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A NOTE ON THE PERIODICITY IN DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions are obtained for the existence of a unique periodic solution of a linear first order difference equation in a Banach space.

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Let $\langle X, \|\cdot\|\rangle$ be a Banach space, $\ell^\infty(X)$ be the Banach space of bounded sequences $x = (x_k)_{k=0}^{+\infty} \subset X$ with the norm $\|x\|_\infty := \sup_{k\geq 0} \|x_k\|$, and $\ell^1(X)$ be the Banach space of summable sequences $x = (x_k)_{k=0}^{+\infty} \subset X$ with the norm $\|x\|_1 := \sum_{k\geq 0} \|x_k\|$.

Consider the linear difference equation

$$x(n+1) - x(n) = (Lx)(n) + f(n), \qquad n = 0, 1, 2, \dots \tag{1}$$

together with the N-periodic ($N \geq 1$) condition

$$x(n+N) = x(n), \qquad n = 0, 1, 2, \dots . \tag{2}$$

In (1), $f \in \ell^1(X)$, and $L : \ell^\infty(X) \to \ell^1(X)$ is a linear continuous operator. Here and below, L is assumed to leave invariant the subspace of sequences having property (2), and f is supposed to satisfy (2).

Remark 1. The use of special sequence spaces when posing problem (1), (2), in fact, can be avoided by restricting the consideration to problem (3), (4) or equation (6); see below. We have began with such a problem setting in order to note at this point that results similar to those to follow can be obtained for problems other than the periodic one.

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The invariance condition above implies that there is a one-to-one correspondence between solutions of (1), (2) and those of the problem

$$x(n+1) - x(n) = \sum_{\nu=0}^{N-1} L_{n,\nu} x(\nu) + f(n), \qquad 0 \le n \le N-1, \tag{3}$$

$$x(N) = x(0), \tag{4}$$

where $(L_{n,\nu})_{n,\nu=0}^{N} \subset \mathcal{B}(X)$ are certain linear operators such that

$$L_{N,\nu} = L_{0,\nu} \text{ for all } \nu \in \{0, 1, \ldots, N-1\}. \tag{5}$$

Here and below, the symbol $\mathcal{B}(X)$ stands for the algebra of all bounded linear operators in X.

Due to property (5), knowing solutions of problem (3), (4), one can reconstruct those of (1), (2) by extending them periodically to all the non-negative integers. Furthermore, the periodic nature of problem (3), (4) allows one to consider it as a single linear equation with operator "matrices" acting in the space X^N of "vectors" $(x(0), x(1), \ldots, x(N-1))$:

$$(\mathit{\Delta} x)(n) = \sum_{\nu=0}^{N-1} L_{n,\nu} x(\nu) + f(n), \qquad 0 \le n \le N-1, \tag{6}$$

where

$$(\mathit{\Delta} x)(n) := \begin{cases} x(n+1) - x(n) & \text{for } 0 \le n < N-1, \\ x(0) - x(n) & \text{for } n = N-1. \end{cases} \tag{7}$$

The latter circumstance will be essentially used below; we shall even identify L with the appropriate mapping $X^N \to X^N$:

$$(Lx)(n) = \sum_{\nu=0}^{N-1} L_{n,\nu} x(\nu), \qquad 0 \le n < N.$$

Lemma 1. *Assume that the operator $\mathit{\Lambda}_{L,N} : X \to X$ defined with the formula*

$$\mathit{\Lambda}_{L,N} := \sum_{n=0}^{N-1} \sum_{\nu=0}^{N-1} L_{n,\nu} \tag{8}$$

is invertible. Then $x = (x(0), x(1), \ldots, x(N-1))$ is a solution of equation (6) *if, and only if there exists some $a \in X$ such that the equalities*

$$x(n) = (H_{L,N,l} Lx)(n) + f_{L,N,l}(n) + a, \quad 0 \le n \le N-1, \tag{9}$$

$$\sum_{n=0}^{N-1} \left[\sum_{\nu=0}^{N-1} L_{n,\nu} x(\nu) + f(n) \right] = 0 \tag{10}$$

