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ON THE RESONANCE PROBLEM FOR THE 4th ORDER ORDINARY DIFFERENTIAL EQUATIONS, FUČÍK'S SPECTRUM

Petr Nečesal*

Centre of Applied Math., Faculty of Applied Sciences, University of West Bohemia
P. O. Box 314, 306 14 Pilsen, Czech Republic
Email: pnecesal@kma.zcu.cz

Abstract. We consider the boundary value problems for the fourth order nonlinear differential equation $u^{\mathrm{IV}} = f(x, u)$ together with three different boundary conditions (the Dirichlet, the periodic and the Navier boundary conditions). We discuss the existence results for these boundary value problems at resonance. Our results contain the Landesman–Lazer type conditions. We also show some numerical results concerning *Fučík's spectrum* for the boundary value problems for the differential equation $u^{\mathrm{IV}} = \mu u^+ - \nu u^-$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, together with our three boundary conditions.

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1. Introduction

In this paper, we introduce some results concerning the boundary value problems for a fourth order differential equation. These results are the main results of the diploma thesis [4] that consists of three parts. The first part deals with the regularity problem of weak solutions, the second one describes *Fučík's spectrum* and the third one concerns the existence of at least one weak solution of our boundary value problems at resonance. This paper covers only the second and the third parts of the thesis [4].

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2. Fučík's spectrum

In this section, we investigate *Fučík's spectrum* of the boundary value problems for a fourth order differential equation. Let us consider a differential operator $L : D(L) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$, where Ω is a bounded domain with a smooth boundary. We define its *Fučík's spectrum* as the following set

$$A_{-1}(L) = \{(\mu,\nu) \in \mathbb{R}^2 \mid Lu = \mu u^+ - \nu u^- \text{ has a nontrivial solution}\},$$

where $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$ are the positive and the negative parts of the function u. Let us denote the spectrum of L by

$$\sigma(L) = \{\lambda \in \mathbb{R} \mid Lu = \lambda u \text{ has a nontrivial solution}\}.$$

Then we have $\{(\lambda,\lambda) \in \mathbb{R}^2 \mid \lambda \in \sigma(L)\} \subseteq A_{-1}(L)$ and therefore we can regard *Fučík's spectrum* $A_{-1}(L)$ as a generalization of the spectrum $\sigma(L)$.

In our case, the differential operator L is defined by

$$Lu(x) = \frac{d^4u}{dx^4} \quad \text{for all } u \in D(L).$$

So, the main goal of our investigation will be the boundary value problems for the fourth order differential equation

$$u^{\mathrm{IV}} = \mu u^+ - \nu u^- \tag{1}$$

together with different type of boundary conditions. The knowledge of *Fučík's spectrum* is essential for studying various mathematical models, especially models with jumping nonlinearities (see e.g. [5] for some concrete applications).

Fučík's spectrum of the boundary value problems for the second order differential equation

$$u'' + \mu u^+ - \nu u^- = 0$$

together with the periodic or the Dirichlet boundary conditions is well known and can be described analytically by some explicit formulas (see [2]). But in the case of the boundary value problems for the fourth order differential equation (1), the situation is absolutely different and much more complicated. First of all, concerning these boundary value problems, we cannot describe corresponding *Fučík's spectrum* by some analytic explicit formulas, and only some kinds of its qualitative properties are known (see the papers [3], [1]). Note that in the recent paper [1], the asymptotic behavior of *Fučík's spectrum* is also studied.

