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ON SOME SPECIFIC NON-LINEAR ORDINARY DIFFERENCE EQUATIONS

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Abstract. It is proved that some specific non-linear ordinary difference equations, which appear in various applications, have a unique solution in the Banach space l_1. Moreover a bound of the solutions and a region of attraction of their equilibrium points are found. The obtained results improve some previous known results.

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1. Introduction

In this paper, we study the homogeneous, non-linear difference equation:

$$f(n+2) = \lambda f(n+1) + pf(n)e^{-\sigma f(n)}, n = 1, 2, ... \tag{1.1}$$

where $0 < \lambda < 1$, $\sigma > 0$, $0 < p < (1-\lambda)e^{\frac{2-\lambda}{1-\lambda}}$, $p \neq 1-\lambda$ and the non-homogeneous, non-linear difference equations:

$$\begin{aligned} f(n+1) = &-\frac{b_1(n+1)}{\alpha_1(n+1)} + \frac{h_1(n+1)}{\alpha_1(n+1)} f(n+2)f(n+1)f(n) + \\ &+\frac{d_1(n+1)}{\alpha_1(n+1)} f(n+2)f(n), n = 1, 2, ... \end{aligned} \tag{1.2}$$

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$$f(n+2) = \frac{\alpha_2(n+1)}{h_2(n+1)} + \frac{b_2(n+1)}{h_2(n+1)}[f(n+1)]^2 - \\ - \frac{1}{h_2(n+1)} f(n+2)[f(n)]^2, n = 1, 2, ... \tag{1.3}$$

$$f(n+1) = h_3(n) + [f(n)]^2, n = 0, 1, ... \tag{1.4}$$

$$f(n+1) = h_4(n) + \mu f(n) \left[1 - \frac{1}{K} f(n)\right], n = 1, 2, ... \tag{1.5}$$

where $\mu \in \mathbb{R}\setminus\{1\}$, $K > 0$ and $\alpha_1(n+1)$, $b_1(n+1)$, $h_1(n+1)$, $d_1(n+1)$, $\alpha_2(n+1)$, $b_2(n+1)$, $h_2(n+1)$, $h_3(n)$ and $h_4(n)$ are suitably defined complex sequences.

Our aim is to prove that the equations (1.1)-(1.5) have a unique solution in the Banach space:

$$l_1 = \{f(n) : \mathbb{N} \to \mathbb{C} / \|f(n)\|_{l_1} = \sum_{n=1}^{\infty} |f(n)| < +\infty\}, \tag{1.6}$$

For the motivation of seeking solutions of non-linear difference equations in l_1 see [1, pp. 84-112], [6]. Also it is known, see [11] and the references therein, that, under various conditions, a positive generated, ordered Banach space is order-isomorphic to l_1. Finally, we would like to point out that, the real space $l_1|_{\mathbb{R}}$, i.e.

$$l_1|_{\mathbb{R}} = \{f(n) : \mathbb{N} \to \mathbb{R} \ / \sum_{n=1}^{\infty} |f(n)| < +\infty\}, \tag{1.7}$$

is suitable for problems of population dynamics, since the condition:

$$\sum_{n=1}^{\infty} |f(n)| < +\infty,$$

represents the realistic fact that the population $f(n)$ is finite in every time instant n.

The method we use is a functional analytic method developed by E. K. Ifantis in [6] and used recently by P. D. Siafarikas and the author in [9], [10] for more general forms of non-linear difference equations. Using this method, equations (1.1)-(1.5) are reduced equivalenlty to operator equations on an abstract Banach space H_1. For our approach we also need the following result of Earle and Hamilton [2]:

If $f : X \to X$ is holomorphic, i.e. its Fréchet derivative exists, and $f(X)$ lies strictly inside X, then f has a unique fixed point in X, where X is a bounded, connected and open subset of a Banach space E.

By saying that a subset X' of X lies strictly inside X we mean that there exists an $\epsilon_1 > 0$ such that $\|x' - y\| > \epsilon_1$ for all $x' \in X'$ and $y \in E - X$.

All our results except those concerning equation (1.5) for $|\mu| > 1$, follow from a general theorem (Theorem 2.1), which was proved in [10] and which we state for the sake of completeness in Section 2.

