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THE l^p TRICHOTOMY FOR DIFFERENCE SYSTEMS AND APPLICATIONS

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Abstract. The notion of l^p trichotomy for a linear difference system is here considered as extension of exponential trichotomy and l^p dichotomy. The main properties are analyzed and necessary and sufficient conditions for the existence are given. The asymptotic behavior of solutions of a quasilinear system $x(n+1) = A(n)x(n) + f(n, x(n))$ is studied under the assumption that the associated linear system possesses a l^p trichotomy.

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1. Introduction

Consider the nonlinear difference system in $\mathbb{R}^m$

$$y(n+1) = A(n)y(n) + f(n, y(n)), \qquad n \in \mathbb{Z} \tag{1}$$

where $A(n)$ is a $m \times m$ invertible matrix for every $n \in \mathbb{Z}$ and f is a continuous function from $\mathbb{Z} \times \mathbb{R}^m$ into $\mathbb{R}^m$. Our aim is to study the existence of bounded solutions of (1) having zero limit as $n \to \pm\infty$, under the assumption that the solutions of the associated linear (homogeneous) system

$$x(n+1) = A(n)x(n), \qquad n \in \mathbb{Z} \tag{2}$$

are not all bounded on $\mathbb{Z}$.

In the continuous case the study of the existence on the whole real line of zero convergent solutions as $t \to \pm\infty$ of a linear differential system often has been accomplished by introducing suitable assumptions on the asymptotic behavior of a fundamental matrix. For instance in [15] the notion of S-S trichotomy is introduced and is employed to study the existence of invariant splittings for linear differential systems. Later a stronger notion of trichotomy, namely exponential trichotomy, was introduced in [8], still in the continuous case. These notions were extended afterwards to the discrete case and during the last years many authors dealt with exponential or ordinary trichotomy of difference systems, giving necessary and sufficient conditions for the existence, proving the roughness and applying these results to nonlinear difference systems, see for instance [2], [9], [11]. We refer the reader to [7] for the basic theory of dichotomies and to [1] for the extension to difference equations.

Here, in section 2, l^p trichotomy for a linear system (2) will be introduced and the main asymptotic properties of the solutions of this system will be analyzed. We point out that l^p trichotomy can be considered as an extension to the l^p spaces of exponential trichotomy, as well as l^p dichotomy is an extension of exponential dichotomy [18].

In section 3 the boundary value problem

$$\begin{cases} y(n+1) = A(n)y(n) + f(n, y(n)), \qquad n \in \mathbb{Z} \\ y(+\infty) = 0, \quad y(-\infty) = 0 \end{cases} \tag{3}$$

will be considered, assuming that the associated linear system has a l^p trichotomy and using a topological approach based on Schauder-Tychonoff fixed point theorem.

The results obtained extend some of the results in [10], [12]–[14], [18] and improve some of those in [9], [2], [11]. A comparison will be made throughout the paper.

2. l^p TRICHOTOMY FOR LINEAR DIFFERENCE SYSTEMS

Let $X(n)$ be a fundamental matrix of (2). We recall the definitions of l^p, exponential and ordinary dichotomy for reader's convenience.

Definition 1 ([18], [10], [12]–[14]). System (2) is said to have a l^p *dichotomy on* $\mathbb{Z}^+ = \{0, 1, 2, \cdots\}$, $1 \le p < \infty$, if there exist a projection P^+ and a constant $K^+ > 0$ such that for every $n \in \mathbb{Z}^+$

$$\begin{gathered} \left[\sum_{s=-1}^{n-1} |X(n)P^+X^{-1}(s+1)|^p\right]^{1/p} < K^+ \\ \left[\sum_{s=n-1}^{\infty} |X(n)(I-P^+)X^{-1}(s+1)|^p\right]^{1/p} < K^+. \end{gathered} \tag{4}$$

