ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 563–573, CDDE 2000 issue

STABILITY ZONES FOR DISCRETE TIME HAMILTONIAN SYSTEMS

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Abstract. The discrete version of the Hamiltonian system

$$\dot{x} = \lambda J H(t) x$$

with $H(t) = H^*(t) = H(t+T)$ is considered. Following the line of M.G. Krein the stability zones with respect to the parameter λ are considered: the side zones have to be estimated from multiplier traffic rules while the central stability zone from the discrete version of the skew - periodic boundary value problem.

AMS Subject Classification. 39A10,39A11,39A12

Keywords. Discrete Hamiltonians, strong stability, λ-zones.

1. Introduction and motivation

The object of this paper is the stability analysis of the discrete version of the linear periodic Hamiltonian system:

$$\dot{x} = \lambda J H(t) x \tag{1}$$

where $H(t) = H^*(t) = H(t+T), T > 0$; $H(t)$ has complex entries and is Hermitian. Also J is defined by

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{2}$$

and λ is, generally speaking, a complex parameter. System (1) is a generalization encompassing a lot of by now classical systems that go back to Sturm, Liapunov and Žukovskii. M.G. Krein [1] give a strong generalization of many classical results and opening new fields of research issued from the interaction of several apparently independent mathematical domains. The long line of research opened by Krein is summarized in the monograph of Yakubovich and Staržinskii [2]. As pointed out by Krein and Yakubovich [3] various problems in contemporary engineering and physics (e.g. dynamic stability of structures, parametric resonance in high-capacity electrical transmission lines, motion of particles in accelerators) lead to the investigation of Hamiltonian systems with periodic coefficients.

Another field of origin for periodic Hamiltonian systems is calculus of variations and optimal control. Here a long list of papers may be mentioned but we mention here only the papers of Yakubovich [4] where linear periodic Hamiltonians are considered in the context of linear optimal feedback (minimizing a quadratic integral performance index) and quadratic Liapunov functions.

A crucial difference between these two directions of research exists. The first one, developed mainly by Krein is concerned with stable Hamiltonian systems whose multipliers are located on the unit circle. On the contrary linear quadratic control requires a dichotomic i.e. totally unstable Hamiltonian system whose multipliers *are not* on the unit circle. This last property is robust (i.e. it is preserved against structural perturbations) while the first one is not robust (generally speaking). The search for robustness of stable Hamiltonian systems led Krein to the introduction of strong stability, to the discovery of "traffic rules" on the unit circle for the multipliers, and to new results about the λ zones of stability. Since the central zone is estimated using the eigenvalues of a certain self adjoint boundary value problem, the research on stability met the old Sturm-Liouville framework which also generates problems for Hamiltonian systems. A good reference on these problems together with variational calculus and optimal control is the book of Kratz [5].

In the last few years a new field of research emerged - discrete time Hamiltonian systems. A basic reference is the book of Ahlbrandt and Peterson [6]. We shall mention here some papers [7], [8], [9], from the long list belonging to Bohner and Došlý. Their topics are oscillation, disconjugacy and transformation of Hamiltonian systems, both continuous and discrete time. The study of discrete-time Hamiltonian systems in connection with linear - quadratic optimal control may be found in the paper of Halanay and Ionescu [10]. Applications of dichotomic periodic linear Hamiltonian systems (i.e. totally unstable), both continuous and discrete-time to forced nonlinear oscillations are to be found in [11].

This paper is concerned with strong stability (in the sense of Krein) of discrete-time Hamiltonian systems. Such systems may arise from sampling (1). *Since stability is, generally speaking, not preserved by sampling* (not always) this problem is of interest. On the other hand, not all results of the continuous time fields may migrate, *mutatis-mutandis*, to the discrete-time field even in the conditions of the new emerging theory on time scales [12], [13],[14]; this will become clear throughout the paper. Let us consider system (1) with $H(t)$ as follows

$$H(t) = \begin{pmatrix} A(t) & B^*(t) \\ B(t) & D(t) \end{pmatrix} \tag{3}$$

with $A(.)$ and $B(.)$ Hermitian matrices. We perform the usual Euler discretization of the derivatives with the step $h = T/N$ but using forward difference in the first equation and the backward difference in the second one; it is necessary to observe this rule if we want to obtain a discrete-time Hamiltonian. We deduce