2. Preliminaries

In the following, H is used to denote an abstract separable Hilbert space with the orthonormal basis $e_n, n = 1, 2, 3, \ldots$. We use the symbols $(\cdot, \cdot)$ and $\| \cdot \|$ to denote scalar product and norm in H respectively. By H_1 we mean the Banach space consisting of those elements f in H which satisfy the condition $\sum_{n=1}^{\infty} |(f, e_n)| < +\infty$. The norm in H_1 is denoted by $\|f\|_1 = \sum_{n=1}^{\infty} |(f, e_n)|$. By $f(n)$ we mean an element of the Banach space l_1 and by $f = \sum_{n=1}^{\infty} f(n)e_n$ we mean that element in H_1 generated by $f(n) \in l_1$. Finaly by V we mean the shift operator on H

$$V : Ve_n = e_{n+1}, n = 1, 2, \ldots$$

and by V^* its adjoint

$$V^* : V^*e_n = e_{n-1}, n = 2, 3, \ldots, V^*e_1 = 0.$$

It can easily be proved that the function

$$\phi : H_1 \to l_1$$

which is defined as follows:

$$\phi(f) = (f, e_n) = f(n)$$

is an isomorphism from H_1 onto l_1. We call f the abstract form of $f(n)$.

In general, if G is a mapping in l_1 and N is a mapping in H_1, we call $N(f)$ the abstract form of $G(f(n))$ if

$$G(f(n) = (N(f), e_n).$$

It follows easily that V^*f is the abstract form of $f(n+1)$.

We state now the basic theorem that we use.

Theorem 2.1. [10] *Consider the $m-th$ order non-homogeneous, nonlinear difference equation:*

$$\begin{aligned} f(n+m) + \sum_{p=1}^{m} (\alpha_p + \beta_p(n)) f(n+m-p) = g(n) + \sum_{s=2}^{\infty} c_s(n)[f(n+q)]^s + \\ + \sum_{i=1}^{N} \sum_{k=1}^{\infty} d_{ik}(n)[f(n+q_{i1}) f(n+q_{i2})]^k + \\ + \sum_{t=1}^{\Lambda} \sum_{k=1}^{\infty} b_{tk}(n)[f(n+q_{t3}) f(n+q_{t4}) f(n+q_{t5})]^k + \\ + \sum_{j=1}^{M} \sum_{k=1}^{\infty} l_{jk}(n)[A_j f(n+q_{j6}) + B_j f(n+q_{j7})]^k f(n+q_{j8}) \end{aligned} \tag{2.1}$$

where m, N, M, Λ positive integers, q, q_{i1}, q_{i2}, $i = 1, ..., N$, q_{t3}, t_{t4}, q_{t5}, $t = 1, ..., \Lambda$, q_{j6}, q_{j7}, q_{j8}, $j = 1, ..., M$ non-negative integers and α_p, $p = 1, ..., m$ in general complex numbers. Assume that $\lim_{n\to\infty} \beta_p(n) = 0$, $\forall p = 1, ..., m$, the complex sequences $c_s(n)$, $d_{ik}(n)$, $b_{tk}(n)$, and $l_{jk}(n)$, $s = 2, 3, ...$, $i = 1, ..., N$, $t = 1, ..., \Lambda$, $j = 1, ..., M$, $k = 1, 2, 3, ...$ satisfy the conditions

$$\sup_n |c_s(n)| \le \gamma_s, \quad \sup_n |d_{ik}(n)| \le \delta_{ik}, \quad \sup_n |b_{tk}(n)| \le \beta_{tk}, \quad \sup_n |l_{jk}(n)| \le \lambda_{jk}$$

and the functions

$$G_0(w) = \sum_{s=2}^{\infty} \gamma_s w^s, \quad G_i(w) = \sum_{k=1}^{\infty} \delta_{ik} w^{2k},$$
$$T_t(w) = \sum_{k=1}^{\infty} \beta_{tk} w^{3k}, \quad F_j(w) = \sum_{k=1}^{\infty} \lambda_{jk}(|A_j| + |B_j|)^k w^{k+1}$$

are entire functions, or they have a sufficiently large radius of convergence. Assume also that the roots of the algebraic equation

$$r^m + \alpha_1 r^{m-1} + ... + \alpha_m = 0$$

satisfy the conditions $|r_p| < 1$, $p = 1, 2, ..., m$. Then there exist positive numbers R_0 and P_0 such that for