2.1. The periodic boundary value problem

Let us consider the periodic boundary value problem of the form

$$\begin{cases} u^{\mathrm{IV}}(x) = \lambda u(x), \quad x \in [0,2\pi], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi). \end{cases} \tag{2}$$

The eigenvalues of this boundary value problem (2) form the sequence

$$\lambda_k = k^4, \qquad k = 0, 1, 2, 3, \dots \tag{3}$$

The eigenvalues λ_k, $k = 1, 2, 3, \dots$ are of multiplicity 2 and two linearly independent orthogonal eigenfunctions correspond to each of them. We denote these orthogonal eigenfunctions by $v_{k,1}$ and $v_{k,2}$. They are of the form

$$v_0(x) = 1, \qquad v_{k,1}(x) = \sin kx, \quad v_{k,2}(x) = \cos kx, \qquad k = 1, 2, 3, \dots \tag{4}$$

2.1.1. Fučík's spectrum Let us consider the periodic boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) = a^4 u^+(x) - b^4 u^-(x), \quad x \in [0, 2\pi], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi). \end{cases} \tag{5}$$

For further considerations, let us consider the direct periodic extension to the whole real line $\mathbb{R}$ of each solution u of this boundary value problem (5). Let us denote $\varphi \in](3/4)\pi, \pi[$ the smallest positive root of the equation $\tan x + \tanh x = 0$. Further, let us define the auxiliary functions f and g by the formulas

$$f(x) = \frac{\cosh x \cos x}{\cosh x \sin x + \sinh x \cos x}, \quad g(x) = \frac{\cosh x \sin x - \sinh x \cos x}{\cosh x \sin x + \sinh x \cos x} \tag{6}$$

for $x \in]0, \varphi[$. The following theorem, which is proved in the paper [3] (some corrections of the analytical bounds for the spectrum is given in [4]), provides the description of the first branch of *Fučík's spectrum.*

Theorem 1. *The set S_1 of all pairs $(a, b) \in]0, +\infty[^2$ such that there exists a nontrivial 2π-periodic solution of the boundary value problem (5), which is composed of two semi-waves, is a curve $(a, b(a))$, where $b(a)$ is a decreasing C^∞-function defined in $]\varphi/\pi, +\infty[$ with $\lim_{a\to+\infty} b(a) = \varphi/\pi$.*

The curve S_1 is symmetric with respect to the straight line $b = a$ and fulfils $S_1 \subset G_1$, where G_1 is the set of all pairs $(a, b) \in]\varphi/\pi, +\infty[^2$ such that

$$\text{for } b \ge a,\ \left[\alpha(a,b) \ge \frac{\pi}{2},\ \xi(a,b) \ge 0\right] \vee \left[\alpha(a,b) < \frac{\pi}{2},\ \xi(a,b) \ge 0 \ge \psi(a,b)\right], \tag{7}$$

$$\text{for } b \le a,\ \left[\beta(a,b) \ge \frac{\pi}{2},\ \psi(a,b) \ge 0\right] \vee \left[\beta(a,b) < \frac{\pi}{2},\ \psi(a,b) \ge 0 \ge \xi(a,b)\right], \tag{8}$$

where

$$\alpha(a,b) = b\pi\left(1 - \frac{1}{2a}\right),\ \beta(a,b) = a\pi\left(1 - \frac{1}{2b}\right),$$

$$\xi(a,b) = \left(\frac{b}{a}\right)^2 - g\left(\pi a\left(1 - \frac{1}{2b}\right)\right),\ \psi(a,b) = \left(\frac{a}{b}\right)^2 - g\left(\pi b\left(1 - \frac{1}{2a}\right)\right).$$

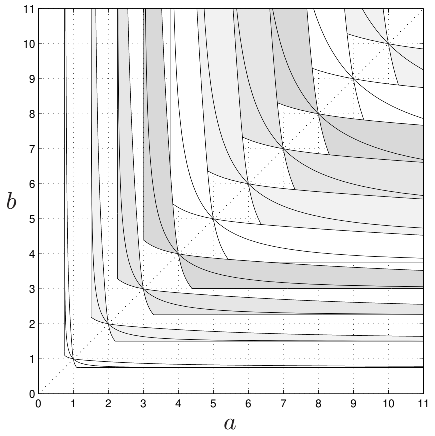

Fig. 1: The correct bounds (7), (8), (9).