Analogously system (2) has a l^p *dichotomy on* $\mathbb{Z}^- = \{0, -1, -2, \cdots\}$, $1 \le p < \infty$, if there exist a projection P^- and a constant $K^- > 0$ such that

$$\begin{aligned} &\left[\sum_{s=-\infty}^{n-1} |X(n)P^- X^{-1}(s+1)|^p\right]^{1/p} < K^- \\ &\left[\sum_{s=n-1}^{-1} |X(n)(I-P^-)X^{-1}(s+1)|^p\right]^{1/p} < K^-. \end{aligned} \tag{5}$$

System (2) has an *exponential dichotomy on* $\mathbb{Z}^+$ if there exist a projection P_0 and constants $M > 0$, $0 < \beta < 1$ such that

$$|X(n)P_0X^{-1}(s)| < M\beta^{n-s}, \quad 0 \le s \le n$$

$$|X(n)(I-P_0)X^{-1}(s)| < M\beta^{s-n}, \quad 0 \le n \le s.$$

The exponential dichotomy on $\mathbb{Z}^-$ is defined in a similar way. If the above two inequalities hold with $\beta = 1$, then system (2) has an *ordinary dichotomy on* $\mathbb{Z}^+$. Clearly ordinary dichotomy is equivalent to l^∞ dichotomy.

The above mentioned notions of dichotomy can be regarded as kinds of conditional stability in future for the linear system (2). In particular system (2) is uniformly stable (in future) if and only if it has an ordinary dichotomy on $\mathbb{Z}^+$ with projection the identity operator, it is asymptotically uniformly stable (in future) if and only if it has an exponential dichotomy on $\mathbb{Z}^+$ with projection the identity operator, it is l^p stable (in future) if and only if it has a l^p dichotomy on $\mathbb{Z}^+$ with projection the identity operator ([17], see also [4], [5]).

If one is interested in the asymptotic behavior of the solutions of (2) both in the future and in the past, then it may be useful to generalize the above kinds of dichotomies. For instance in [9], [11] the exponential trichotomy is considered as a generalization of exponential dichotomy on $\mathbb{Z}$ and it is employed to study the asymptotic behavior in the future and in the past of the solutions of perturbed difference systems. Analogously it is possible to generalize the l^p dichotomy on $\mathbb{Z}$ in the following way:

Definition 2. System (2) is said to have a l^p trichotomy on $\mathbb{Z}$ with $1 \le p < \infty$, if there exist three mutually orthogonal projections P_1, P_2, P_3, with $P_1+P_2+P_3 = I$, and a constant $K > 0$, such that

$$\begin{aligned} &\left[\sum_{s=-\infty}^{n-1} |X(n)P_1X^{-1}(s+1)|^p\right]^{1/p} < K \\ &\left[\sum_{s=n-1}^{\infty} |X(n)P_2X^{-1}(s+1)|^p\right]^{1/p} < K \\ &\left[\sum_{s=-1}^{n-1} |X(n)P_3X^{-1}(s+1)|^p\right]^{1/p} < K \qquad \text{for } n \ge 0 \end{aligned} \tag{6}$$

$$\left[\sum_{s=n-1}^{-1} |X(n)P_3X^{-1}(s+1)|^p\right]^{1/p} < K \qquad \text{for } n \leq 0.$$

It is worth to remark that the l^p trichotomy is a property that does not depend on the fixed fundamental matrix. Indeed, if $Y(n)$ is another fundamental matrix of (2), then there exists a nonsingular matrix C such that $X(n) = Y(n)C$ and $|X(n)P_jX^{-1}(s+1)| = |Y(n)CP_jC^{-1}Y^{-1}(s+1)|$. Only the projections depend on the fixed fundamental matrix.