hold with some $l \in \{0, 1, \dots, N-1\}$, *where the linear mapping* $H_{L,N,l} : X^N \to X^N$ *is defined with the formula*

$$(H_{L,N,l}x)(n) := \begin{cases} \sum_{k=l}^{N-1} \Big[x(k) - \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{j=0}^{N-1} x(j) \Big] & \text{for } n = 0, \\ \sum_{k=l}^{n-1} \Big[x(k) - \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{j=0}^{N-1} x(j) \Big] & \text{for } 0 < n < N, \end{cases} \tag{11}$$

and

$$f_{L,N,l} := H_{L,N,l} f. \tag{12}$$

Proof. Assume that $x = (x(0), x(1), \dots, x(N-1))$ satisfies (9) and (10). Then, for $1 \le n < N-1$, we have

$$\begin{aligned} x(n) &= a + f_{L,N,l}(n) + \sum_{k=l}^{n-1} \sum_{\nu=0}^{N-1} L_{k,\nu} x(\nu) \\ &\quad - \sum_{k=l}^{n-1} \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} \sum_{\nu=0}^{N-1} L_{k,\nu} x(\nu) \qquad (13) \\ &= a + f_{L,N,l}(n) + \sum_{k=l}^{n-1} \sum_{\nu=0}^{N-1} L_{k,\nu} x(\nu) + \sum_{k=l}^{n-1} \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k), \qquad (14) \end{aligned}$$

whence

$$\begin{aligned} x(n+1) - x(n) = \sum_{\nu=0}^{N-1} L_{n,\nu} x(\nu) + \sum_{\nu=0}^{N-1} L_{n,\nu} \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k) \\ + f_{L,N,l}(n+1) - f_{L,N,l}(n). \end{aligned} \tag{15}$$

It is easy to see from definition (11) that, when $1 \le n \le N-1$, (12) is equivalent to the relation

$$f_{L,N,l}(n) = \sum_{k=l}^{n-1} \Big[f(k) - \sum_{\nu=0}^{N-1} L_{k,\nu} \Lambda_{L,N}^{-1} \sum_{j=0}^{N-1} f(j) \Big], \tag{16}$$

whence

$$f_{L,N,l}(n+1) - f_{L,N,l}(n) = f(n) - \sum_{\nu=0}^{N-1} L_{n,\nu} \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k) \tag{17}$$

for $0 < n < N$. Combining (15) and (17), we show that (6) holds for $1 \le n < N-1$. The case $n = 0$ is considered analogously.

Let us now suppose that equality (6) holds. Then, we have

$$\sum_{n=0}^{N-2}\left(\sum_{\nu=0}^{N-1} L_{n,\nu}x(\nu)+f(n)\right)=\sum_{n=0}^{N-2}[x(n+1)-x(n)]=x(N-1)-x(0). \tag{18}$$

According to definition (7), equation (6) for $n=N-1$ means that

$$\sum_{\nu=0}^{N-1} L_{N-1,\nu}x(\nu)+f(N-1)=x(0)-x(N-1),$$

which, combined with (18), implies (10).

Furthermore, in view of (11) and (10), for $n\in\{1,2,\ldots,N-1\}$, we have

$$\begin{aligned}(H_{L,N,l}Lx)(n)&=\sum_{k=l}^{n-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu}x(\nu)-\sum_{\mu=0}^{N-1} L_{k,\mu}\Lambda_{L,N}^{-1}\sum_{j=0}^{N-1}\sum_{\nu=0}^{N-1} L_{j,\nu}x(\nu)\right]\\ &=\sum_{k=l}^{n-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu}x(\nu)+\sum_{\mu=0}^{N-1} L_{k,\mu}\Lambda_{L,N}^{-1}\sum_{j=0}^{N-1} f(j)\right].\end{aligned} \tag{19}$$

Carrying out the manipulations marked as (13), (14), and (15) in the reverse order and taking into account (19), we find that equality (9) holds for $0<n\leq N-1$. When $n=0$, in view of (11), identity (19) is replaced by the relation

$$(H_{L,N,l}Lx)(0)=\sum_{k=l}^{N-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu}x(\nu)+\sum_{\mu=0}^{N-1} L_{k,\mu}\Lambda_{L,N}^{-1}\sum_{j=0}^{N-1} f(j)\right],$$

and a similar argument leads one to (9) in this case as well. □

Remark 2. Lemma 1 is similar to some statements from [3], [4], and [5].