$$\begin{aligned} \tfrac{y((k+1)h)-y(kh)}{h} &= \lambda B(kh)y(kh) + \lambda D(kh)z(kh) \\ \tfrac{z(kh)-z((k-1)h)}{h} &= -\lambda A(kh)y(kh) - \lambda B^*(kh)z(kh) \end{aligned} \tag{4}$$

where y,z are the m-dimensional sub-vectors of the $2m$ vector x. Denoting $y(kh) = y_k$, $z(kh) = z_{k+1}$, $A(kh) = A_k$, $B(kh) = B_k$, $D(kh) = D_k$ and, with an abuse of notations, λh by λ we obtain the discrete-time linear periodic Hamiltonian system:

$$\begin{aligned} y_{k+1} - y_k &= \lambda B_k y_k + \lambda D_k z_{k+1} \\ z_{k+1} - z_k &= -\lambda A_k - \lambda B_k^* z_{k+1} \end{aligned} \tag{5}$$

with A_k, B_k, D_k being N-periodic. Remark that this system may be also written as:

$$\begin{pmatrix} y_{k+1} - y_k \\ z_{k+1} - z_k \end{pmatrix} = \lambda J H_k \begin{pmatrix} y_k \\ z_{k+1} \end{pmatrix} \tag{6}$$

with $H_k = \begin{pmatrix} A_k & B_k^* \\ B_k & D_k \end{pmatrix}$ and J as previously. Also system (5) may be given the Cauchy form

$$x_{k+1} = C_k(\lambda) x_k \tag{7}$$

with

$$C_k(\lambda) = \begin{pmatrix} I & -\lambda D_k \\ 0 & I + \lambda B_k^* \end{pmatrix}^{-1} \begin{pmatrix} I + \lambda B_k & 0 \\ -\lambda A_k & I \end{pmatrix} \tag{8}$$

and this is true for any $\lambda \in \mathbb{C}$ except a finite member of eigenvalues of B_k^*. If the eigenvalues of B_k are also excluded, then $C_k(\lambda)$ is invertible and the solution of (5) may be constructed for all integers $k \in \mathbb{Z}$ (i.e. forward and backward); only in this case stability and strong stability have sense.

Definition 1. A point λ_0 is called a λ-point of stability of system (5) if for $\lambda = \lambda_0$ all solutions of the system are bounded on $\mathbb{Z}$. If, moreover, for $\lambda = \lambda_0$, all solutions of any system of (6) type having H_k replaced by $\hat{H}_k$ (N-periodic and Hermitian) sufficiently close to H_k (in some well-defined sense) are also bounded on $\mathbb{Z}$, then we call $\lambda = \lambda_0$ a λ-point of strong stability of (6).

It will be shown in the paper that, as in the continuous time case [1] the set of λ-points of strong stability of (6) is an open set and thus if it is nonempty it decomposes into a finite or infinite system of disjoint intervals that are called *λ-zones of stability*.

In the following we shall deal with the theory of the λ-zones of stability for system (6) following the line of [1], relating the existence and estimation of these zones to the multiplier problem (as in the pioneering papers of Liapunov).

2. The monodromy matrix and the multipliers

We may compute $C_k(\lambda)$ from (8) and find that

$$C_k^*(\lambda)JC_k(\lambda) - J = (\overline{\lambda} - \lambda)Q_k(\lambda) \tag{9}$$

where $Q_k(\lambda)$ is Hermitian. We deduce that $C_k(\lambda)$ is J-unitary for real λ and Hermitian H_k and J - orthogonal (symplectic) if H_k is symmetric. We may also write

$$x_k(\lambda) = C_{k-1}(\lambda)...C_0(\lambda)x_0 = U_k(\lambda)x_0 \tag{10}$$

thus defining the transition matrix(fundamental matrix of solutions) which results J-unitary or symplectic accordingly. It follows that the monodromy matrix $U_N(\lambda)$ will be also J-unitary or symplectic. In the terminology of [2] systems with complex coefficients and J-unitary matrix $C_k(\lambda)$ are called *Hamiltonian* while systems with real coefficients and symplectic matrix $C_k(\lambda)$ are called canonical.

The eigenvalues of the monodromy matrix i.e. the roots $\rho_i(\lambda)$ of the characteristic equation

$$det(U_N(\lambda) - \rho I) = 0 \tag{11}$$

are called multipliers of (5) (or (6)). The following result of Poincaré-Liapunov type may be proved following, e.g.,[2].