From Definition 2 l^p trichotomy on $\mathbb{Z}$ implies l^p dichotomy on $\mathbb{Z}^+$ and on $\mathbb{Z}^-$, and l^p dichotomy on $\mathbb{Z}$ implies a trivial l^p trichotomy, with the projection $P_3 = 0$. In particular the following holds:

Proposition 1. *The following statements are equivalent:*

i) *System* (2) *has a l^p trichotomy on $\mathbb{Z}$, with projections P_1, P_2, P_3.*

ii) *There exist two projections P, Q, such that $PQ = QP$, $P + Q - PQ = I$ and a positive constant N, such that*

$$(7)\qquad \begin{aligned}
&\left[\sum_{s=-1}^{n-1} |X(n)PX^{-1}(s+1)|^p\right]^{1/p} < N, \qquad n \geq 0\\
&\left[\sum_{s=n-1}^{\infty} |X(n)(I-P)X^{-1}(s+1)|^p\right]^{1/p} < N\\
&\left[\sum_{s=n-1}^{-1} |X(n)QX^{-1}(s+1)|^p\right]^{1/p} < N, \qquad n \leq 0\\
&\left[\sum_{s=-\infty}^{n-1} |X(n)(I-Q)X^{-1}(s+1)|^p\right]^{1/p} < N.
\end{aligned}$$

iii) *System* (2) *has a l^p dichotomy on $\mathbb{Z}^+$ with projection P^+ and a l^p dichotomy on $\mathbb{Z}^-$ with projection P^-, such that $P^+P^- = P^-P^+ = P^-$. In addition the second inequality in* (4) *and the first one in* (5) *hold for every $n \in \mathbb{Z}$.*

Proof. i) $\Longrightarrow$ ii). Let $P = I - P_2$ and $Q = I - P_1$. It is trivial to check that $PQ = P_3 = QP$ and $P + Q - PQ = I$. The second and the fourth inequalities in (7) are immediately verified. With regard to the first one in (7) we have for $n \geq 0$

$$\begin{aligned}
\sum_{s=-1}^{n-1} |X(n)PX^{-1}(s+1)|^p &= \sum_{s=-1}^{n-1} |X(n)(P_1+P_3)X^{-1}(s+1)|^p\\
&\leq 2^{p-1}\sum_{s=-1}^{n-1}\Big(|X(n)P_1X^{-1}(s+1)|^p + |X(n)P_3X^{-1}(s+1)|^p\Big)\\
&< 2^pK^p.
\end{aligned}$$

Similarly the last one in (7) can be proved
ii) $\implies$ iii). Let $P^+ = P$ and $P^- = (I - Q)$. Then system (2) has a l^p dichotomy on $\mathbb{Z}^+$ with projection P^+, and a l^p dichotomy on $\mathbb{Z}^-$ with projection P^-. Further $P^+P^- = P(I-Q) = I - Q = P^- = P^-P^+$ and both the second inequality in (4) and the first one in (5) hold for every $n \in \mathbb{Z}$.
iii) $\implies$ i). Let $P_1 = P^-$, $P_2 = I - P^+$, $P_3 = P^+ - P^- = P^+(I - P^-) = (I - P^-)P^+$. Then clearly $P_1 + P_2 + P_3 = I$ and $P_iP_j = 0$ if $i \neq j$. The proof of the inequalities (6) is quite immediate; for the last two inequalities it is sufficient to observe that $P_3 = (I - P^-)P^+$ for the first one, and that $P_3 = P^+(I - P^-)$ for the second one. □

The equivalence between conditions (i), (iii) in Proposition 1 gives the following:

Corollary 1. *System (2) has a l^p trichotomy if and only if the following two conditions are satisfied:*

a) system (2) has a l^p dichotomy both on $\mathbb{Z}^+$ and on $\mathbb{Z}^-$;
b) every solution is the sum of two solutions, one bounded on $\mathbb{Z}^+$ and the other bounded on $\mathbb{Z}^-$.