Lemma 2. *The identity*

$$(H_{L,N,l}Lx)(n)=\Omega_{L,N,l}\begin{pmatrix}x(0)\\ x(1)\\ \vdots\\ x(N-1)\end{pmatrix} \tag{20}$$

holds for $0 \le n < N$, where $\Omega_{L,N,l} : X^N \to X^N$ is given by the matrix

$$
\Omega_{L,N,l} = \begin{bmatrix}
\sum_{k=l}^{N-1}\Big(L_{k,0} - L_k^{\#}\sum_{j=0}^{N-1} L_{j,0}\Big) & \dots & \sum_{k=l}^{N-1}\Big(L_{k,N-1} - L_k^{\#}\sum_{j=0}^{N-1} L_{j,N-1}\Big) \\
\sum_{k=l}^{0}\Big(L_{k,0} - L_k^{\#}\sum_{j=0}^{N-1} L_{j,0}\Big) & \dots & \sum_{k=l}^{0}\Big(L_{k,N-1} - L_k^{\#}\sum_{j=0}^{N-1} L_{j,N-1}\Big) \\
\sum_{k=l}^{1}\Big(L_{k,0} - L_k^{\#}\sum_{j=0}^{N-1} L_{j,0}\Big) & \dots & \sum_{k=l}^{1}\Big(L_{k,N-1} - L_k^{\#}\sum_{j=0}^{N-1} L_{j,N-1}\Big) \\
\cdots & \cdots & \cdots \\
\sum_{k=l}^{N-2}\Big(L_{k,0} - L_k^{\#}\sum_{j=0}^{N-1} L_{j,0}\Big) & \dots & \sum_{k=l}^{N-2}\Big(L_{k,N-1} - L_k^{\#}\sum_{j=0}^{N-1} L_{j,N-1}\Big)
\end{bmatrix} \tag{21}
$$

and

$$
L_k^{\#} := \sum_{\nu=0}^{N-1} L_{k,\nu}\Lambda_{L,N}^{-1}, \quad 0 \le k \le N-1. \tag{22}
$$

Proof. Considering (11), it is not difficult to verify by computation that, for $1 \le n \le N-1$,

$$
(H_{L,N,l}Lx)(n) = \sum_{\nu=0}^{N-1}\left[\sum_{k=l}^{n-1} L_{k,\nu} - \sum_{j=0}^{N-1}\sum_{k=l}^{n-1} L_k^{\#} L_{j,\nu}\right] x(\nu), \tag{23}
$$

where $L_k^{\#}$ $(0 \le k \le N-1)$ are the linear operators given by (22) and (8). This, together with a similar observation for $n = 0$, leads one to formula (21) for the operator "matrix" $\Omega_{L,N,l}$ in equality (20). □

Introduce the notation

$$
\operatorname{diag} X^N := \big\{(\underbrace{a, a, \dots, a}_{N}) \;:\; a \in X\big\}. \tag{24}
$$

Lemma 3. $\operatorname{diag} X^N \subset \ker H_{L,N,l}L$.

Proof. According to equality (23) established in the proof of Lemma 2, we have

$$
\begin{aligned}
(H_{L,N,l}La)(n) &= \sum_{\nu=0}^{N-1}\left[\sum_{k=l}^{n-1} L_{k,\nu} - \sum_{j=0}^{N-1}\sum_{k=l}^{n-1} L_k^{\#} L_{j,\nu}\right] a \\
&= \sum_{k=l}^{n-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu} - L_k^{\#}\sum_{j=0}^{N-1}\sum_{\nu=0}^{N-1} L_{j,\nu}\right] a,
\end{aligned}
$$

whence, by definitions (8) and (22),

$$\left(H_{L,N,l}La\right)(n) = \sum_{k=l}^{n-1}\left[\sum_{\nu=0}^{N-1} L_{k,\nu} - L_k^{\#}\Lambda_{L,N}\right]a = 0$$

for all $a \in X$ and $n \in \{1, 2, \ldots, N-1\}$.