Theorem 1. *a) If H_k is Hermitian the spectrum of $U_N(\lambda)$ is located symmetrically with respect to the unit circle i.e. the multipliers occur in pairs $(\rho, \overline{\rho}^{-1})$ including their multiplicities as roots of* (11)*.*

b) If H_k is symmetric the spectrum of $U_N(\lambda)$ is located skew-symmetrically with respect to the unit circle, i.e., the multipliers occur in pairs (ρ, ρ^{-1}).

c) If H_k and λ are real and H_k is symmetric the multipliers occur in groups of four, being symmetric with respect to both unit circle and imaginary axis.

From here we may deduce:

Proposition 1. *All solutions of* (5) *are bounded on $\mathbb{Z}$ iff all multipliers of the system are of modulus one (located on the unit circle) and are of simple type (its root space coincides with its eigenspace) or, equivalently, have simple elementary divisors.*

Since we are concerned with robust(strong) stability, it is useful to analyze parameter dependence (on λ) of the multipliers. *Unlike in the continuous-time case* $U_N(\lambda)$ is not of entire but of meromorphic type being rational with respect to λ. For λ sufficiently close to the origin we may consider the McLaurin expansion of $C_k(\lambda)$

$$C_k(\lambda) = I_{2m} + \lambda J H_k + o(\lambda)$$

and of $U_N(\lambda)$

$$U_N(\lambda) = I_{2m} + \lambda J \sum_0^{N-1} H_k + o(\lambda)$$

It follows that in a sufficiently small neighborhood of $\lambda = 0$ the holomorphic matrix-valued logarithm is well defined

$$\Gamma(\lambda) = ln\, U_N(\lambda) = \Gamma_0 + \Gamma_1 \lambda + o(\lambda)$$

such that $U_N(\lambda) = e^{\Gamma(\lambda)}$. We deduce that $\Gamma_0 = 0, \Gamma_1 = J\sum_0^{N-1} H_k$. With an appropriate indexing we shall have $\rho_j(\lambda) = exp(\gamma_j(\lambda)), j = \overline{1,n}$, with $\rho_j(\lambda)$ being system's multipliers and $\gamma_j(\lambda)$ the eigenvalues of $\Gamma(\lambda)$. Following the line of [1] and [15] we may prove.

Theorem 2. *Assume that* $\sum_0^{N-1} H_k > 0$ *and has distinct eigenvalues. Then there exists an interval* $(-l, l)$ *such that for* $\lambda \in (-l, l)$ *all solutions of* (6) *are bounded on* Z.

Remark that this is the first result asserting existence of a central λ-zone of stability for (6). In the following we shall extend the result to the case of non-distinct eigenvalues and obtain estimates for l.

3. Self-adjoint boundary value problems for the canonical system

In this section we shall consider the boundary value problem for (6) defined by the boundary condition

$$x_N - Gx_0 = 0 \tag{12}$$

with G some J-unitary matrix ($G^*JG = J$). Following [1] and [15] it can be proved.

Theorem 3. *Let* $H_k \geq 0, k = \overline{0, N-1}, \sum_0^{N-1} H_k > 0$. *Then the eigenvalues (characteristic numbers) of the boundary value problem defined by* (6) *and* (12) *are real.*

We point out also the following facts

1. Any root of the equation

$$det(U_N(\lambda) - G) = 0 \tag{13}$$

   is a characteristic number of the boundary value problem and is real. Therefore all roots of (13), if any, are real.
2. The number $\lambda = 0$ is a characteristic number iff $det(I - G) = 0$ (iff G has 1 as eigenvalue).

We may also prove

Theorem 4. *The multiplicity k_j of any characteristic number of* (6), (12) *coincides with the number of linearly independent associated solutions of the problem.*

The proof of this theorem follows the line of Theorem 3.4 in [1] and Theorem 3.3 in [15] but in this case $U_N(\lambda) - G$ is, generally speaking, rational and we need the Smith-McMillan form of a rational matrix in order to obtain the result.

In order to obtain strong (i.e. robust) stability using the properties of the boundary value problem we shall need a result concerning the dependence of the characteristic numbers λ_j on the matrix H_k, dependence that is symbolized by $\lambda_j(H)$.

Theorem 5. *Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the positive characteristic numbers of the boundary value problem and let $0 > \lambda_{-1} > \lambda_{-2} > \ldots$ be the negative ones (it is assumed that each characteristic number occurs in the corresponding sequence the number of times equal to its multiplicity as a root of* (13)*). Let H_k^1, H_k^2 be such that $H_k^i \geq 0, \sum_0^{N-1} H_k^i > 0, i = 1, 2$ and assume that $H_k^1 \leq H_k^2, k = \overline{0, N-1}$. Then $\lambda_j(H^1) \geq \lambda_j(H^2), \lambda_{-j}(H^1) \leq \lambda_{-j}(H^2)$.*

The proof follows the line of [1] and [15].