$$\begin{aligned} |u| + \|g(n)\|_{l_1} =& |u_1| + |\alpha_1 u_1 + u_2| + ... + \\ &+ |\alpha_{m-1} u_1 + \alpha_{m-2} u_2 + ... + u_m| + \|g(n)\|_{l_1} < P_0, \end{aligned} \tag{2.2}$$

where

$$f(p) = u_p, \quad p = 1, ..., m \tag{2.3}$$

the equation (2.1) *together with the initial conditions* (2.3) *has a unique solution $f(n)$ in l_1. Moreover*

$$\sum_{n=1}^{\infty} |f(n)| < R_0. \tag{2.4}$$

Remark 1. The numbers R_0 and P_0 predicted by the above theorem are precisely determined due to the constructive character of Theorem 2.1. In particular R_0 is the point at which the function

$$P_1(R) = L^{-1} R \left[1 - LR \left(M_0(R) + \sum_{i=1}^{N} M_i(R) + R \sum_{t=1}^{\Lambda} \Delta_t(R) + \sum_{j=1}^{M} Q_j(R) \right) \right], \tag{2.5}$$

attains a maximum and $P_0 = P_1(R_0)$. In (2.5)

$$M_0(R) = \sum_{s=2}^{\infty} \gamma_s R^{s-2}, M_i(R) = \sum_{k=1}^{\infty} \delta_{ik} R^{2k-2}, \tag{2.6}$$

$$\Delta_t(R) = \sum_{k=1}^{\infty} \beta_{tk} R^{3k-3}, Q_j(R) = \sum_{k=1}^{\infty} \lambda_{jk}(|A_j| + |B_j|)^k R^{k-1}, \tag{2.7}$$

$1 \le i \le N$, $1 \le t \le \Lambda$, $1 \le j \le M$ are positive, continuous and increasing functions of R in an open interval suitably defined and L is the norm or a bound of the norm of the operator Γ^{-1}, where

$$\Gamma = (I - r_1 V)(I - r_2 V)...(I - r_m V) + V^m \sum_{p=1}^{m} B_p V^{*m-p}.$$

Remark 2. From (2.4) it follows that:

$$|f(n)| < R_0.$$

3. Applications

1) Consider the difference equation:

$$f(n+2) = \lambda f(n+1) + p f(n) e^{-\sigma f(n)}, n = 1, 2, ... \tag{3.1}$$

where $0 < \lambda < 1$, $\sigma > 0$, $0 < p < (1-\lambda)e^{\frac{2-\lambda}{1-\lambda}}$, $p \neq 1 - \lambda$. Equation (3.1) is the discrete version of a population model described by a differential equation [7].

The equilibrium points of (3.1) are:

$$\varrho_1 = 0, \quad \varrho_2 = \frac{1}{\sigma} \ln \frac{p}{1-\lambda} > 0.$$

For the equilibrium point $\varrho_1 = 0$ equation (3.1) can also be written as follows:

$$f(n+2) - \lambda f(n+1) - p f(n) = \sum_{s=2}^{\infty} \frac{(-1)^{s-1} p \sigma^{s-1}}{(s-1)!} [f(n)]^s. \tag{3.2}$$

Equation (3.2) results from equation (2.1) for:

$$m = 2, \quad \alpha_1 = -\lambda, \quad \alpha_2 = -p, \quad \beta_1(n) \equiv \beta_2(n) \equiv 0, \quad g(n) \equiv 0,$$
$$d_{ik}(n) \equiv b_{tk}(n) \equiv l_{jk}(n) \equiv 0, \quad c_s(n) = \frac{(-1)^{s-1} p \sigma^{s-1}}{(s-1)!}, \quad q = 0.$$

In this case $\gamma_s = \dfrac{p\sigma^{s-1}}{(s-1)!}$ and $G_0(s) = \displaystyle\sum_{s=2}^{\infty} \frac{p\sigma^{s-1}}{(s-1)!} w^s$ is an entire function. Also the roots of the algebraic equation $r^2 - \lambda r - p = 0$ are

$$r_1 = \frac{\lambda + \sqrt{\lambda^2 + 4p}}{2} \in (0, 1), \quad r_2 = \frac{\lambda - \sqrt{\lambda^2 + 4p}}{2} \in (-1, 0),$$

for $0 < p < 1-\lambda$. Then

$$\Gamma = (I - r_1 V)(I - r_2 V), \quad L = \frac{1}{1 + p - \sqrt{\lambda^2 + 4p}},$$
$$P_1(R) = \frac{R}{L} - R^2 \sum_{s=2}^{\infty} \frac{p\sigma^{s-1}}{(s-1)!} R^{s-2}.$$