The remaining case when $n = 0$ is considered in a similar way. □

Let us now put $\rho_L(N) := r\left(\Omega_{L,N,l}\right)$, the spectral radius of the linear operator $\Omega_{L,N,l} : X^N \to X^N$ defined with formula (21). The notation is justified by the following

Lemma 4. *$\rho_L(N)$ is independent of l.*

Proof. Let us first prove the following CLAIM: *If $A : X^N \to X^N$ and $B : X^N \to X^N$ are bounded linear mappings such that $\sigma(B) \subset \sigma(A)$ and $\operatorname{im} B \subset \ker A$, then $\sigma(A + B) = \sigma(A)$.*

Indeed, let $\lambda \notin \sigma(A)$ be a regular point for A. Then the equation

$$Ax - \lambda x = y - \phi$$

has the unique solution $x(y - \phi, \lambda) := -\lambda^{-1}[y - \phi + \lambda^{-1}A(y - \phi) + \ldots]$ for all y and ϕ. Consider the equation

$$\phi = Bx(y - \phi, \lambda), \tag{25}$$

or, which is the same,

$$\phi = \lambda^{-1}B\sum_{\nu=0}^{+\infty}\lambda^{-\nu}A^{\nu}(\phi - y).$$

Since, obviously, we are seeking for a ϕ in $\operatorname{im} B$, the assumption that $\operatorname{im} B \subset \ker A$ yields $\sum_{\nu=0}^{+\infty}\lambda^{-\nu}A^{\nu}\phi = \phi$ and, therefore, equation (25) rewrites as

$$B\phi - \lambda\phi = B\sum_{\nu=0}^{+\infty}\lambda^{-\nu}A^{\nu}y. \tag{26}$$

Since $\lambda \not\ni \sigma(A) \supset \sigma(B)$, we see that (26), and hence (25), has a unique solution, say $\phi(y, \lambda)$. Thus, for every y, the equation

$$Ax - \lambda x = y - \phi(y, \lambda) \tag{27}$$

has a unique solution and, moreover, by virtue of the form of equation (25), the solution $\Xi(y, \lambda) := x\left(y - \phi(y, \lambda), \lambda\right)$ of (27) also satisfies the equation

$$Ax - \lambda x = y - Bx. \tag{28}$$

Let us prove that (28) cannot have any other solutions. Indeed, in the contrary case, when (28) has another solution, say z, the difference $\delta := \Xi(y,\lambda) - z$ satisfies the equality

$$A\delta - \lambda\delta = -B\delta. \tag{29}$$

Since, by assumption, $\operatorname{im} B$ is contained in $\ker A$, relation (29) implies that $A^2\delta = \lambda A\delta$. Therefore, $A\delta = 0$, because otherwise $A\delta$ would be an eigen-vector of A with the eigen-value λ, which has been assumed to be regular for A. The same equality (29) then yields $B\delta = \lambda\delta$, which can be the case only when $\delta = 0$, because $\lambda \notin \sigma(B)$. Hence, z and $\Xi(y,\lambda)$ coincide.

The argument above shows that, for $\lambda \notin \sigma(A)$ and arbitrary y, equation (28) has a unique solution, whose continuous dependence upon y is obvious. Therefore, $\sigma(A) \supset \sigma(A+B)$.