Proposition 1 permits us to give a complete description of the asymptotic behavior of the solutions of (2), both in the future and in the past. More precisely we have

Theorem 1. *If system (2) has a l^p trichotomy, $1 \leq p < \infty$, with projections P_1, P_2, P_3 corresponding to the fundamental matrix $X(n)$ s.t. $X(0) = I$, then the m-dimensional space S of all the solutions of (2) can be written as direct sum*

$$S = B_k^+ \oplus B_r^- \oplus B_{m-k-r}^\pm$$

where

B_k^+ *is the k-dimensional subspace of solutions x such that $x(0) = \eta \in Range(P_1)$, where $k = Rank(P_1)$. If $x \in B_k^+$ then $x(+\infty) = 0$ and x is unbounded for $n \to -\infty$.*
B_r^- *is the r-dimensional subspace of solutions x such that $x(0) = \nu \in Range(P_2)$, where $r = Rank(P_2)$. If $x \in B_r^-$ then $x(-\infty) = 0$ and x is unbounded for $n \to +\infty$.*
$B_{m-k-r}^\pm$ *is the subspace of solutions x such that $x(0) = \mu \in Range(P_3)$, where $m - k - r = Rank(P_3)$. If $x \in B_{m-k-r}^\pm$ then $x(\pm\infty) = 0$.*

In particular a solution of (2) is bounded for all $n \in \mathbb{Z}$ if and only if it has zero limit as $n \to \pm\infty$.

Proof. If system (2) has a l^p trichotomy on $\mathbb{Z}$, with projections P_1, P_2, P_3, from Proposition 1, (2) has also a l^p dichotomy on $\mathbb{Z}^+$ with projection $P^+ = I - P_2$ and a l^p dichotomy on $\mathbb{Z}^-$ with projection $P^- = P_1$. In addition $P^+P^- = P^-P^+ = P^-$ and also $(I - P^-)(I - P^+) = (I - P^+)(I - P^-) = I - P^+$.

Let x be a solution of (2). Then $x(n) = X(n)P_1x(0)+X(n)P_2x(0)+X(n)P_3x(0)$. From ([16], [18]) and using the fact that (2) has a l^p dichotomy on $\mathbb{Z}^+$ and on $\mathbb{Z}^-$ we obtain

$$\lim_{n\to+\infty} |X(n)P_1x(0)| = \lim_{n\to+\infty} |X(n)P^-x(0)| = \lim_{n\to+\infty} |X(n)P^+P^-x(0)| = 0$$

$$\lim_{n\to-\infty} |X(n)P_1x(0)| = \lim_{n\to-\infty} |X(n)P^-x(0)| = +\infty \quad \text{ if } P_1x(0) \neq 0$$

$$\lim_{n\to+\infty} |X(n)P_2x(0)| = \lim_{n\to+\infty} |X(n)(I-P^+)x(0)| = +\infty \quad \text{ if } P_2x(0) \neq 0$$

$$\begin{aligned}\lim_{n\to-\infty} |X(n)P_2x(0)| &= \lim_{n\to-\infty} |X(n)(I-P^+)x(0)| \\ &= \lim_{n\to-\infty} |X(n)(I-P^-)(I-P^+)x(0)| = 0\end{aligned}$$

$$\lim_{n\to+\infty} |X(n)P_3x(0)| = \lim_{n\to+\infty} |X(n)P^+(I-P^-)x(0)| = 0$$

$$\lim_{n\to-\infty} |X(n)P_3x(0)| = \lim_{n\to-\infty} |X(n)(I-P^-)P^+x(0)| = 0.$$

This ends the proof of the first part of the proposition. To prove the second assertion it is sufficient to observe that necessarily $P_1x(0) = P_2x(0) = 0$ in order to have a solution of (2) bounded on all $\mathbb{Z}$. □

As l^p trichotomy is more general than exponential trichotomy, the previous results extend the correspondent ones in [9], [2], [11]. Further the notion of trichotomy allows to consider the behavior of the solutions of (2) on the whole set $\mathbb{Z}$, therefore the results in Theorem 1 imply the corresponding ones in [18].

Remark 1. It is also possible to give an estimate of the rate of convergence towards zero of the various terms, see [18], [16].

