ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 543–548, CDDE 2000 issue

SUFFICIENT CONDITIONS FOR NONOSCILLATION OF NEUTRAL EQUATIONS

Jan Ohriska*

Department of Mathematics, Faculty of Science, P. J. Šafarik University
Jesenná 5, 041 54 Košice, Slovak Republic
Email: ohriska@kosice.upjs.sk

Abstract. Sufficient conditions are given under which the first order neutral differential equation with constant coefficients has a nonoscillatory solution.

AMS Subject Classification. 34K40

Keywords. Neutral equations, nonoscillatory solutions.

1. Introduction

Consider the neutral differential equation

$$\frac{d}{dt}[x(t) + px(t-\tau)] + qx(t-\sigma) = 0, \quad t \geq t_0, \tag{1}$$

where

(i) p, q, τ, σ are positive real numbers.

Note that a nontrivial solution of an equation we call oscillatory if it has arbitrarily large zeros, and call it nonoscillatory otherwise, and next we shall say that an equation is oscillatory provided all its (nontrivial) solutions are oscillatory, and call it nonoscillatory otherwise.

A basic result on the oscillation of equation (1) says that every solution of equation (1) is oscillatory if and only if its characteristic equation

$$\lambda + p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma} = 0 \tag{2}$$

* Research supported by grant VGA of Slovak Republic No. 1/7466/20

has no real roots. Such result we can find in the book [1] and for more general equations in the book [2] and in the paper [3]. But to determine if equation (2) has a real root is quite a problem itself. Therefore an effort of many authors is to derive other conditions for oscillation and nonoscillation of considered equation which can be easily applied than previous one. In a literature we can find several sufficient conditions for every solution of equation (1) to be oscillatory (see e.g. [1] and [4]) but less conditions for the existence of nonoscillatory solution of (1).

The aim of this contribution is to present new well-applicable conditions for the existence of nonoscillatory solution of (1). The method is based on a transformation of the equation (1) by a transformation of the independent variable.

The straight consideration about the existence of a real root of characteristic equation (2) enables us to obtain the following result.

Theorem 1. *Assume the condition* (*i*) *holds true and* $\tau \geq \sigma$. *Then equation* (1) *has nonoscillatory solution* $x(t) = e^{\lambda t}, \ \lambda \in (-\frac{q}{p}, 0)$.

Proof. According to assumptions it is clear that if the equation (2) has a real root so it must be negative. Thus we define

$$F(\lambda) = \lambda + p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma} \quad \text{for} \quad \lambda \leq 0$$

and put $F(\lambda) = H_1(\lambda) + H_2(\lambda)$, where $H_1(\lambda) = \lambda + p\lambda e^{-\lambda\tau}$, $H_2(\lambda) = qe^{-\lambda\sigma}$. Then we have

$$\lim_{\lambda\to 0^-} H_1(\lambda) = 0, \quad \lim_{\lambda\to -\infty} H_1(\lambda) = -\infty, \quad H_1'(\lambda) = 1 + pe^{-\lambda\tau}(1 - \lambda\tau) > 0$$

$$\text{and} \quad \lim_{\lambda\to 0^-} H_2(\lambda) = q, \quad \lim_{\lambda\to -\infty} H_2(\lambda) = \infty, \quad H_2'(\lambda) = -q\sigma e^{-\lambda\sigma} < 0$$

from which we see that for $\tau \geq \sigma$ we have $F(-\frac{q}{p}) = -\frac{q}{p} + q(e^{\frac{q}{p}\sigma} - e^{\frac{q}{p}\tau}) < 0$. Since $F(0) = q > 0$ so we know that the equation (2) has the root $\lambda \in (-\frac{q}{p}, 0)$, the function $x(t) = e^{\lambda t}$ is the solution of (1) and the proof is complete.

Another way how to gain sufficient conditions for the existence of nonoscillatory solution of equation (1) we present in the following sections.

2. Preliminaries

Consider the equation (1) but instead of condition (i) we suppose that

(ii) p, q, τ, σ are real numbers different from zero.