Conversely, if $\lambda \notin \sigma(A+B)$, then there exists a bounded inverse operator $(A+B-\lambda I)^{-1}$, where I stands for the unity in $\mathcal{B}(X)$. Since, by assumption, $AB = 0$, we have

$$(A-\lambda I)(B-\lambda I) = -\lambda[A+B-\lambda I], \tag{30}$$

an invertible operator. Assume that $B - \lambda I$ is non-invertible. Then, according to a well-known criterion (see, e. g., Theorem 2 in [1, p. 209]), there is some sequence $(u_k)_{k=1}^{+\infty}$ such that $\|u_k\| = 1$ and $\|Bu_k - \lambda u_k\| \le \frac{1}{k}$ for all $k \ge 1$. On the other hand, since operator (30) is invertible, the same reasoning shows the existence of a constant $c \in (0,+\infty)$ such that $\|(A-\lambda I)(B-\lambda I)x\| \ge c\|x\|$ for all x. Combining these two statements, we obtain that, for all $k \ge 1$,

$$c \le \|(A-\lambda I)(B-\lambda I)u_k\| \le \|A-\lambda I\| \cdot \|Bu_k - \lambda u_k\| \le \frac{\|A-\lambda I\|}{k},$$

which is impossible. Therefore, $B - \lambda I$ is invertible and, by (30), so does $A - \lambda I$, i. e., $\lambda \notin \sigma(A)$. Hence, $\sigma(A+B) \supset \sigma(A)$, and the proof of the CLAIM is complete.

Returning to our lemma, one can readily check that matrix (21) corresponding to operator (11) has the property

$$[\Omega_{L,N,l_1}x - \Omega_{L,N,l_2}x](n) = \sum_{k=l_1}^{l_2}\sum_{\nu=0}^{N-1}\left[L_{k,\nu} - L_k^{\#}\sum_{j=0}^{N-1}L_{j,\nu}\right]x(\nu)$$

for all $n \in \{0,1,\dots,N-1\}$. It is then easy to verify that $\sigma\left(\Omega_{L,N,l_1} - \Omega_{L,N,l_2}\right) = \sigma(\beta)$, where $\beta := \sum_{k=l_1}^{l_2}\sum_{\nu=0}^{N-1}[L_{k,\nu} - L_k^{\#}\sum_{j=0}^{N-1}L_{j,\nu}]$. Recalling notations (8) and (22), we see that, in fact, $\beta = 0$.

Finally, putting $A := \Omega_{L,N,l_1}$ and $B := \Omega_{L,N,l_2} - \Omega_{L,N,l_1}$ in the CLAIM above, we obtain that $\sigma(\Omega_{L,N,l_1}) = \sigma(\Omega_{L,N,l_2})$ for all l_1 and l_2 in $\{0,1,\dots,N-1\}$. □

Lemma 5. *$\rho_L(N) = r(Q_{L,N})$, where $Q_{L,N} : X^{N-1} \to X^{N-1}$ is given by*

$$\left(Q_{L,N}x\right)(n) := \sum_{k=0}^{n-1}\sum_{\nu=0}^{N-1}\left(L_{k,\nu} - L_k^{\#}\sum_{j=0}^{N-1}L_{j,\nu}\right)x(\nu), \qquad 1 \le n \le N-1. \tag{31}$$

Proof. By virtue of Lemma 4, we can put $l = 0$ in (11), in which case, as is easy to see, the first row of matrix (21) is filled with zeroes. Thus, $\Omega_{L,N,0} = \left[\begin{smallmatrix} 0 & 0 \\ M & Q_{L,N} \end{smallmatrix}\right]$ with a certain M and, obviously, $r(\Omega_{L,N,0}) = r(Q_{L,N})$. □

Now we can apply the above lemmata to obtain the following theorem.

Theorem 1. *Assume that operator* (8) *is invertible and, moreover,* $\rho_L(N) < 1$. *Then equation* (6) *has a unique solution for every* $f : \{0, 1, \ldots, N-1\} \to X$.

Proof. By Lemma 1, every solution of (6), if there are any, satisfies relations (9) and (10) for some $a \in X$ and, conversely, a solution of (9) is also that of (6) whenever a is such that (10) holds. Let us fix some $a \in X$ and consider the corresponding equation (9).