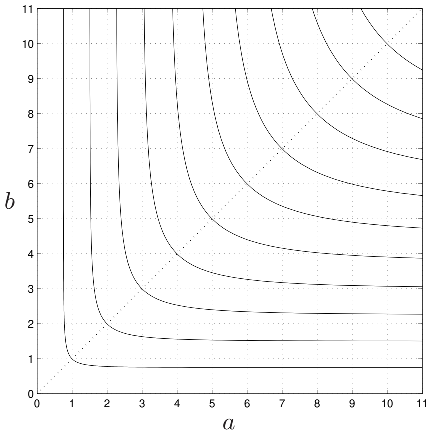

Fig. 2: *Fučík's spectrum* for the BVP (5).

The analytical bounds (7) and (8) are shown in the Figure 1. By virtue of previous Theorem 1, we can summarize the actual knowledge of *Fučík's spectrum* for the periodic boundary value problem (5) into the following items (see also [3]):

1. The set S of all pairs $(a,b) \in \,]0,+\infty[^2$, for which there exists a nontrivial 2π-periodic solution of the boundary value problem (5), is the countable set $\{S_k, k \in \mathbb{N}\}$ of C^∞-curves, where $S_k = \{(a,b) \in \,]0,+\infty[^2, (a/k, b/k) \in S_1\}$ for $k = 2,3,\dots$, the description of the curve S_1 is given by Theorem 1.
2. The inclusion $S_k \subset G_k$ holds for all $k \in \mathbb{N}$, where

   $$G_k = \{(a,b) \in \,]0,+\infty[^2, (a/k, b/k) \in G_1\}. \tag{9}$$

   The set G_1 is defined in Theorem 1. Thus we obtain

   $$S \subset \bigcup_{k=1}^{+\infty} G_k.$$

3. For the pair $(a,b) \in S_k$, the corresponding 2π-periodic nontrivial solutions of the boundary value problem (5) have exactly $2k$ semi-waves in an interval of the length 2π. This solution is unique if the translation in the direction of the x-axes and positive multiples are not considered.

Then *Fučík's spectrum* for the periodic boundary value problem (5) is the set

$$A_{-1} = \{(a^4, b^4) \in \mathbb{R}^2 \mid (a,b) \in S\} \cup \{S_0^x, S_0^y\},$$

where S_0^x (or S_0^y, respectively) is just x-axes (y-axes, respectively). The corresponding nontrivial solutions of the boundary value problems (5) for the pairs $(a,b) \in S_0^x$ (S_0^y) are arbitrary constants $c < 0$ ($c > 0$).

2.1.2. The description of the algorithm The algorithm how to generate the points of *Fučík's spectrum* A_{-1} with some specific accuracy is in details described in [4]. It is obvious from the previous considerations that if we are able to generate the points of the set S_1 that determine the first branch of *Fučík's spectrum*, then we are able to generate automatically the other branches of *Fučík's spectrum*. It can be shown (see [3]) that the set S_1 is described by the system of two nonlinear equations

$$\begin{aligned} af(ar) + bf(b(\pi - r)) &= 0, \\ a^2 g(ar) - b^2 g(b(\pi - r)) &= 0. \end{aligned} \tag{10}$$

The principle of the algorithm is such that for the chosen fixed $r \in (\pi/2, \pi)$ we compute the parameters a and b of the system (10) numerically with some accuracy. This provides the approximation of one pair $(a, b) \in S_1$. For the complete description of the algorithm see thesis [4].

2.2. The Navier boundary value problem

Let us consider the boundary value problem of the form

$$\begin{cases} u^{\mathrm{IV}}(x) = \lambda u(x), \quad x \in [0, \pi], \\ u(0) = u''(0) = u(\pi) = u''(\pi) = 0. \end{cases} \tag{11}$$

The eigenvalues of this boundary value problem (11) and the corresponding eigenfunctions are

$$\lambda_k = k^4, \qquad v_k(x) = \sin kx, \qquad k = 1, 2, 3, \dots.$$

2.2.1. Fučík's spectrum Let us consider the boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) = a^4 u^+(x) - b^4 u^-(x), \quad x \in [0, \pi], \\ u(0) = u''(0) = u(\pi) = u''(\pi) = 0. \end{cases} \tag{12}$$