It follows easily from Theorem 2.1 that for

$$|f(1)| + |f(2) - \lambda f(1)| < P_1(R_0), \tag{3.3}$$

equation (3.2) has a unique solution in l_1, where R_0 is the point at which $P_1(R)$ attains a maximum. Thus $\lim_{n\to\infty} f(n) = 0$ and $\varrho_1 = 0$ is a locally asymptotically stable equilibrium point of (3.2) with region of attraction given by (3.3). Also

$$|f(n)| < R_0.$$

For the equilibrium point $\varrho_2 = \dfrac{1}{\sigma} \ln \dfrac{p}{1-\lambda}$ we set

$$f(n) = F(n) + \varrho_2$$

and (3.2) becomes:

$$\begin{aligned} F(n+2) - \lambda F(n+1) + p(\varrho_2\sigma - 1)e^{-\sigma\varrho_2} F(n) = \\ = \sum_{s=2}^{\infty} \frac{(-1)^{s-1} p e^{-\sigma\varrho_2} \sigma^{s-1}(s-\sigma)}{s!} [F(n)]^s. \end{aligned} \tag{3.4}$$

Equation (3.4) results from equation (2.1) for:

$$m = 2, \quad \alpha_1 = -\lambda, \quad \alpha_2 = p(\varrho_2\sigma - 1)e^{-\sigma\varrho_2}, \quad \beta_1(n) \equiv \beta_2(n) \equiv 0, \quad g(n) \equiv 0,$$
$$d_{ik}(n) \equiv b_{tk}(n) \equiv l_{jk}(n) \equiv 0, \quad c_s(n) = \frac{(-1)^{s-1} p e^{-\sigma\varrho_2} \sigma^{s-1}(s-\sigma)}{s!}, \quad q = 0.$$

In this case

$$\gamma_s = \frac{p e^{-\sigma\varrho_2} \sigma^{s-1} |s-\sigma|}{s!} = \frac{(1-\lambda)\sigma^{s-1}|s-\sigma|}{s!}$$

and $G_0(s) = \displaystyle\sum_{s=2}^{\infty} \frac{(1-\lambda)\sigma^{s-1}|s-\sigma|}{s!} w^s$ is an entire function. Also the roots of the algebraic equation

$$r^2 - \lambda r + p(\varrho_2\sigma - 1)e^{-\sigma\varrho_2} = 0 \Leftrightarrow r^2 - \lambda r + (1-\lambda)(\ln \frac{p}{1-\lambda} - 1) = 0$$

are

i)

$$r_1 = \frac{\lambda + \sqrt{\lambda^2 + 4(1-\lambda)(1 - \ln \frac{p}{1-\lambda})}}{2} \in (0,1),$$
$$r_2 = \frac{\lambda - \sqrt{\lambda^2 + 4(1-\lambda)(1 - \ln \frac{p}{1-\lambda})}}{2} \in (-1,0)$$

for $1 - \lambda < p < e(1-\lambda)$,
ii)

$$r_1 = \frac{\lambda + \sqrt{\lambda^2 + 4(1-\lambda)(1 - \ln \frac{p}{1-\lambda})}}{2} \in (0,1),$$
$$r_2 = \frac{\lambda - \sqrt{\lambda^2 + 4(1-\lambda)(1 - \ln \frac{p}{1-\lambda})}}{2} \in (0,1)$$

for $e(1-\lambda) \le p < (1-\lambda)e^{1+\frac{\lambda^2}{4(1-\lambda)}}$,
iii) $r_1 = r_2 = \dfrac{\lambda}{2} \in (0,1)$ for $p = (1-\lambda)e^{1+\frac{\lambda^2}{4(1-\lambda)}}$ and
iv)

$$r_{1,2} = \frac{\lambda \pm i\sqrt{-\lambda^2 - 4(1-\lambda)(1 - \ln \frac{p}{1-\lambda})}}{2} \text{ and}$$
$$|r_{1,2}| = \sqrt{(1-\lambda)(\ln \frac{p}{1-\lambda} - 1)} < 1$$