3. Applications to nonlinear boundary value problems

Suppose that (2) has a l^p trichotomy and consider the associated nonlinear system (1). The following holds:

Proposition 2. *Assume:*

i) system (2) *has a l^p trichotomy, $1 \leq p < \infty$, with projections P_1, P_2, P_3 associated with the fundamental matrix $X(n)$ s.t. $X(0) = I$;*
ii) there exists a function $g : \mathbb{Z} \times \mathbb{R}^+ \mapsto \mathbb{R}^+$, continuous with respect to the second variable $\forall n \in \mathbb{Z}$ and such that

$$|f(n,c)| \leq g(n,|c|), \quad n \in \mathbb{Z}, c \in \mathbb{R}^m \tag{8}$$

$$\max_{v\in[0,r]} g(n,v) = g_r(n) \in l^q, \quad r \in \mathbb{R}^+,\ 1/p+1/q = 1,\ (p = 1, q = \infty). \tag{9}$$

Then every bounded solution of (1) *is solution of*

$$\begin{aligned} y(n) =& X(n)P_3y(0) + \sum_{s=-\infty}^{n-1} X(n)P_1X^{-1}(s+1)f(s,y(s)) \\ &- \sum_{s=n}^{+\infty} X(n)P_2X^{-1}(s+1)f(s,y(s)) + \sum_{s=0}^{n-1} X(n)P_3X^{-1}(s+1)f(s,y(s)) \\ &- \sum_{s=n}^{-1} X(n)P_3X^{-1}(s+1)f(s,y(s)) \end{aligned} \tag{10}$$

(with the convention $\sum_{s=a}^{b} g(s) = 0$ *if* $a > b$*) and vice versa.*

Proof. The assertion is an easy consequence of the variation of constants formula. We only sketch the proof.

Let u be a bounded solution of (1). From i) and ii) we get

$$\left| \sum_{s=n}^{+\infty} X(n)P_2X^{-1}(s+1)f(s,u(s)) \right| \le K\|g_{\|u\|_\infty}\|_q, \quad 1 < q \le \infty$$

where K is the trichotomy constant (see Definition 2). Then for $n \ge 0$ we can write

$$\begin{aligned} u(n) =& X(n)P_1u(0) + X(n)P_2u(0) + X(n)P_3u(0) \\ &+ \sum_{s=0}^{n-1} X(n)(P_1+P_3)X^{-1}(s+1)f(s,u(s)) \\ &+ \sum_{s=0}^{+\infty} X(n)P_2X^{-1}(s+1)f(s,u(s)) - \sum_{s=n}^{+\infty} X(n)P_2X^{-1}(s+1)f(s,u(s)). \end{aligned}$$

The sequence

$$\left\{ \sum_{s=0}^{n-1} X(n)(P_1+P_3)X^{-1}(s+1)f(s,u(s)) \right\}$$

is bounded by the constant $2K\|g_{\|u\|_\infty}\|_q$, $1 < q \le \infty$. As $\lim_{n\to+\infty} |X(n)P_ju(0)| = 0$, $j = 1, 3$ (see Theorem 1) and u is bounded, the sequence

$$\left\{ X(n)P_2\left[u(0) + \sum_{s=0}^{+\infty} X^{-1}(s+1)f(s,u(s)) \right] \right\}$$

is bounded too. From Theorem 1 it follows that

$$P_2\left[u(0) + \sum_{s=0}^{+\infty} X^{-1}(s+1)f(s,u(s)) \right] = 0. \tag{11}$$

Now to show that u satisfies (10) it is sufficient to prove that

$$P_1\left[u(0) - \sum_{s=-\infty}^{-1} X^{-1}(s+1)f(s,u(s))\right] = 0. \tag{12}$$

Since u is a solution of (1), for $n \le -1$ we have

$$\begin{aligned} u(n) =& X(n)P_1u(0) + X(n)P_2u(0) + X(n)P_3u(0) \\ &- \sum_{s=-\infty}^{-1} X(n)P_1X^{-1}(s+1)f(s,u(s)) + \sum_{s=-\infty}^{n-1} X(n)P_1X^{-1}(s+1)f(s,u(s)) \\ &- \sum_{s=n}^{-1} X(n)(P_2+P_3)X^{-1}(s+1)f(s,u(s)). \end{aligned}$$

Following an argument similar to that above given and taking into account that u is bounded, we obtain (12), and so u satisfies (10) for $n \ge 0$. Starting from the variation of constants formula for $n \le -1$ and taking into account (11) and (12) we obtain that u satisfies (10) for $n \le -1$ too.