Fučík's spectrum of this boundary value problem (12) is the set

$$A_{-1} = \{(a^4, b^4) \in \mathbb{R}^2 \mid (a, b) \in S\},$$

where S is the system of continuous curves $S = \{S_i^+, S_i^-, i \in \mathbb{N}\}$ with the following properties (see [3]):

1. Let $(a, b) \in S_i^+$ (S_i^-), then the solution u of the boundary value problem (12) is the solution of the initial value problem

$$\begin{cases} u^{\mathrm{IV}}(x) = a^4 u^+(x) - b^4 u^-(x), \quad x \in [0, +\infty[, \\ u(0) = 0, \quad u'(0) = \alpha, \quad u''(0) = 0, \quad u'''(0) = t, \end{cases} \tag{13}$$

   with $\alpha > 0$ $(\alpha < 0)$ and with some $t \in \mathbb{R}$. This solution u is uniquely determined by the choice of the constant α and has exactly $(i + 1)$ zeros in the interval $[0, \pi]$.

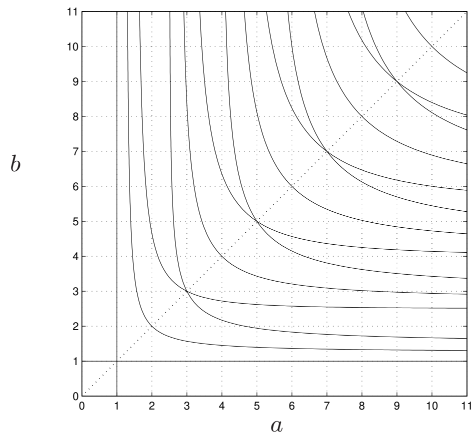

Fig. 3. *Fučík's spectrum* for the Navier BVP (12).

2. The curves S_i^+ and S_i^- are mutually symmetric with respect to the straight line $a = b$. If i is even, then $S_i^+ = S_i^-$.
3. For all $i \in \mathbb{N}$, $(S_i^+ \cup S_i^-) \cap (S_{i+1}^+ \cup S_{i+1}^-) = \emptyset$ holds.

2.2.2. The description of the algorithm We will try to explain the main idea of the algorithm for generating *Fučík's spectrum* for the easiest case. This means, we consider the second branch S_2^+ that merges in the curve S_2^-, which follows from the properties of the spectrum that we mentioned in the previous Section 2.2.1. If we restrict our attention only to the second branch S_2^+, then we know that the corresponding solutions u of the boundary value problem (12) will have exactly 3 zeros in the interval $[0, \pi]$. Further, we know that the curve S_2^+ is passing through the point $(\sqrt[4]{\lambda_2}, \sqrt[4]{\lambda_2}) = (2, 2)$ and the corresponding nontrivial solution of the boundary value problem (12) is then $v_2(x) = \sin 2x$. Due to the symmetry of *Fučík's spectrum* with respect to the straight line $a = b$, we can concentrate hereafter only on the case $a \geq b$.

We will try to find the inspiration in the classical shooting methods, which are based on a transformation of a boundary value problem into a sequence of some initial value problems. Our attention will be therefore concentrated on the initial value problem (13). There are four parameters a, b, α and t in the initial value problem (13). We will try to determine these parameters in such a way that the corresponding solution u of the problem (13) will be the solution of the boundary value problem (12) and in the interval $[0, \pi]$ will have exactly 3 zeros. If u is the solution of the boundary value problem (12), then an arbitrary positive multiple of u is also its solution. This fact can be expressed just by the parameter α. Let us therefore choose an arbitrary, but fixed value of the parameter α such that $\alpha > 0$, because we are studying the curve S_2^+. Our goal is now to find the corresponding values of the parameters b and t (for the chosen parameter