We transform the equation (1) by the transformation of the independent variable. We put $s = at, \quad y(s) = x(\frac{1}{a}s)$ where $a > 0$. Then the equation (1) acquires the form

$$\frac{d}{ds}[y(s) + py(s - a\tau)] + \frac{1}{a}qy(s - a\sigma) = 0, \quad s \geq s_0, \tag{3}$$

where $s_0 = at_0$.

It is clear the following holds true.

Note 1. A function $x(t)$ is a solution of the equation (1) for $t \geq t_0$ if and only if the function $y(s) = x(\frac{1}{a}s)$ is a solution of the equation (3) for $s \geq s_0$ and thus the equation (1) is oscillatory if and only if equation (3) is oscillatory.

Since equation (3) is of the same form as equation (1) is so it is oscillatory if and only if its characteristic equation

$$a\eta + pa\eta e^{-a\eta\tau} + qe^{-a\eta\sigma} = 0 \tag{4}$$

has no real roots and we can decide about solutions of (1) by the roots of the equation (4).

Now we analyse this position.

(a) First of all we see that the number $\lambda = 0$ is not the root of equation (2).

(b) Suppose that equation (2) has a positive root λ. Then we can take $a = \lambda$ and equation (4) will be of the form $\lambda\eta + p\lambda\eta e^{-\lambda\eta\tau} + qe^{-\lambda\eta\sigma} = 0$ and we see that $\eta = 1$ is the root of this equation. It means that equation

$$\frac{d}{ds}[y(s) + py(s - \lambda\tau)] + \frac{q}{\lambda}y(s - \lambda\sigma) = 0, \quad s \geq s_0,$$

has nonoscillatory solution $y(s) = e^s$.

(c) Now suppose that equation (2) has a negative root λ. So if we take $a = -\lambda$, equation (4) will be of the form $-\lambda\eta - p\lambda\eta e^{\lambda\eta\tau} + qe^{\lambda\eta\sigma} = 0$ and we see that $\eta = -1$ is the root of this equation. It means that equation

$$\frac{d}{ds}[y(s) + py(s + \lambda\tau)] - \frac{q}{\lambda}y(s + \lambda\sigma) = 0, \quad s \geq s_0,$$

has nonoscillatory solution $y(s) = e^{-s}$.

We conclude this consideration in the following note.

Note 2. To every equation of the form (1), the characteristic equation of which has a positive (negative) root, we can coordinate an equation of the same form with the characteristic root 1 (-1). On the other hand, if we take an equation of the form (1) with the solution $y(s) = e^s$ (similarly with the solution $y(s) = e^{-s}$) and we choose some positive number λ (a negative number λ) so we can write the equation of the same form with the solution $x(t) = e^{\lambda t}$ ($x(t) = e^{\lambda t}$).

3. Conditions for nonoscillatory solutions

Theorem 2. *Assume that* $p \neq 0,\ q > 0,\ \tau > 0,\ \sigma > 0.$

(I) Let there exist numbers $q_1 > 0,\ \tau_1 > 0,\ \sigma_1 > 0$ *such that the conditions*

$$1 + pe^{-\tau_1} + q_1 e^{-\sigma_1} = 0 \quad \textit{and} \quad \frac{\tau_1}{\tau} = \frac{\sigma_1}{\sigma} = \frac{q}{q_1} = \frac{1}{a} \tag{5}$$

are satisfied. Then equation (1) *has nonoscillatory solution* $x(t) = e^{\frac{1}{a}t}$.

(II) Let there exist numbers $q_2 > 0,\ \tau_2 > 0,\ \sigma_2 > 0$ *such that the conditions*

$$-1 - pe^{\tau_2} + q_2 e^{\sigma_2} = 0 \quad \text{and} \quad \frac{\tau_2}{\tau} = \frac{\sigma_2}{\sigma} = \frac{q}{q_2} = \frac{1}{a}, \tag{6}$$

are satisfied. Then equation (1) *has nonoscillatory solution* $x(t) = e^{-\frac{1}{a}t}$.