Vice versa let u be a bounded solution of (10). A standard calculation shows that u satisfies (1). $\square$

Denote $l_0^\infty = \{u \in l^\infty : \lim_{n\to\pm\infty} u(n) = 0\}$. From the above proposition we have

Corollary 2. *Assume conditions i) and ii) of Proposition 2 hold, with* $1 \le p < \infty$. *Assume also for* $p = 1\,(q = \infty)$

iii) $g(n,|c|) \le \gamma|c| + \lambda(n)$, *for every* $n \in \mathbb{Z}$, $c \in \mathbb{R}^m$, *where* $\gamma > 0$, $2K\gamma < 1$ *and* $\lambda \in l_0^\infty$.

Then every bounded solution of (1) *belongs to* l_0^∞.

Proof. Let u be a bounded solution of (1). From Proposition 2 u is solution of (10). Let $1 < p < \infty$ and $n \ge n_1 > 0$, n_1 fixed; from (11) we get

$$\begin{aligned} |u(n)| \le& |X(n)(P_1+P_3)|\left\{|u(0)| + \sum_{s=0}^{n_1-1} |X^{-1}(s+1)f(s,u(s))|\right\} \\ &+ \sum_{s=n}^{+\infty} |X(n)P_2X^{-1}(s+1)f(s,u(s))| + \sum_{s=n_1}^{n} |X(n)(P_1+P_3)X^{-1}(s+1)f(s,u(s))| \\ \le& |X(n)(P_1+P_3)|\left\{|u(0)| + \sum_{s=0}^{n_1-1} |X^{-1}(s+1)f(s,u(s))|\right\} \\ &+ 3K\left(\sum_{s=n_1}^{+\infty} (g_{\|u\|_\infty}(s))^q\right)^{1/q} \end{aligned}$$

Choosing n_1 sufficiently large, in view of Theorem 1 we obtain $\lim_{n\to+\infty} u(n) = 0$. The assertion $\lim_{n\to-\infty} u(n) = 0$ can be proved in a similar way taking into account (12).

When $p = 1$ the proof comes using similar arguments to those in [6] (Th. 8 p. 68 and Th. 10 p. 7) with slight modifications; see also [18], Prop. 3.2. □

Remark 2. When $p = 1$ conditions i) and ii) in Proposition 2 are not sufficient to assure that every bounded solution of (1) belongs to l_0^∞. It is possible to find conditions different from iii) in Corollary 2 that, together with conditions i) and ii), assure the decaying of all the bounded solutions towards zero; for instance this happens by assuming

iv) $g_\alpha \in l_0^\infty$ *for every* $\alpha > 0$.

Finally consider the boundary value problem (3). The method here used for solving (3) is to reduce it to a fixed point problem in the Fréchet space $\mathcal{X}$ of all the sequences from $\mathbb{Z}$ into $\mathbb{R}^m$

$$\mathcal{X} := \{q : \mathbb{Z} \mapsto \mathbb{R}^m\}$$

and then to apply the Schauder-Tychonoff fixed point theorem.

Theorem 2 (Existence). *Let* $\xi \in Range(P_3)$ *be fixed. If conditions i) and ii) in Proposition 2 and, for* $p = 1$*, also condition iii) in Corollary 2 hold, and if in addition*

v) there exists a constant $\beta > 0$ *such that*

$$\sup_{n\in\mathbb{Z}} |X(n)\xi| + 3K\|g_\beta\|_q \le \beta,$$

then the boundary value problem

$$\begin{cases} y(n+1) = A(n)y(n) + f(n, y(n)), & n \in \mathbb{Z} \\ y(+\infty) = 0, \quad y(-\infty) = 0 \\ y(0) = \xi \end{cases} \tag{13}$$

has at least a solution.