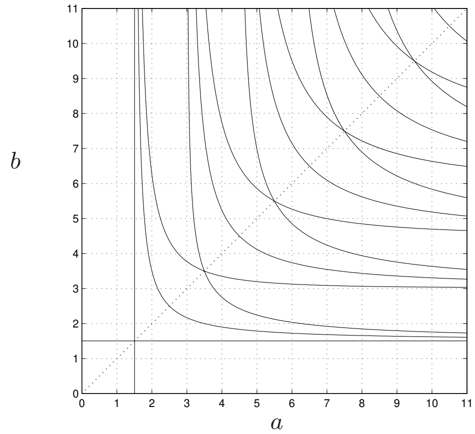

Fig. 4. *Fučík's spectrum* for the Dirichlet BVP (17).

a) such that the point (a, b) will be the point of the curve S_2^+. For more details see the second part of the thesis [4], where the complete form of the algorithm can be also found.

2.3. The Dirichlet boundary value problem

Let us consider the eigenvalue problem for the Dirichlet boundary value problem of the form

$$\begin{cases} u^{\mathrm{IV}}(x) = \lambda u(x), \quad x \in [0, \pi], \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0. \end{cases} \tag{14}$$

The eigenvalues λ_k of this boundary value problem (14) are given by

$$\lambda_k = \varphi_k^4, \quad \text{where} \quad \cos\varphi_k\pi \cosh\varphi_k\pi = 1, \quad \varphi_k \neq 0, \qquad k = 1, 2, 3, \ldots \tag{15}$$

and the corresponding eigenfunctions are

$$\begin{aligned} v_k(x) = {} & [\cosh\varphi_k\pi - \cos\varphi_k\pi][\sinh\varphi_k x - \sin\varphi_k x] - \\ & -[\sinh\varphi_k\pi - \sin\varphi_k\pi][\cosh\varphi_k x - \cos\varphi_k x]. \end{aligned} \tag{16}$$

2.3.1. Fučík's spectrum Let us consider the boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) = a^4 u^+(x) - b^4 u^-(x), \quad x \in [0, \pi], \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0. \end{cases} \tag{17}$$

Fučík's spectrum of the Dirichlet boundary value problem (17) has similar properties as *Fučík's spectrum* of the previous Navier boundary value problem (12). This can be also observed if we compare the Figures 3 and 4. The algorithm for generating *Fučík's spectrum* of the Dirichlet boundary value problem (17) is analogous to the algorithm for the previous problem (12) (see [4]).

2.4. The implementation of the algorithms

The algorithms stated in this paper can be easily modified for the problems with other boundary conditions. In general, it is possible to say that for realization of the algorithms for generating *Fučík's spectrum* it is necessary to perform the individual steps of the computations with relatively high accuracy; the higher accuracy, the higher number of the branches of *Fučík's spectrum* we would like to generate.

The mentioned algorithms for generating *Fučík's spectrum* of our three boundary value problems (the periodic boundary value problem (5), the Navier boundary value problem (12) and the Dirichlet boundary value problem (17)) were implemented in Fortran 77 on the parallel computer cluster Lyra.

Due to the required higher accuracy, for the computations generating the higher branches of *Fučík's spectrum*, the mentioned algorithms were implemented also in the system Mathematica 3.0. The algorithms were included into the system of procedures for modelling of bifurcations (MBx). For more results of our numerical experiments visit the internet site

`http://cam.zcu.cz/members/necesal/index.cz.shtml`.

3. Existence results

Let us consider the boundary value problems for the fourth order nonlinear differential equation

$$u^{\mathrm{IV}} = f(x, u)$$

together with three different boundary conditions (the Dirichlet, the periodic and the Navier boundary conditions). In this section, we discuss the existence results for these boundary value problems at resonance. Our results rely on the Landesman–Lazer type conditions.

3.1. The Dirichlet boundary value problem

Let us consider the boundary value problem of the form

$$\begin{cases} u^{\mathrm{IV}}(x) - \lambda_m u(x) + g(x, u(x)) = f(x), & x \in [0, \pi], \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0, \end{cases} \tag{18}$$

where $g : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ is the *Carathéodory function*, the right hand side $f \in \mathrm{L}^1(0, \pi)$, λ_m is the eigenvalue of the boundary value problem (14) (see the relation (15)).