4. Multipliers of 1st and 2nd kind: analytic properties and the strong (robust) stability

We shall return to Proposition 1 which states that (6) is stable provided all its multipliers are located on the unit circle and are of simple type. Generally speaking, a *J-unitary matrix with the eigenvalues on the unit circle and of simple type is called of stable type*. The matrices of stable type have an interesting property: *all J-unitary matrices that belong to a δ-neighborhood of a matrix of stable type are also of stable type* ([1],Theorem 1.2). This property suggests the approach to be token in the analysis of strong stability for Hamiltonian systems.

Definition 2. A Hamiltonian system is said to be strongly stable if it is stable and all Hamiltonian systems belonging to a neighborhood of it are also stable.

In fact we may follow [1] and [15] and use some arguments of [16] to show that if the Hamiltonian system

$$x_{k+1} - x_k = JH_k \begin{pmatrix} y_k \\ z_{k+1} \end{pmatrix}$$

is stable (of stable type) then there exists some $\delta > 0$ such that all Hamiltonian systems with H_k replaced by $\tilde{H}_k$ with $\sum_0^{N-1} |H_k - \tilde{H}_k| < \delta$ are also of stable type.

This robustness result has the following consequences:

A. If we consider (6) we may obtain neighboring Hamiltonian systems by modifying the parameter λ; but λ has to take real values in order that monodromy matrices be J-unitary.
B. Since stability is expressed through the properties of the multipliers and strong stability means preservation of this property with respect to parameter λ variations (among other perturbations that preserve the Hamiltonian character of the system) it would be of interest to discuss multiplier properties with respect to λ.

The first remark hints to the λ-zones of stability for real λ. The other one indicates that multiplier dependence on λ may help in strong stability studies even for complex λ. Indeed, for complex λ we may state and prove

Theorem 6. *Consider* (6) *with complex* λ *i.e. with* $Im\ \lambda \neq 0$. *Then half of system's multipliers have moduli less than 1 and the other half have them larger than 1 provided* $H_k \geq 0, \sum_0^{N-1} H_k > 0$.

The proof relies on the fact that $U_N(\lambda)$ is nonsingular and also either J-increasing (for $Im\ \lambda > 0$) or J-decreasing (for $Im\ \lambda < 0$); then Theorem 1.1 of [1] is used.

Definition 3. a) Let ρ_0 with $|\rho_0| = 1$ be a simple eigenvalue of a J-unitary matrix and e_0 the associated eigenvector. If e_0 is a plus-vector (with $i(Je_0, e_0) > 0$) the eigenvalue is called of 1st kind and if e_0 is a minus-vector (with $i(Je_0, e_0) < 0$) the eigenvalue is called of 2nd kind.

b) Let ρ_0 with $|\rho_0| = 1$ be a non-simple eigenvalue of a J-unitary matrix and let $\mathcal{L}_{\rho_0}$ be the corresponding proper subspace. If $\mathcal{L}_{\rho_0}$ contains plus-vectors only, then ρ_0 is of 1st kind and if $\mathcal{L}_{\rho_0}$ contains minus-vectors only, then ρ_0 is of 2nd kind. If $\mathcal{L}_{\rho_0}$ contains at least a null-vector (with $i(Je_0, e_0) = 0$) then ρ_0 is of mixed (indefinite type).

c) Let ρ_0 with $|\rho_0| \neq 1$ be a non simple eigenvalue of a J-unitary matrix: if $|\rho_0| > 1$ it is called of 1st kind and if $|\rho_0| < 1$ it is called of 2nd kind.

The main feature of this classification is the fact that it relies on the sign of the associated eigenvectors. This allows the extension of the notions to matrices that *are not* J-unitary. Indeed we already known [1], [15] that $U_N(\lambda)$ - the monodromy matrix - whose eigenvalues, the multipliers, are of interest - is J-increasing for

$Im\ \lambda > 0$ and J-decreasing for $Im\ \lambda < 0$. It is also known [1], that for J-increasing matrices an eigenvalue with modulus larger than 1 has its eigenvectors plus vectors thus being of 1st kind; accordingly, the eigenvalues with modulus lower than 1 (located inside the unit disk) are of 2nd kind. For J-decreasing matrices, the eigenvalues inside the unit disk are of 1st kind etc.