for $(1-\lambda)e^{1+\frac{\lambda^2}{4(1-\lambda)}} < p < (1-\lambda)e^{\frac{2-\lambda}{1-\lambda}}$. Then

$$\Gamma = (I - r_1 V)(I - r_2 V),$$

and the corresponding bounds of Γ^{-1} are
i) $L = \dfrac{1}{(1-\lambda)(1 - \ln \frac{p}{1-\lambda})}$, ii) $L = \dfrac{1}{(1-\lambda)(\ln \frac{p}{1-\lambda} - 1)}$,
iii) $L = \dfrac{4}{(2-\lambda)^2}$, iv) $L = \dfrac{1}{\left(1 - \sqrt{(1-\lambda)(\ln \frac{p}{1-\lambda} - 1)}\right)^2}$, respectively.

Thus

$$P_1(R) = \frac{R}{L} - R^2 \sum_{s=2}^{\infty} \frac{pe^{-\sigma \varrho_2} \sigma^{s-1} |s - \sigma|}{s!} R^{s-2}.$$

It follows easily from Theorem 2.1 that for

$$|F(1)| + |F(2) - \lambda F(1)| < P_1(R_0), \tag{3.5}$$

equation (3.4) has a unique solution in l_1, where R_0 is the point at which $P_1(R)$ attains a maximum. Thus $\lim_{n\to\infty} F(n) = 0$ and 0 is a locally asymptotically stable equilibrium point of (3.4) with region of attraction given by (3.5). Thus $\varrho_2 = \frac{1}{\sigma}\ln\frac{p}{1-\lambda}$ is a locally asymptotically stable equilibrium point of (3.1) with region of attraction given by:

$$|f(1)-\frac{1}{\sigma}\ln\frac{p}{1-\lambda}|+|f(2)-\lambda f(1)+\frac{\lambda-1}{\sigma}\ln\frac{p}{1-\lambda}|<P_1(R_0), \tag{3.6}$$

Also

$$|f(n)|\le|F(n)|+\varrho_2\Leftrightarrow|f(n)|<R_0+\frac{1}{\sigma}\ln\frac{p}{1-\lambda}$$

and equation (3.1) has a unique solution in $l_1+\left\{\frac{1}{\sigma}\ln\frac{p}{1-\lambda}\right\}$.

Remark 3. Equation (3.1) is a particular case (for $\nu=1$) of the equation:

$$f(n+\nu+1)=\lambda f(n+\nu)+pf(n)e^{-\sigma f(n)}, \tag{3.7}$$

which was studied, among other things, in [7]. It was shown there that any solution of (3.7) converges to its positive equilibrium point ϱ_2 as $n\to\infty$ if $p\in(1-\lambda,(1-\lambda)e]$. Notice that this is a subset of $(1-\lambda,(1-\lambda)e^{\frac{2-\lambda}{1-\lambda}}]$.

Remark 4. Relations (3.3) and (3.5) describe the region of attraction for the equilibrium points ϱ_1 and ϱ_2 respectively. Note that these inequalities do not give explicitly the regions of attraction, because we do not know the point R_0, but we can achieve that by truncating the power series, of which $P_1(R)$ is consisted.

Remark 5. If the initial conditions $f(1)$, $f(2)$ are positive numbers then every real solution of (3.1) is positive.

2) Consider the difference equation:

$$\begin{aligned} f(n+1)=&-\frac{b_1(n+1)}{\alpha_1(n+1)}+\frac{h_1(n+1)}{\alpha_1(n+1)}f(n+2)f(n+1)f(n)+\\ &+\frac{d_1(n+1)}{\alpha_1(n+1)}f(n+2)f(n), n=1,2,... \end{aligned} \tag{3.8}$$

where $\frac{b_1(n+1)}{\alpha_1(n+1)}\in l_1$, $\sup_n|\frac{h_1(n+1)}{\alpha_1(n+1)}|\le\beta$ and $\sup_n|\frac{d_1(n+1)}{\alpha_1(n+1)}|\le\delta$.