Introduce the sequence

$$y_{m+1}(n) = a + f_{L,N,l}(n) + (H_{L,N,l}Ly_m)(n), \quad 0 \le n < N,\ m \ge 0,$$

where $f_{L,N,l} : \{0, 1, \ldots, N-1\} \to X$ is defined by (12) and the starting member is arbitrary. We have:

$$\begin{aligned} y_{m+1} &= a + f_{L,N,l} + H_{L,N,l}Ly_m \\ &= a + f_{L,N,l} + H_{L,N,l}L\left[a + f_{L,N,l} + H_{L,N,l}Ly_{m-1}\right], \end{aligned}$$

which, by Lemma 3, yields

$$y_{m+1} = a + f_{L,N,l} + H_{L,N,l}Lf_{L,N,l} + (H_{L,N,l}L)^2\, y_{m-1}.$$

Proceeding similarly, we arrive at the equality

$$y_{m+1} = a + \sum_{\nu=0}^{m} (H_{L,N,l}L)^{\nu} f_{L,N,l} + (H_{L,N,l}L)^{m+1} y_0.$$

It follows immediately from Lemma 2 that $r(H_{L,N,l}L) = \rho_L(N)$ and, therefore, our assumption implies the convergence of the series $\sum_{\nu=0}^{+\infty} (H_{L,N,l}L)^{\nu} f_{L,N,l}$, which means that equation (9) has a unique solution for every $a \in X$.

Furthermore, according to Lemma 1, a certain $x : \{0, 1, \ldots, N-1\} \to X$ is a solution of equation (6) if, and only if

$$x = a + \sum_{\nu=0}^{+\infty} (H_{L,N,l}L)^{\nu} f_{L,N,l} \tag{32}$$

with some $a \in X$ such that (10) holds. However, it is easy to see that, for x given by (32), relation (10) is equivalent to the equality

$$a = -\Lambda_{L,N}^{-1} \sum_{n=0}^{N-1} \left(f(n) + \left[L \sum_{\nu=0}^{+\infty} (H_{L,N,l}L)^{\nu} f_{L,N,l} \right](n) \right). \tag{33}$$

Inserting (33) into (32) and expanding notation (12), we obtain the unique solution of equation (6) in the form of the series

$$x = \sum_{\nu=0}^{+\infty} \Big[(H_{L,N,l}L)^{\nu} H_{L,N,l}f - \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} [L (H_{L,N,l}L)^{\nu} H_{L,N,l}f](k) \Big] - \Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k), \tag{34}$$

and the proof of the theorem is thus complete. □

Remark 3. Theorem 1 is in the spirit of Corollary 5.2 from [2] and Corollary 4.2.1 from [6] established for linear systems of ordinary differential equations.

Let us say that some problem *does not possess uniqueness property* if it either has no solutions or has more than one solution.

Corollary 1. *Assume that* $\{L_{k,\nu}\}_{k,\nu=0}^{N-1} \subset \mathcal{B}(X)$ *are some linear operators such that the corresponding mapping* (8) *is invertible. Then, for the boundary value problem*

$$x(n+1) - x(n) = \lambda \sum_{\nu=0}^{N-1} L_{n,\nu} x(\nu) + f(n), \qquad 0 \le n \le N-1, \tag{35}$$

$$x(N) = x(0) \tag{36}$$

not to possess the uniqueness property for some $f : \{0, 1, 2, \ldots, N-1\} \to X$, *it is necessary that the parameter* $\lambda \in (-\infty, +\infty)$ *satisfy the inequality*

$$|\lambda| \ge 1/\rho_L(N).$$

Proof. It suffices to replace system (35), (36) by an equation of type (6) and apply Theorem 1. □

Corollary 2. *Assume that the operators* $\{L_{k,\nu}\}_{k,\nu=0}^{N-1} \subset \mathcal{B}(X)$ *satisfy the condition*

$$\sum_{\nu=0}^{N-1} L_{n,\nu} = A \quad \textit{for all } n \in \{0, 1, \ldots, N-1\} \tag{37}$$

with some invertible $A \in \mathcal{B}(X)$ *and, moreover, the spectral radius of the operator*