Henceforth we will assume that the function $g = g(x, s)$ satisfies the following *growth condition*. Let us suppose that there exist the function $p \in \mathrm{L}^1(0, \pi)$ and the constant $C > 0$ such that the inequality

$$|g(x, s)| \le p(x) + C|s| \tag{19}$$

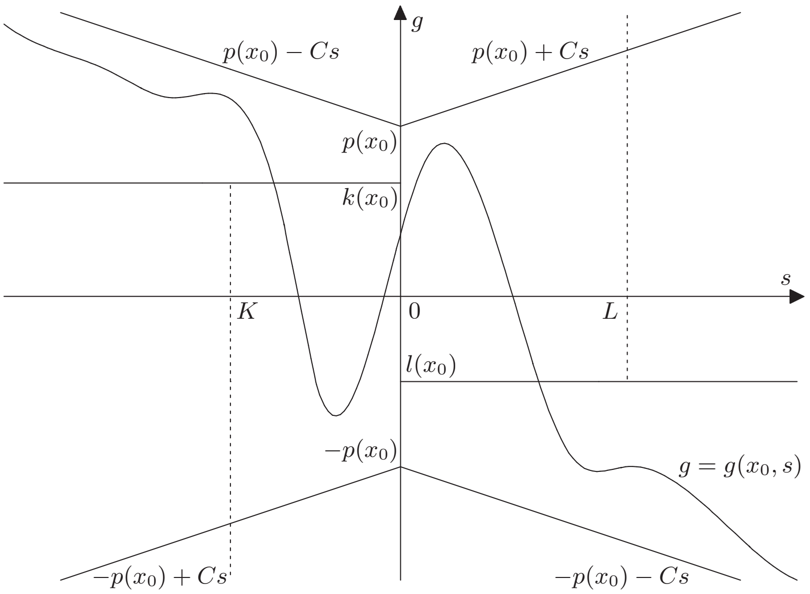

Fig. 5. The illustration of the conditions (19), (20) and (21) for fixed $x_0 \in [0, \pi]$.

holds for all $s \in \mathbb{R}$ and for a. a. $x \in [0, \pi]$. Moreover, let us suppose that there exist the functions $k, l \in \mathrm{L}^1(0, \pi)$ and the constants $K, L \in \mathbb{R}$, $K < 0 < L$, such that

$$g(x, s) \geq k(x) \quad \text{for all } s \leq K \text{ and for a. a. } x \in [0, \pi], \tag{20}$$

$$g(x, s) \leq l(x) \quad \text{for all } s \geq L \text{ and for a. a. } x \in [0, \pi]. \tag{21}$$

Let us denote $\mathrm{H} = \mathrm{W}_0^{2,2}(0, \pi)$ the Sobolev space on the interval $]0, \pi[$ with the inner product and the norm

$$(u, v) = \int_0^\pi u''(x)v''(x)\,dx \quad \text{and} \quad \|u\| = \sqrt{(u, u)}, \quad \text{respectively.}$$

We say that u is the *weak solution* of the boundary value problem (18), if $u \in \mathrm{H}$ and the integral identity

$$\int_0^\pi u''(x)v''(x)\,dx - \lambda_m \int_0^\pi u(x)v(x)\,dx + \int_0^\pi g(x, u(x))v(x)\,dx = \int_0^\pi f(x)v(x)\,dx$$

holds for all $v \in \mathrm{H}$.

Theorem 2 (Sublinear growth). *Let us suppose that the function $g = g(x, s)$ satisfies all assumptions stated above and, moreover,*

$$\lim_{s \to \pm\infty} \frac{g(x, s)}{s} = 0 \tag{22}$$

uniformly for a. a. $x \in [0,\pi]$. *Denote*

$$g^{+\infty}(x) = \limsup_{s\to+\infty} g(x,s), \qquad g_{-\infty}(x) = \liminf_{s\to-\infty} g(x,s).$$

Then the boundary value problem (18) *has at least one weak solution provided the Landesman–Lazer type condition*

$$\int_0^\pi g^{+\infty}(x)v_m^+(x)\,dx - \int_0^\pi g_{-\infty}(x)v_m^-(x)\,dx < \int_0^\pi f(x)v_m(x)\,dx <$$

$$< \int_0^\pi g_{-\infty}(x)v_m^+(x)\,dx - \int_0^\pi g^{+\infty}(x)v_m^-(x)\,dx$$

holds.