Proof. Let $\Omega := \{q \in \mathcal{X} : q \in l_0^\infty,\ q(0) = \xi,\ \|q\|_\infty \le \beta\}$. Clearly Ω is a nonempty, closed, convex and bounded subset of $\mathcal{X}$. Consider the operator $F : \Omega \mapsto \mathcal{X}$ defined by (see the right end side of (10))

$$\begin{aligned}(Fq)(n) =& X(n)\xi + \sum_{s=-\infty}^{n-1} X(n)P_1X^{-1}(s+1)f(s,q(s)) \\ &- \sum_{s=n}^{+\infty} X(n)P_2X^{-1}(s+1)f(s,q(s)) + \sum_{s=0}^{n-1} X(n)P_3X^{-1}(s+1)f(s,q(s)) \\ &- \sum_{s=n}^{-1} X(n)P_3X^{-1}(s+1)f(s,q(s))\end{aligned}$$

(with the convention $\sum_{s=a}^{b} g(s) = 0$ if $a > b$).

Let $1 < p < \infty$. Assumptions i) and ii) in Proposition 2 assure that this operator is well defined, being $\Omega \subset l^\infty$. Let us show that $F(\Omega) \subseteq \Omega$. For every $q \in \Omega$, taking into account assumption v), we have

$$|(Fq)(n)| \leq \sup_{n\in\mathbb{Z}} |X(n)\xi| + 3K\|g_\beta\|_q \leq \beta.$$

Moreover from Proposition 2 and Corollary 2 it follows $(Fq)(n) \to 0$ as $n \to \pm\infty$ and $(Fq)(0) = \xi$, for every $q \in \Omega$. Thus $F(\Omega) \subseteq \Omega$. This also implies that $F(\Omega)$ is a relatively compact subset of $\mathcal{X}$, because in such a Fréchet space a subset is relatively compact if and only if it is bounded. Finally F is a continuous operator in Ω: let $\{q_k\}_{k\in\mathbb{N}}$ a sequence in Ω such that $q_k \to \bar{q}$ in $\mathcal{X}$, and consider the sequence $\{Fq_k\}_{k\in\mathbb{N}}$. We have

$$|(Fq_k)(n) - (F\bar{q})(n)| \leq 3K\left(\sum_{s=-\infty}^{+\infty} |f(s, q_k(s)) - f(s, \bar{q}(s))|^q\right)^{1/q}.$$

Note that $|f(s, q_k(s)) - f(s, \bar{q}(s))| \leq 2g_\beta(s) \in l^q$, for every $k \in n$. The continuity of f with respect to the second argument and the fact that the convergence in $\mathcal{X}$ implies the pointwise convergence, allow us to apply the dominated convergence theorem (see [3] for the formulation in the space $\mathcal{X}$). Thus $Fq_k \to F\bar{q}$ in $\mathcal{X}$ being the convergence of $(Fq_k)(n) - (F\bar{q})(n)$ towards zero uniform with respect to n. By the Schauder-Tychonoff fixed point theorem the operator F has at least a fixed point y in Ω and Proposition 2 assures that y is a solution of problem (13).

The case $p = 1$ can be treated by means of similar arguments. □

Remark 3. If $\xi \in Range(P_3)$ then $\sup_{n\in\mathbb{Z}} |X(n)\xi| = \max_{n\in\mathbb{Z}} |X(n)\xi| < \infty$. Indeed $\lim_{n\to\pm\infty} |X(n)\xi| = 0$.

Remark 4. Assumption v) in Theorem 2 is trivially satisfied if $\sup_{\alpha>0} \|g_\alpha\| < \infty$.

It is worth to remark that the choice of the Fréchet space $\mathcal{X}$ makes the proof of the compactness of $F(\Omega)$ quite immediate, while the proof of the continuity of F is not more difficult than working in a Banach space like l^∞.

The results of this section extend those in [18] and generalize those in [4], because the nonlinear discrete boundary value problem is completely solved. They also generalize some of the results in [2], [9], [11] because exponential trichotomy implies l^p trichotomy.

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