Equation (3.2) appears often in various applications. In this case $\Delta_1(R)=\beta$, $M_1(R)=\delta$ are entire functions and $\Gamma=I$, $L=1$. Thus

$$P_1(R)=R-\delta R^2-\beta R^3.$$

It follows easily that $R_0=\frac{\sqrt{\delta^2+3\beta}-\delta}{2}$ and $P_0=\frac{(2\delta^2+6\beta)(\sqrt{\delta^2+3\beta}-\delta)}{27\beta^2}-\frac{\delta}{9\beta}$. By applying Theorem 2.1 to equation (3.8) we find that for

$$|f(1)|+\|\frac{b_1(n+1)}{\alpha_1(n+1)}\|_{l_1}<\frac{(2\delta^2+6\beta)(\sqrt{\delta^2+3\beta}-\delta)}{27\beta^2}-\frac{\delta}{9\beta},$$

equation (3.8) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{\sqrt{\delta^2 + 3\beta} - \delta}{2}.$$

In the special case where $d_1(n+1) \equiv 1$ and $h_1(n+1) \equiv 0$, equation (3.8) becomes:

$$f(n+1) = -\frac{b_1(n+1)}{\alpha_1(n+1)} + \frac{1}{\alpha_1(n+1)} f(n+2) f(n), \tag{3.9}$$

which is the well-known non-autonomous Lyness equation. As before, we find that $\Gamma = I$, $L = 1$ and $P_1(R) = R - \delta R^2$. Thus $R_0 = \frac{1}{2\delta}$ and $P_0 = \frac{1}{4\delta}$. By applying Theorem 2.1 to equation (3.3) we find that for

$$|f(1)| + \|\frac{b_1(n+1)}{\alpha_1(n+1)}\|_{l_1} < \frac{1}{4\delta},$$

equation (3.9) has a unique bounded solution in l_1 with bound:

$$|f(n)| < R_0 = \frac{1}{2\delta}.$$

Remark 6. In the case when equation (3.8) has positive solutions and $\alpha_1(n+1)$, $b_1(n+1)$, $h_1(n+1)$, $d_1(n+1)$ are constants, equation (3.8) was studied in [4]. Invariants for equation (3.8) have been found in [3], in the case when $\alpha_1(n+1)$, $b_1(n+1)$, $h_1(n+1)$, $d_1(n+1)$, are periodic sequences of positive numbers and the initial conditions are positive numbers. The non-autonomous Lyness equation (3.9) was studied, among other things, in [5]. In particular it was shown there that under some different, than those we used, but more complicated conditions on the sequences $\alpha_1(n+1)$ and $b_1(n+1)$, every positive solution of (3.9) is bounded.

3) Consider the difference equation:

$$\begin{aligned} f(n+2) =& \frac{\alpha_2(n+1)}{h_2(n+1)} + \frac{b_2(n+1)}{h_2(n+1)}[f(n+1)]^2 - \\ & - \frac{1}{h_2(n+1)} f(n+2)[f(n)]^2, n = 1, 2, ... \end{aligned} \tag{3.10}$$

where $\frac{\alpha_2(n+1)}{h_2(n+1)} \in l_1$, $\sup_n |\frac{b_2(n+1)}{h_2(n+1)}| \leq \gamma$ and $\sup_n |\frac{1}{h_2(n+1)}| \leq \lambda$.

In this case $M_0(R) = \gamma$, $Q_1(R) = \lambda R$ are entire functions and $\Gamma = I^2 = I$, $L = 1$. Thus

$$P_1(R) = R - \gamma R^2 - \lambda R^3.$$

It follows easily that $R_0 = \frac{\sqrt{\gamma^2 + 3\lambda} - \gamma}{2}$ and $P_0 = \frac{(2\gamma^2 + 6\lambda)(\sqrt{\gamma^2 + 3\lambda} - \gamma)}{27\lambda^2} - \frac{\gamma}{9\lambda}$. By applying Theorem 2.1 to equation (3.10) we find that for

$$|f(1)| + |f(2)| + \|\frac{\alpha_2(n+1)}{h_2(n+1)}\|_{l_1} < \frac{(2\gamma^2 + 6\lambda)(\sqrt{\gamma^2 + 3\lambda} - \gamma)}{27\lambda^2} - \frac{\gamma}{9\lambda},$$

equation (3.10) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{\sqrt{\gamma^2 + 3\lambda} - \gamma}{2}.$$

Remark 7. Equation (3.10) has been studied in [8] for $\alpha_2(n+1)$, $b_2(n+1)$ and $h_2(n+1)$ constants.

