - w_2, u_1, u_2, \ldots, u_{n-2}) \in D_\beta\big\}$$

and introduce the sequence of functions

$$\begin{aligned} x_{m+1}(t,w,u,\lambda_1,\lambda_2) := z &+ \int_0^t f\left(s, x_m(s,w,u,\lambda_1,\lambda_2) + w, \lambda_1, \lambda_2\right) ds \\ &- \frac{t}{T}\int_0^T f\left(s, x_m(s,w,u,\lambda_1,\lambda_2) + w, \lambda_1, \lambda_2\right) ds \\ &- \frac{t}{T}[B^{-1}A + E_n]z, \end{aligned} \tag{18}$$

where $m \geq 0$ and $x_0(t,w,u,\lambda_1,\lambda_2) \equiv z$.

Note that $x_m(0,w,u,\lambda_1,\lambda_2) = z$ for all m.

It can be verified that all the members of sequence (18) satisfy conditions (13) and (14) for arbitrary $u \in D_1$, $w \in \Omega$, and $\lambda_k \in I_k$ $(k = 1,2)$.

By virtue of (13), every solution, x, of (12)–(14) satisfies

$$x(T) = -B^{-1}Ax(0).$$

Therefore, Eq. (10) can be rewritten as

$$g(x(0) + w, -B^{-1}Ax(0) + w, \lambda_1, \lambda_2) = 0. \tag{19}$$

So, we conclude that problem (9)–(14) is equivalent to the following family of boundary value problems with linear conditions:

$$x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2), \qquad t \in [0,T], \tag{20}$$

$$Ax(0) + Bx(T) = 0, \tag{21}$$

$$x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2 \tag{22}$$

considered together with the determining equation (19).

We suggest to solve the latter system sequentially: first solve (20)–(22), and then try to find out whether (19) can simultaneously be fulfilled.

Theorem 1. *Assume conditions* (4), (15), *and* (17). *Then:*

1. *Sequence* (18) *converges to the function* $x^* = x^*(\cdot, w, u, \lambda_1, \lambda_2)$ *as* $m \to +\infty$ *uniformly in* $(w, u, \lambda_1, \lambda_2) \in \Omega \times D_1 \times I_1 \times I_2$.
2. *The limit function* $x^*(\cdot, w, u, \lambda_1, \lambda_2)$ *is the unique solution of the "perturbed" parametrised boundary value problem*

$$\begin{gathered} x'(t) = f(t, x(t) + w, \lambda_1, \lambda_2) + \Delta(w, u, \lambda_1, \lambda_2), \qquad t \in [0, T], \\ Ax(0) + Bx(T) = 0, \\ x_1(0) = y_{10} - w_1, \quad x_2(0) = x_{20} - w_2 \end{gathered} \tag{23}$$

having the initial value $x^*(0, w, u, \lambda_1, \lambda_2) = z$, *where*

$$\begin{aligned} \Delta(w, u, \lambda_1, \lambda_2) := & -\frac{1}{T}[B^{-1}A + E_n]z \\ & - \frac{1}{T}\int_0^T f\left(s, x^*(s, w, u, \lambda_1, \lambda_2) + w, \lambda_1, \lambda_2\right) ds. \end{aligned}$$

3. *The following error estimate holds:*

$$|x_m(t, w, u, \lambda_1, \lambda_2) - x^*(t, w, u, \lambda_1, \lambda_2)| \leq h(t, w, u, \lambda_1, \lambda_2), \tag{24}$$

where

$$\begin{aligned} h(t, w, u, \lambda_1, \lambda_2) := & \frac{20t}{9}\left(1 - \frac{t}{T}\right) Q^{m-1}\left(E_n - Q\right)^{-1}\Big[Q\delta_G(f) \\ & + K\left|\left(B^{-1}A + E_n\right) z\right|\Big], \end{aligned}$$

the vector $\delta_G(f)$ *is given by* (16), *and* $Q := \frac{3T}{10}K$.

Proof. It can be carried out similarly to that of Theorem 2.1 from [5, p. 34].

The following statement shows the relation of the function $x^*(\cdot, w, u, \lambda_1, \lambda_2)$ to the solution of problem (20)–(22).

Theorem 2. *Under the assumptions of Theorem 1, the function*

$$x^*(\cdot, w^*, u^*, \lambda_1^*, \lambda_2^*)$$

is a solution of the parametrised boundary value problem (20)–(22) *if, and only if the triplet* $\{u^*, \lambda_1^*, \lambda_2^*\}$ *satisfies the system of determining equations*

$$[B^{-1}A + E_n]z + \int_0^T f\left(s, x^*(s, w, u, \lambda_1, \lambda_2) + w, \lambda_1, \lambda_2\right) ds = 0,$$

where w *is considered as a parameter.*

Proof. Analogous to that of Theorem 2.3 from [5, p. 40].