$$\begin{bmatrix} L_{1,1} - \dfrac{1}{N}\displaystyle\sum_{j=0}^{N-1} L_{j,1} & \ldots & L_{1,N-1} - \dfrac{1}{N}\displaystyle\sum_{j=0}^{N-1} L_{j,N-1} \\ \displaystyle\sum_{k=0}^{1} L_{k,1} - \dfrac{2}{N}\displaystyle\sum_{j=0}^{N-1} L_{j,1} & \ldots & \displaystyle\sum_{k=0}^{1} L_{k,N-1} - \dfrac{2}{N}\displaystyle\sum_{j=0}^{N-1} L_{j,N-1} \\ \cdots & \cdots & \cdots \\ \displaystyle\sum_{k=0}^{N-2} L_{k,1} - \dfrac{N-1}{N}\displaystyle\sum_{j=0}^{N-1} L_{j,1} & \ldots & \displaystyle\sum_{k=0}^{N-2} L_{k,N-1} - \dfrac{N-1}{N}\displaystyle\sum_{j=0}^{N-1} L_{j,N-1} \end{bmatrix} \tag{38}$$

is less than one. Then, for every $f \in \operatorname{diag} X^{N+1}$, *problem* (3), (4) *has a unique solution, and this solution belongs to* $\operatorname{diag} X^{N+1}$*:*

$$x(n) = -A^{-1}f \quad \textit{for all } n \in \{0, 1, 2, \dots, N\}.$$

Proof. As before, instead of (3), (4), we consider equation (6).

Taking into account notations (22) and (8), it is not difficult to verify that, under assumption (37), $\Lambda_{L,N} = N \cdot A$ and $L_k^{\#} = \frac{1}{N} I$ $(0 \le k \le N-1)$, whence we see that the operator defined by matrix (38) is nothing but $Q_{L,N}$ given by (31). Theorem 1, together with Lemma 5, then guarantees the unique solvability of equation (6), whose solution can be represented as series (34).

By Lemma 3, the relation $f \in \operatorname{diag} X^N$ yields $H_{L,N,l} f = 0$, whence, considering (34), we conclude that the solution of (6) is equal identically to $-\Lambda_{L,N}^{-1} \sum_{k=0}^{N-1} f(k)$. Returning to problem (3), (4), we obtain the conclusion desired. □

Remark 4. The condition imposed on $\rho_L(N)$ in Theorem 1, generally speaking, cannot be weakened. Indeed, consider the simplest scalar difference equation

$$x(n+1) = -x(n) \quad (n \ge 0). \tag{39}$$

The 2-periodic boundary value problem for equation (39) can be interpreted as (6) with $N = 2$, $f(0) = f(1) = 0$, $L_{0,1} = L_{1,0} = 0$, and $L_{0,0} = L_{1,1} = -2$. It is obvious that, in this case, $\Omega_{L,N,0} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ and, thus, $\rho_L(2) = 1$. On the ther hand, every non-trivial solution of (39) is periodic with period 2. Hence, the corresponding inhomogeneous problem does not have uniqueness property and, therefore, the inequality $\rho_L(2) < 1$ in Theorem 1 [resp., $|\lambda| \ge \rho_L(2)$ in Corollary 1] cannot be replaced by $\rho_L(2) \le 1$ [resp., $|\lambda| > \rho_L(2)$].

One can also construct similar examples for an arbitrary period $N \ge 2$ (this is not done here).

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References

1. Kantorovich L. V. and Akilov G. P., *Functional Analysis* (in Russian), 2nd rev. ed., Nauka, Moscow, 1977.
2. Kiguradze I. T., *Initial and Boundary Value Problems for Systems of Ordinary Differential Equations. I* (in Russian), Metsniereba, Tbilisi, 1997.
3. Ronto A., *On boundary value problems with linear multipoint restrictions*, Publ. Univ. Miskolc, Ser. D., Natur. Sc. Math., **36**, No. 1 (1995), 80-89.
4. Ronto A. and Samoilenko A., *Conditionally additive operators and non-linear differential equations* (submitted).
5. Ronto A., Rontó M., Samoilenko A., and Trofimchuk S., *On periodic solutions of autonomous difference equations* (submitted).
6. Samoilenko A. M., Kenzhebaev K., and Laptinskii V. N., *Constructive Methods of Investigating Periodic and Multipoint Boundary Value Problems* (in Russian), Institute of Mathematics, Kiev, 1999.