4) Consider the difference equation:

$$f(n+1) = h_3(n) + [f(n)]^2, n = 1, 2, \dots \tag{3.11}$$

where $h_3(n) \in l_1$.

In this case $M_0(R) = 1$ is an entire function and $\Gamma = I$, $L = 1$. Thus

$$P_1(R) = R - R^2.$$

It follows easily that $R_0 = \dfrac{1}{2}$ and $P_0 = \dfrac{1}{4}$. By applying Theorem 2.1 to equation (3.11) we find that for

$$|f(1)| + \|h_3(n)\|_{l_1} < \frac{1}{4}, \tag{3.12}$$

equation (3.11) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{1}{2}.$$

Also notice that (3.11) can also be written as:

$$\frac{f(n+1)}{f(n)} = \frac{h_3(n)}{f(n)} + f(n).$$

Thus if $K = \lim\limits_{n\to\infty} \dfrac{h_3(n)}{f(n)}$ exists then $\lim\limits_{n\to\infty} \dfrac{f(n+1)}{f(n)} = K$ and the generating analytic function $f(z) = \sum\limits_{n=1}^{\infty} f(n) z^{n-1}$ converges absolutely for $|z| < \dfrac{1}{K}$.

Remark 8. In the case where $h_3(n) \equiv h \notin l_1$, equation (3.11) becomes the well-known equation from which the Mandlebrot and the Julia sets are deduced. More particularly, the set of all points h for which the solution $f(n)$ of (3.11) with $f(1) = 0$ stays bounded as $n \to \infty$ is called the Mandlebrot set (M) and for a given parameter $h =$ constant, the set of initial values f(0) for which $f(n)$ stays bounded is the so-called filled-in Julia set (J_c). (The Julia set proper consists of the boundary points of J_c.)

Thus for $f(1) = 0$ we obtain from (3.12):

$$\|h_3(n)\|_{l_1} < \frac{1}{4},$$

which can be considered as a generalized Mandelbrot set.

Also for $h_3(n)$ a given sequence of l_1, relation (3.12) can be considered as a generalized Julia set.

Notice that when $h_3(n) \equiv h$ = constant, our method can not be applied, because h does not belong in l_1.

5) Consider the difference equation:

$$f(n+1) = h_4(n) + \mu f(n)\left[1 - \frac{1}{K} f(n)\right], n = 1, 2, ... \tag{3.13}$$

where $\mu \in \mathbb{R} \setminus \{1\}$, $K > 0$ and $h_4(n) \in l_1$.

Equation (3.13) describes the development of a single species population $f(n)$, where μ is the parameter related to the growth or death rate, $K > 0$ is the carrying capacity and $h_4(n)$ represents the harvest/stock [12].

We shall distinguish the following two cases:

1) First case: $|\mu| < 1$.

Here $M_0(R) = \frac{|\mu|}{K}$ is an entire function and $\Gamma = I - \mu V$, $L = \frac{1}{1-|\mu|}$. Thus

$$P_1(R) = (1 - |\mu|)R - \frac{|\mu|}{K} R^2.$$

It follows easily that $R_0 = \dfrac{(1 - |\mu|)K}{2|\mu|}$ and $P_0 = \dfrac{(1 - |\mu|)^2 K}{4|\mu|}$. By applying Theorem 2.1 to equation (3.13) we find that for

$$|f(1)| + \|h_4(n)\|_{l_1} < \frac{(1 - |\mu|)^2 K}{4|\mu|}, \quad |\mu| < 1$$

equation (3.13) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{(1 - |\mu|)K}{2|\mu|}, \quad |\mu| < 1.$$

2) Second case: $|\mu| > 1$.

In this case, Theorem 2.1 can not be applied to equation (3.13) because the unique solution of the algebraic equation

$$r - \mu = 0$$

is $r = \mu$ and $|\mu| > 1$.

Notice that equation (3.13) can also be written as:

$$f(n) - \frac{1}{\mu} f(n+1) = -\frac{1}{\mu} h_4(n) + \frac{1}{K}[f(n)]^2, n = 1, 2, ... \tag{3.14}$$

According to the representation presented in Section 2, the abstract form of (3.14) in H_1 is:

$$\left(I - \frac{1}{\mu}V^*\right) f = N(f) - \frac{1}{\mu}h_4, \tag{3.15}$$

where h_4 is the abstract form of $h_4(n)$ and $N(f) = \frac{1}{K}(f, e_n)(f, e_n)e_n$, is a Fréchet differentiable operator defined on all H_1 with $\|N(f)\|_1 \leq \|f\|_1^2$ ([9] or [10]).