Proof. The proof is based on the Leray-Schauder degree theory (see [4]).

3.2. The Navier boundary value problem

Let us consider the boundary value problem

$$\begin{cases} u^{\mathrm{IV}}(x) - \lambda_m u(x) + g(x,u(x)) = f(x), & x \in [0,\pi],\\ u(0) = u''(0) = u(\pi) = u''(\pi) = 0, \end{cases} \tag{23}$$

where $g : [0,\pi]\times\mathbb{R} \to \mathbb{R}$ is the *Carathéodory function* satisfying the assumptions from Section 3.1, the right hand side $f \in \mathrm{L}^1(0,\pi)$, $\lambda_m = m^4$ for $m \in \mathbb{N}$ is the eigenvalue of the boundary value problem (11).

Let us denote $\mathrm{H} = \{u \in \mathrm{W}^{2,2}(0,\pi);\ u(0) = u(\pi) = 0\} = \mathrm{W}^{2,2}(0,\pi) \cap \mathrm{W}_0^{1,2}(0,\pi)$ the space with the inner product and the norm

$$(u,v) = \int_0^\pi [u''(x)v''(x) + u(x)v(x)]\,dx, \quad \text{and} \quad \|u\| = \sqrt{(u,u)}, \quad \text{respectively.}$$

We say that u is the *weak solution* of the boundary value problem (23), if $u \in \mathrm{H}$ and the integral identity

$$\int_0^\pi u''(x)v''(x)\,dx - \lambda_m \int_0^\pi u(x)v(x)\,dx + \int_0^\pi g(x,u(x))v(x)\,dx = \int_0^\pi f(x)v(x)\,dx$$

holds for all $v \in \mathrm{H}$.

Theorem 3 (Sublinear growth). *Let us suppose that the Carathéodory function* $g = g(x,s)$ *satisfies* (19) – (22). *Then the boundary value problem* (23) *has at least one weak solution provided the Landesman–Lazer type condition*

$$\int_0^\pi g^{+\infty}(x)(\sin mx)^+\,dx - \int_0^\pi g_{-\infty}(x)(\sin mx)^-\,dx < \int_0^\pi f(x)\sin mx\,dx <$$

$$< \int_0^\pi g_{-\infty}(x)(\sin mx)^+\,dx - \int_0^\pi g^{+\infty}(x)(\sin mx)^-\,dx$$

holds.

Proof. The proof is analogous to the proof of Theorem 2 (see [4]).

3.3. The periodic boundary value problem

In this section, we will consider the periodic boundary value problem

$$\text{(24)}\qquad \begin{cases} u^{\mathrm{IV}}(x) - \lambda_m u(x) + g(x,u(x)) = f(x), \quad x \in [0,2\pi], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\ u''(0) = u''(2\pi),\ u'''(0) = u'''(2\pi), \end{cases}$$

where $g : [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ is the *Carathéodory function*, the right hand side $f \in \mathrm{L}^1(0,2\pi)$, $\lambda_m = m^4$ for $m \in \mathbb{N}$ is the eigenvalue of the boundary value problem (2). Moreover, let us suppose that the function $g = g(x,s)$ satisfies all assumptions for the function g in the Section 3.1 with $[0,\pi]$ replaced by $[0,2\pi]$. In particular, this means that the growth condition (19) and the conditions (20), (21) hold with the replacement of the interval $[0,\pi]$ by $[0,2\pi]$.