Theorem 3. *Assume conditions* (4), (15), *and* (17). *Then, for the function*

$$y^* := x^*(\cdot, w^*, u^*, \lambda_1^*, \lambda_2^*) + w^* \tag{25}$$

to be a solution of the given parametrised problem (1)–(3), *it is necessary and sufficient that* $\{w^*, u^*, \lambda_1^*, \lambda_2^*\}$ *satisfy the system of determining equations*

$$\begin{gathered} z + \int_0^T f\left(s, x^*(s, w, u, \lambda_1, \lambda_2) + w, \lambda_1, \lambda_2\right) ds = 0, \\ g\left(z + w, -B^{-1}Az + w, \lambda_1, \lambda_2\right) = 0. \end{gathered} \tag{26}$$

Proof. It is easily seen from the form substitution (5) that Eqns. (26) hold whenever the transformed boundary value problem (23) is equivalent to the original problem (1)–(3).

Remark 1. Considering function (25), one can set

$$y_m := x_m(\cdot, w_m, u_m, \lambda_{1,m}, \lambda_{2,m}) + w_m \tag{27}$$

and regard (27) as the mth approximation to function (25), which solves the boundary value problem (1)–(3).

In Eq. (27), x_m is given by (18), whereas w_m, u_m, $\lambda_{1,m}$, and $\lambda_{2,m}$ are solutions of

$$\begin{gathered} z + \int_0^T f\left(s, x_m(s, w, u, \lambda_1, \lambda_2) + w, \lambda_1, \lambda_2\right) ds = 0, \\ g\left(z + w, -B^{-1}Az + w, \lambda_1, \lambda_2\right) = 0. \end{gathered} \tag{28}$$

We do not consider the strict substantiation of the above idea, referring to [5] where similar techniques are described.

Example 1. Let us consider the third order parametrised differential equation

$$y'''(t) + \frac{1}{2}(y''(t))^2 + \lambda_1 y(t) = \left(\lambda_2 + \frac{3}{4}\right)\frac{t^2}{16}, \qquad t \in [0,1] \tag{29}$$

with the following non-linear boundary conditions containing parameters:

$$\begin{gathered} y'(1)y'(0) + \lambda_1 y(1) = \frac{1}{32}, \\ y(1)y'(0) + \lambda_2 y'(0) + \lambda_2 y''(1) = \frac{1}{16}, \\ \frac{1}{2}y'(0) + \left(\frac{1}{2} - \lambda_1\right) y'(1) = 0, \\ y(0) = -\frac{1}{16}, \qquad y'(0) = 0. \end{gathered} \tag{30}$$

Equivalently, equation (29) can be rewritten as

$$
\begin{aligned}
y_1'(t) &= y_2(t),\\
y_2'(t) &= y_3(t),\\
y_3'(t) &= \frac{t^2}{16} - \frac{1}{2}y_3^2(t) - \lambda_1 y_1(t)
\end{aligned}
\tag{31}
$$

together with the boundary conditions

$$
\begin{aligned}
y_2(1)y_2(0) + \lambda_1 y_1(1) &= \frac{1}{32},\\
y_1(1)y_2(0) + \lambda_2 y_2(0) + \lambda_2 y_3(1) &= \frac{1}{16},\\
\frac{1}{2}y_2(0) + \left(\frac{1}{2} - \lambda_1\right) y_2(1) &= 0,\\
y_1(0) = -\frac{1}{16}, \qquad y_2(0) &= 0.
\end{aligned}
\tag{32}
$$

One can verify that, for problem (31), (32), conditions (4), (15), and (17) are fulfilled with $(t, y_2, y_2, \lambda_1, \lambda_2) \in [0,1]\times G\times I_1\times I_2$, $\lambda_1 \in I_1 := [0,1]$, $\lambda_2 \in I_2 := [0,1]$, $A := B := E_3 := \operatorname{diag}(1,1,1)$, $K := \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & \frac{1}{3}\end{bmatrix}$, and

$$
G := \left\{ (y_1, y_2, y_3) \quad : \quad |y_1| \le \frac{1}{2},\ |y_2| \le \frac{1}{2},\ |y_3| \le \frac{1}{3}, \right\},
$$

because, in this case, $r(K) = 0.9$,

$$
\delta_G(f) \le \begin{pmatrix} \frac{1}{2}\\ \frac{1}{3}\\ \frac{53}{144}\end{pmatrix},
$$

and

$$
\beta(x) = \frac{T}{2}\delta_G(f) + \left|(B^{-1}A + E_3)x\right| \le \begin{pmatrix} \frac{1}{4}\\ \frac{1}{6}\\ \frac{53}{288}\end{pmatrix} + 2|x|.
$$

Substitution (5) brings (31) to the form

$$
\begin{aligned}
x_1'(t) &= x_2(t) + w_2,\\
x_2'(t) &= x_3(t) + w_3,\\
x_3'(t) &= \frac{t^2}{16} - \frac{1}{2}(x_3 + w_3)^2(t) - \lambda_1\left(x_1(t) + w_1\right),\\
x_1(0) &= -\frac{1}{16} - w_1, \qquad x_2(0) = -w_2.
\end{aligned}
$$

The computation performed according to (18) shows that the components of the first iteration have the form

$$x_{1,1}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = -\frac{1}{16} - w_1 + \frac{1}{8}\,t + 2\,t\,w_1,$$
$$x_{1,2}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = -w_2 + 2\,t\,w_2,$$

and

$$x_{1,3}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = u + \frac{1}{48}t^3\lambda_2 - \frac{1}{48}t\lambda_2 + \frac{1}{64}t^3 - \frac{1}{64}t - 2\,u\,t,$$

where $x_m = \operatorname{col}(x_{m,1}, x_{m,2}, x_{m,3})$.