Since $|\mu| > 1$, the operator $\left(I - \frac{1}{\mu}V^*\right)^{-1}$ is uniquely determined on H_1 and bounded, with bound:

$$\| \left(I - \frac{1}{\mu}V^*\right) \|_1 < \frac{|\mu|}{|\mu| - 1}.$$

Thus (3.15) becomes

$$f = \left(I - \frac{1}{\mu}V^*\right)^{-1} \left[N(f) - \frac{1}{\mu}h_4\right]. \tag{3.16}$$

Following a technique similar to the one used in [6], [9], [10] we define the function:

$$\phi(f) = \left(I - \frac{1}{\mu}V^*\right)^{-1} \left[N(f) - \frac{1}{\mu}h_4\right].$$

Let $\|f\|_1 \leq R < \bar{R} < +\infty$, where $\bar{R}$ is as large as we want, but finite. Then from (3.16) we obtain:

$$\|\phi(f)\|_1 \leq \frac{|\mu|}{|\mu| - 1} \left[\frac{R^2}{K} + \frac{1}{|\mu|}\|h_4\|_1\right]. \tag{3.17}$$

Since $\bar{R}$ is sufficienlty large, there exists an $\bar{R}_1 \in [0, \bar{R}]$ such that

$$\frac{|\mu|}{|\mu| - 1}\frac{\bar{R}_1}{K} > 1.$$

Thus the value $\bar{R}_2 = \dfrac{(|\mu| - 1)K}{|\mu|}$ is a zero of the function

$$P(R) = 1 - \frac{|\mu|}{|\mu| - 1}\frac{\bar{R}_1}{K}.$$

So the continuous function

$$P_1(R) = \frac{|\mu| - 1}{|\mu|}RP(R)$$

satisfies $P_1(0) = P_1(\bar{R}_2) = 0$ and therefore attains a maximum at the point

$$R_0 = \frac{(|\mu| - 1)K}{2|\mu|} \in (0, \bar{R}_2).$$

Now for every $\epsilon > 0$, $R = R_0$ and

$$\|h_4\|_1 \leq \frac{(|\mu|-1)^2 K}{4|\mu|} - (|\mu|-1)\epsilon$$

we find from (3.17)

$$\|\phi(f)\|_1 \leq \frac{(|\mu|-1)K}{2|\mu|} - \epsilon = R_0 - \epsilon < R_0$$

for $\|f\|_1 < R_0$. This means that for

$$\|h_4\|_1 < \frac{(|\mu|-1)^2 K}{4|\mu|}$$

ϕ is a holomorphic map from $B\left(0, \frac{(|\mu|-1)K}{2|\mu|}\right)$ strictly inside $B\left(0, \frac{(|\mu|-1)K}{2|\mu|}\right)$. Thus applying the fixed point theorem of Earle and Hamilton [2] we find that equation $\phi(f) = f$ has a unique fixed point in H_1. This means equivalently that for

$$\|h_4(n)\|_{l_1} < \frac{(|\mu|-1)^2 K}{4|\mu|}, \quad |\mu| > 1$$

equation (3.14) has a unique bounded solution in l_1 with bound:

$$|f(n)| < \frac{(|\mu|-1)K}{2|\mu|}, \quad |\mu| > 1.$$

Remark 9. In [12] the real periodic solutions of (3.14) have been investigated for $\mu \in (1,2)$ and $h_4(n) : \mathbb{N} \to \mathbb{R}$ an ω periodic number sequence with $\omega \geq 1$ which satisfies the relation:

$$\|h_4\| < \frac{(|\mu|-1)^2 K}{4|\mu|}, \quad \mu \in (1,2)$$

where $\|h_4\| = \max_n |h_4(n)|$. Moreover it was found in [12] that the predicted periodic solution satisfies:

$$|f(n)| < \left(1 - \frac{1}{\mu}\right) K r_0, \quad r_0 \in (0, 1/2), \quad \mu \in (1,2).$$

Remark 10. Our results, concerning all five applications hold also, if we consider the Banach space $l_1|_{\mathbb{R}}$ instead of l_1.

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