The dependence of multipliers' properties on λ may be followed using arguments from analytic function theory as in [1] and especially in [2]. The multipliers equation:

$$\Delta(\rho;\lambda) \equiv det(U_N(\lambda) - \rho I) = 0$$

takes the form

$$\rho^{2m} + A_{2m-1}(\lambda)\rho^{2m-1} + \cdots + A_1(\lambda)\rho + A_0(\lambda) = 0$$

where $A_k(\lambda)$ are rational functions and $A_0(\lambda) = det\ U_N(\lambda)$. From a basic representation lemma of Weierstrass it follows that in a neighborhood of $\lambda_0 \in R$ the multipliers $\rho_j(\lambda)$ that coincide for $\lambda \to \lambda_0$ with a multiplier ρ_0 of definite kind (1st or 2nd but not mixed) are analytic functions of λ i.e. the expansion of $\rho_j(\lambda)$ contains only integer powers of $(\lambda - \lambda_0)$. Further, we may follow [2] and obtain more specific information on the expansions of $\rho_j(\lambda)$, $\rho_j(\lambda)$ being considered branches of some analytic function coinciding in ρ_0 for $\lambda \to \lambda_0$.

From this information on expansion's coefficients we may deduce the so-called *Krein traffic rules* for the multipliers on the unit circle. We shall give below an account on these traffic rules that remain unchanged in the discrete-time case.

1. Let $\lambda_0 \in R$ and ρ_0 be a multiplier i.e an eigenvalue of $U_N(\lambda_0)$ with $|\rho_0| = 1$ and of multiplicity r. Consider a sufficiently small disk $\gamma : \{\rho : |\rho - \rho_0| < \epsilon\}$ such that there are no other eigenvalues of $U_N(\lambda_0)$ inside it. There will then exist some $\delta(\epsilon) > 0$ such that for all λ satisfying $|\lambda - \lambda_0| < \delta$ there will exist exactly r multipliers (eigenvalues of $U_N(\lambda)$ with their multiplicities) which are located inside the disk γ considered above. If $\lambda = \lambda_0 + ih, 0 < h < \delta$, $U_N(\lambda)$ is J-increasing and, therefore, the multipliers which are in γ and inside the unit disk are of 2nd kind while those which are in γ and outside the unit disk are of 1st kind. It was shown [1], [2] that this distribution of multipliers does not change as long as λ does not cross the real axis of the (λ) plane.
   Consequently we may say that in ρ_0 coincide for $\lambda = \lambda_0$ e.g. r_1 of 1st kind and $r - r_1$ of 2nd kind. The multiplier ρ_0 is thus of mixed type.
2. Consider a multiplier of definite type on the unit circle e.g. a multiplier of 1st kind (with its eigenvectors - plus-vectors) with multiplicity r, corresponding to λ_0. In its neighborhood one may find only multipliers of 1st kind. Let us assume that λ takes real values on the interval $(\lambda_0-\delta, \lambda_0+\delta)$. For $\lambda \neq \lambda_0$ the multipliers that coincided in ρ_0 *split off* in r multipliers describing r branches of the corresponding analytic function. Nevertheless the resulting multipliers remain on the unit circle and *move clockwise for increasing* λ. Were this not true, if a multiplier of 1st kind occurs (outside the unit disk) it will be accompanied by the occurrence of a multiplier of 2nd kind due to multipliers' symmetry; but in this case ρ_0 would not be of definite kind.

Obviously *for multipliers of 2nd kind the motion for increasing* λ *is counter-clockwise* when the multiplier splits off.

3. The multipliers of mixed type from the unit circle split off in multipliers of different kinds and they may, for some real λ to leave the circle in a symmetrical way: one outside and one inside.

We may now represent the multiplier traffic on the unit circle. The multipliers of definite kinds split and move clockwise and counter-clockwise, they met and separate, but do not leave the circle as $\lambda \in R$ increases or decreases. When two multipliers of different kind met they generate a multiplier of mixed kind which will split in multipliers of different kind again leaving the circle symmetrically (an equal number entering the unit disk and leaving it) thus generating instability.

5. The stability zones of the Hamiltonian system with parameter

In this section we shall consider that the neighboring Hamiltonians of the strong stability problem are generated by modifying the parameter λ.

Theorem 7. *The strong stability points of* (6) *form an open set which is not empty when* (6) *is of positive type, i.e., when* $H_k \geq 0, \sum_0^{N-1} H_k > 0$.