Let us denote $\mathrm{H} = \{u \in \mathrm{W}^{2,2}(0,2\pi);\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}$ the space with the inner product and the norm

$$(u,v) = \int_0^{2\pi} [u''(x)v''(x) + u(x)v(x)]\,dx \quad \text{and} \quad \|u\| = \sqrt{(u,u)}, \quad \text{respectively.}$$

We say that u is the *weak solution* of the boundary value problem (24), if $u \in \mathrm{H}$ and the integral identity

$$\int_0^{2\pi} u''(x)v''(x)\,dx - \lambda_m \int_0^{2\pi} u(x)v(x)\,dx + \int_0^{2\pi} g(x,u(x))v(x)\,dx = \int_0^{2\pi} f(x)v(x)\,dx$$

holds for all $v \in \mathrm{H}$.

Theorem 4 (Sublinear growth). *Let us suppose that the function $g = g(x,s)$ satisfies all assumptions stated above and, moreover, the growth condition* (22) *holds uniformly for a. a. $x \in [0,2\pi]$. Then the boundary value problem* (24) *has at least one weak solution provided the Landesman–Lazer type condition*

$$\int_{v>0} g^{+\infty}(x)v(x)\,dx + \int_{v<0} g_{-\infty}(x)v(x)\,dx < \int_0^{2\pi} f(x)v(x)\,dx$$

holds for all $v \in \mathrm{Span}\{\cos mx, \sin mx\} \setminus \{0\}$.

Proof. The proof is analogous to the proof of Theorem 2 (see [4]).

3.4. The reverse growth of the nonlinearity

Let us suppose that in the case of the Dirichlet boundary value problem (18) the function $g = g(x, s)$ satisfies

$$g(x, s) \le k(x) \quad \text{for all } s \le K \text{ and for a. a. } x \in [0, \pi], \tag{25}$$

$$g(x, s) \ge l(x) \quad \text{for all } s \ge L \text{ and for a. a. } x \in [0, \pi], \tag{26}$$

instead of the conditions (20) and (21). The meaning of k, K, l and L is the same as in the Section 3.1. Note that the hypotheses (25), (26) are in a certain sense *dual* to the assumptions (20), (21). In this case we can formulate the *dual version* of Theorem 2.

Theorem 5 (Sublinear growth). *Let us suppose that the function $g = g(x, s)$ satisfies* (19), (22) *and the conditions* (25), (26). *Then the boundary value problem* (18) *has at least one weak solution provided the Landesman–Lazer type condition*

$$\int_0^\pi g^{-\infty}(x) v_m^+(x)\, dx - \int_0^\pi g_{+\infty}(x) v_m^-(x)\, dx < \int_0^\pi f(x) v_m(x)\, dx <$$

$$< \int_0^\pi g_{+\infty}(x) v_m^+(x)\, dx - \int_0^\pi g^{-\infty}(x) v_m^-(x)\, dx$$

holds, where

$$g^{-\infty}(x) = \limsup_{s \to -\infty} g(x, s), \qquad g_{+\infty}(x) = \liminf_{s \to +\infty} g(x, s).$$

Proof. The proof follows the lines of that of Theorem 2 (see [4]).

The main difference between Theorem 2 and its *dual version* Theorem 5 is in different form of the Landesman-Lazer type condition. For the *dual* formulations in the cases of our two remaining boundary value problems see thesis [4].

References

1. Campos, J., Dancer, E. N. : On the Resonance Set in a Fourth Order Equation with Jumping Nonlinearity, preprint.
2. Drábek, P. : Solvability and Bifurcations of Nonlinear Equations. Pitman Research Notes in Mathematics Series 264, Longman, Harlow 1992.
3. Krejčí, P. : On Solvability of Equations of the 4^{th} Order with Jumping Nonlinearities. Časopis pro pěstování matematiky, 29 – 39, roč. 108 , Prague 1983.
4. Nečesal, P. : Nonlinear Differential Equations of the 4^{th} Order – Solvability, Regularity of Solutions, Fučík's spectrum and Landesman-Lazer Type Conditions. Diploma thesis (in Czech), University of West Bohemia, Pilsen 2000.
5. Tajčová, G. : Mathematical models of suspension bridges. Appl. Math. 42 (1997), No. 6, 451–480.