Proof. Consider the equation

$$\frac{d}{dz}[u(z) + p_1 u(z - \tau_1)] + q_1 u(z - \sigma_1) = 0, \quad z \geq z_0, \tag{7}$$

with $p_1 \neq 0, q_1 > 0, \tau_1 > 0, \sigma_1 > 0$, which has the solution $u(z) = e^z$, i.e. such that its characteristic equation $\mu + p_1 \mu e^{-\mu\tau_1} + q_1 e^{-\mu\sigma_1} = 0$ has the root $\mu = 1$, i.e. such that $1 + p_1 e^{-\tau_1} + q_1 e^{-\sigma_1} = 0$. The equation (7) we can transform to the equation (1) by a suitable $a > 0$. In other words, there exists a number $a > 0$ such that the transformation of (7) by $t = az$, $x(t) = u(\frac{1}{a}t)$ gives the equation (1) in the formal form

$$\frac{d}{dt}[x(t) + p_1 x(t - a\tau_1)] + \frac{1}{a} q_1 x(t - a\sigma_1) = 0.$$

So we have $p = p_1$, and next

$$q = \frac{1}{a} q_1, \quad \tau = a\tau_1, \quad \sigma = a\sigma_1. \tag{8}$$

The conditions (8) we can write in the form

$$\frac{\tau_1}{\tau} = \frac{\sigma_1}{\sigma} = \frac{q}{q_1} = \frac{1}{a}.$$

The straight computation shows that the number $\frac{1}{a}$ is the root of the equation (2).

The similar arguments hold true if we take the equation

$$\frac{d}{dz}[u(z) + p_2 u(z - \tau_2)] + q_2 u(z - \sigma_2) = 0, \quad z \geq z_0, \tag{9}$$

where $p_2 \neq 0, q_2 > 0, \tau_2 > 0, \sigma_2 > 0$ with the solution $u(z) = e^{-z}$. The theorem is proved.

Now using Theorem 2 we study the problem of the existence of nonoscillatory solutions of the equation (1) under the condition (i).

The assumption (i) ensures that the equation (2) has not nonnegative root i.e. the equation (1) has not the solution of the form $x(t) = e^{\lambda t},\ \lambda \geq 0$ and thus there do not exist positive numbers q_1, τ_1, σ_1 satisfying the first condition from (5). Therefore we devote our attention to the case (II) of Theorem 2.

Let the numbers $q_2 > 0,\ \tau_2 > 0,\ \sigma_2 > 0$ be such that the first condition from (6) is satisfied (note that such numbers always exist) and for some $\sigma_2 > 0$ we choose $q_2 > 0$ and $\tau_2 > 0$ such that

$$q_2 = \frac{q\sigma}{\sigma_2} \quad \text{and} \quad \tau_2 = \frac{\tau\sigma_2}{\sigma}. \tag{10}$$

Then the numbers $q_2,\ \tau_2,\ \sigma_2$ satisfy the second condition from (6) and the problem of the existence of trinity of numbers for which the first condition from (6) is satisfied is reduced to the problem of the existence of one such number.

Now we define the function $G(\sigma_2) = \frac{1}{\sigma_2}e^{\sigma_2}, \quad \sigma_2 > 0$. Then

$$G'(\sigma_2) = \frac{1}{\sigma_2^2}e^{\sigma_2}(\sigma_2 - 1), \quad G''(\sigma_2) = \frac{1}{\sigma_2^3}e^{\sigma_2}((\sigma_2 - 1)^2 + 1),$$

from which we see that for every $\sigma_2 > 0$ we have $G(\sigma_2) \geq e$.

Now suppose that $q\sigma > \frac{1}{e}$. Then for every $\sigma_2 > 0$ we have

$$\frac{1}{q\sigma} < e \leq \frac{1}{\sigma_2}e^{\sigma_2}.$$

Therefore, according to (10) we have $-1 + q_2e^{\sigma_2} > 0$ and the first condition from (6) will be satisfied if and only if $\frac{\tau}{\sigma}\sigma_2 = \ln\frac{q_2e^{\sigma_2}-1}{p}$ or

$$\frac{\tau}{\sigma}\sigma_2 + \ln p = \ln\left(\frac{q\sigma}{\sigma_2}e^{\sigma_2} - 1\right) \tag{11}$$

for some $\sigma_2 > 0$.