The proof goes as in [1] and [15] with $\lambda_0 H_k$ and λH_k as $\tilde{H}_k$: if $|\lambda - \lambda_0| < \delta$ then we are in the basic case of neighboring Hamiltonians.

If $\lambda_0 \in R$ is a point of strong stability, the set of strong stability points is open: we start with the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$ and afterwards we consider neighborhoods of the points of this interval ("continuations"). The open intervals thus obtained are the λ-stability zones.

Non emptiness is connected with the central stability zone (around $\lambda = 0$) which is nonempty at least in the case of Theorem 2. The central stability zone will be again considered in the next section. Now we shall focus on side zones in the positive type case, when $H_k \geq 0, \sum_0^{N-1} H_k > 0$.

The main tool of the analysis is an inequality that follows from the analytic properties of the multipliers:

$$-\frac{d}{d\lambda} arg\ \rho_j(\lambda)|_{\lambda=\lambda_0} \geq \sum_{k=0}^{N-1} \sigma_k^{min}(\lambda_0) \tag{14}$$

where $\rho_j(\lambda)$ is any branch of the analytic functions defined by multiplier dependence on λ [1] and σ_k^{min} is the lowest eigenvalue of a nonnegative matrix. It has been shown by a simple example that, *unlike in the continuous-time case*, a strictly positive lower bound that is independent of λ_0 does not exist. Therefore it is not possible to obtain, even in the simplest case, an estimate of the width of any side zone that is independent of its position with respect to the central zone [15].

We may however choose some interval $(-\Lambda_0, \Lambda_0)$ and compute a lower bound for the smallest eigenvalue that is independent of λ_0 but depends on the chosen interval i.e. on Λ_0. Let $\chi_k(\Lambda_0)$ be this lower bound. Since the system is of positive type, $\chi_k(\Lambda_0) \geq 0$ but $\sum_0^{N-1} \chi_k(\Lambda_0) > 0$ and (14) becomes

$$-\frac{d}{d\lambda} arg\ \rho_j(\lambda)|_{\lambda=\lambda_0} \geq \sum_0^{N-1} \chi_k(\Lambda_0) \tag{15}$$

This inequality is similar to (5.12) of [1]; the dependence on some interval width Λ_0 that may include the central zone and, possibly, some side zones, is not very restrictive: any numerical results are obtained for finite intervals, finite sums etc.

Theorem 8. *If $H_k \geq 0, \sum_0^{N-1} H_k > 0$ then the width of any λ-zone of stability included in some interval $(-\Lambda_0, \Lambda_0)$ does not exceed $\pi(\sum_0^{N-1} \chi_k(\Lambda_0))^{-1}$ where $\chi_k(\Lambda_0) = inf_{|\lambda| \leq \Lambda_0} \sigma_k^{min}(\lambda)$.*

The proof follows at once by applying the "traffic rules" [1],[15]. Note that the width of any of two parts of the central zone also does not exceed the above estimate.

6. The central zone of stability for a Hamiltonian system of positive type

We shall consider here the boundary value problem for (6) defined by (12) with $G = -I$. Its characteristic numbers are real: their existence follows from the fundamental theorem of Algebra provided $det(U_N(\lambda) + I) \neq$ const. and their number is finite. Let Λ_+ be the smallest (first) positive characteristic number and Λ_- the largest (first) negative one. We shall have

Theorem 9. *Assume that $H_k \geq 0, \sum_0^{N-1} H_k > 0$. The open interval (Λ_-, Λ_+) belongs to the central zone of stability of* (6)*; moreover, if H_k are real, this interval and the central zone of stability coincide.*

The proof of this result goes as in [1], [15] and relies on Theorem 2.3; the restriction on distinct eigenvalues is removed by a perturbation argument.

The only remaining point of the entire construction is existence of the characteristic numbers of opposite sign for the skew-symmetric (with $G = -I$) boundary value problem. The complex function argument of [1] was valid in the case of [15] but it can not be used in general since $U_N(\lambda)$ is not, generally speaking, of entire type and the contradiction obtained in [1] which proved existence of characteristic numbers of opposite signs fails. Krein himself was aware of the fact that complex function arguments were perhaps too strong [1] and suggested to apply the theory of weighted integral equations [17]; later this theory was incorporated in the theory of Volterra operators on Hilbert spaces [18]. In the discrete-time case this may reduce to some (possibly less) known results on determinants. Application of the theory on time scales [12], [13], [14] may be of great interest.

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