Similarly, for the second iteration, we have the first

$$x_{2,1}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = -\frac{1}{16} - w_1 + w_2\,t^2 - t\,w_2 + \frac{1}{8}\,t + 2\,t\,w_1,$$

the second

$$\begin{aligned} x_{2,2}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) = -w_2 + \frac{1}{192}\,t^4\,\lambda_2 + \frac{1}{256}\,t^4 - t^2\,u - \frac{1}{96}\,t^2\,\lambda_2 \\ - \frac{1}{128}\,t^2 + u\,t + 2\,t\,w_2, \end{aligned}$$

and the third

$$\begin{aligned} x_{2,3}(t,\, w_1,\, w_2,\, w_3,\, u,\, \lambda_1,\, \lambda_2) := -\frac{1}{256}\,w_3\,t - \frac{1679}{107520}\,t + u + t\,\lambda_1\,w_1 \\ - \frac{1}{192}\,t\,w_3\,\lambda_2 + \frac{1}{2880}\,t\,u\,\lambda_2 - t\,w_3\,u + \frac{1}{96}\,t^2\,w_3\,\lambda_2 + t^2\,w_3\,u \\ + \frac{1}{96}\,t^2\,u\,\lambda_2 - t^2\,\lambda_1\,w_1 - \frac{1}{72}\,t^3\,u\,\lambda_2 - \frac{1}{192}\,t^4\,w_3\,\lambda_2 \\ - \frac{1}{192}\,t^4\,u\,\lambda_2 + \frac{1}{60480}\,t\,{\lambda_2}^2 + \frac{1}{120}\,t^5\,u\,\lambda_2 - \frac{1}{32256}\,t^7\,{\lambda_2}^2 \\ - \frac{1}{21504}\,t^7\,\lambda_2 - \frac{1}{256}\,t^4\,u - \frac{1}{256}\,t^4\,w_3 + t^2\,u^2 + \frac{1}{128}\,t^2\,w_3 \\ - \frac{1}{3}\,t\,u^2 + \frac{1}{16}\,\lambda_1\,t - \frac{7679}{3840}\,u\,t - \frac{839}{40320}\,t\,\lambda_2 + \frac{191}{9216}\,t^3\,\lambda_2 \\ + \frac{383}{24576}\,t^3 + \frac{1}{128}\,t^2\,u + \frac{1}{20480}\,t^5 - \frac{1}{57344}\,t^7 + \frac{1}{160}\,t^5\,u \\ + \frac{1}{11520}\,t^5\,{\lambda_2}^2 + \frac{1}{7680}\,t^5\,\lambda_2 - \frac{1}{13824}\,t^3\,{\lambda_2}^2 - \frac{2}{3}\,t^3\,u^2 - \frac{1}{96}\,t^3\,u \\ - \frac{1}{16}\,t^2\,\lambda_1 \end{aligned}$$

components of the function x_2.

Solving the approximate determining equations (28) gives us the approximate values of the unknown parameters. More precisely, we have

$$w_1 = 0,\ w_2 \approx .1250000000,\ w_3 \approx .2552083572$$
$$\lambda_1 = \frac{1}{2},\ \lambda_2 \approx .2500045836,\ u = \frac{-1 + 16\,w_3\,\lambda_2}{16\lambda_2} \approx .005212940674$$

for $m = 1$ and

$$w_1 = 0,\ w_2 \approx .127331555,\ w_3 \approx .2547074002$$
$$\lambda_1 = \frac{1}{2},\ \lambda_2 \approx .2458952578,\ u = \frac{-1 + 16\, w_3\, \lambda_2}{16\lambda_2} \approx .2458952578$$

for $m = 2$.

Therefore, in the first approximation, the solution of parametrised problem (29), (30) is

$$\begin{gathered} y_{1,1}(t) = -\frac{1}{16} + \frac{1}{8}\, t, \quad t \in [0, 1], \\ \lambda_1 = \frac{1}{2},\ \lambda_2 \approx .2500045836 \end{gathered} \tag{33}$$

and, in the second approximation,

$$\begin{gathered} y_{2,1}(t) \approx -\frac{1}{16} + .1273315558\, t^2 - .0023315558\, t, \quad t \in [0, 1], \\ \lambda_1 = \frac{1}{2},\ \lambda_2 \approx .2458952578. \end{gathered} \tag{34}$$

Note that

$$\begin{gathered} y(t) = \frac{t^2}{8} - \frac{1}{16}, \quad t \in [0, 1], \\ \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{4} \end{gathered} \tag{35}$$

is an exact solution of problem (29), (30). Computation by using `Maple` shows that (33) and (34) provide good enough approximations to (35).

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