The existence of a positive root of the equation (11) we investigate now by the auxiliary function

$$F(\sigma_2) = \frac{\ln(q\sigma\frac{1}{\sigma_2}e^{\sigma_2} - 1)}{\frac{\tau}{\sigma}\sigma_2 + \ln p},$$

defined

- for $\sigma_2 \in (0, \infty)$ if $p \geq 1$
- for $\sigma_2 \in ((0, -\frac{\sigma}{\tau}\ln p) \cup (-\frac{\sigma}{\tau}\ln p, \infty))$ if $0 < p < 1$.

Then for $p > 0$ we have $\lim_{\sigma_2\to\infty} F(\sigma_2) = \frac{\sigma}{\tau}$, and

$$\lim_{\sigma_2\to 0^+} F(\sigma_2) = \begin{cases} \infty & \text{if} \quad p \geq 1 \\ -\infty & \text{if} \quad 0 < p < 1. \end{cases}$$

In the case $0 < p < 1$ we compute one-side limits of the function F at the point $-\frac{\sigma}{\tau}\ln p$ and we obtain

$$\lim_{\sigma_2\to -\frac{\sigma}{\tau}\ln p^-} F(\sigma_2) = \begin{cases} -\infty & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p > 0 \\ \infty & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p < 0 \\ c \in \mathbb{R} & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p = 0 \end{cases}$$

and

$$\lim_{\sigma_2\to -\frac{\sigma}{\tau}\ln p^+} F(\sigma_2) = \begin{cases} \infty & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p > 0 \\ -\infty & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p < 0 \\ c \in \mathbb{R} & \text{if} \quad q\tau + 2p^{\frac{\sigma}{\tau}}\ln p = 0. \end{cases}$$

This investigation and the continuity of F enables us to formulate the following results.

Theorem 3. *Let the condition (i) hold true and let*

$$0 < p < 1, \quad q\sigma > \frac{1}{e}, \quad q\tau + 2p^{\frac{\sigma}{\tau}} \ln p < 0.$$

Then there exists $\sigma_2 \in (0, -\frac{\sigma}{\tau} \ln p)$ *such that* (11) *holds true, i.e. the equation* (9) *has the solution* $x(t) = e^{-t}$ *and the equation* (1) *has the nonoscillatory solution* $x(t) = e^{-\frac{\sigma_2}{\sigma} t}$.

Theorem 4. *Let the condition (i) hold true and let*

$$0 < p < 1, \quad q\sigma > \frac{1}{e}, \quad q\tau + 2p^{\frac{\sigma}{\tau}} \ln p < 0 \quad \text{and} \quad \frac{\sigma}{\tau} > 1.$$

Then there exists $\sigma_2 \in (-\frac{\sigma}{\tau} \ln p, \infty)$ *such that* (11) *holds true i.e. the equation* (9) *has the solution* $x(t) = e^{-t}$ *and the equation* (1) *has the nonoscillatory solution* $x(t) = e^{-\frac{\sigma_2}{\sigma} t}$.

Theorem 5. *Let the condition (i) hold true and let*

$$0 < p < 1, \quad q\sigma > \frac{1}{e}, \quad q\tau + 2p^{\frac{\sigma}{\tau}} \ln p > 0 \quad \text{and} \quad \frac{\sigma}{\tau} < 1.$$

Then there exists $\sigma_2 \in (-\frac{\sigma}{\tau} \ln p, \infty)$ *such that* (11) *holds true i.e. the equation* (9) *has the solution* $x(t) = e^{-t}$ *and the equation* (1) *has nonoscillatory solution* $x(t) = e^{-\frac{\sigma_2}{\sigma} t}$.

Remark 1. One can see that the above presented method can be used in many other cases not only in the case when the condition (i) is satisfied.

References

1. D. D. Bainov, D. P. Mishev, *Oscillation theory for neutral differential equations with delay*, Adam Hilger; Bristol, Philadephia and New York 1991.
2. L. H. Erbe, Q. Kong, B. G. Zhang, *Oscillation theory for functional differential equations*, Marcel Dekker Inc., New York, Basel, Hong Kong 1994.
3. M. K. Grammatikopoulos, I. P. Stavroulakis, *Oscillation of neutral differential equations*, Radovi Matematicky, 47–71, 1991.
4. B. G. Zhang, *Oscillation of first order neutral functional differential equations*, J. Math. Anal. Appl., 311–318, 1989.