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COMPARISON THEOREMS FOR HALF–LINEAR SECOND ORDER DIFFERENCE EQUATIONS

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Abstract. In the paper new comparison theorems for half–linear difference equation

$$\Delta\Big(R_k\Phi(\Delta z_k)\Big) + C_k\Phi(z_{k+1}) = 0$$

are derived. The main tool is variational technique developed for half–linear difference equations in Řehák [5].

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1. Introduction

The well–known result from the calculus of variations states, that there is an equivalence between disconjugacy of second order ordinary differential equation

$$(r(t)y')' + c(t)y = 0$$

on the interval (a, b) and nonnegativity of quadratic functional

$$\int_a^b \Big(r(t)\eta'^2(t) - c(t)\eta^2(t)\Big)\mathrm{d}t$$

defined on the class of functions $\eta \in W_0^{1,2}(a, b)$. This classical result has been later extended in various directions. The generalizations include $n-$dimensional problem with general boundary conditions [1], singular functional [2], p-degree

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functionals [3,4] and also discrete p-degree functionals [5]. The relationship between the functional and the corresponding equation is frequently used in the comparison and oscillation theory of differential equations, see. e.g. [6].

In this paper we will study the second order half–linear difference equation

$$L[z_k] = \Delta\Big(R_k\Phi(\Delta z_k)\Big) + C_k\Phi(z_{k+1}) = 0 \tag{E}$$

for $k \in [0, n]$ and the corresponding discrete scalar p-degree functional

$$J(x) = A|x_0|^p + \sum_{k=0}^{n}\Big(R_k|\Delta x_k|^p - C_k|x_{k+1}|^p\Big). \tag{J}$$

The relationship between Eq. (E) and (J) will be used further in the proof of comparison theorems, which compare Eq. (E) with another half–linear discrete differential equation

$$l[y_k] = \Delta\Big(r_k\Phi(\Delta y_k)\Big) + c_k\Phi(y_{k+1}) = 0. \tag{e}$$

Remark that, unless stated explicitly, under the interval $[m, n]$ we actually mean the discrete set $\{m, m+1, \dots n\}$. Similarly under the term function we actually mean the sequence.

The following lemma presents our main tool – Picone-type identity for half–linear difference equations. It is a simplified version of the Picone identity published for Eq. (E) in Řehák [5].

Lemma 1 ([5]). *If $L[z_k] = 0$ for $k \in [0, n]$ and $z_k \neq 0$ for $k \in [0, n+1]$, then for $k \in [0, n]$*

$$\Delta\left\{-|x_k|^p R_k \frac{\Phi(\Delta z_k)}{\Phi(z_k)}\right\} = C_k|x_{k+1}|^p - R_k|\Delta x_k|^p + \frac{R_k z_k}{z_{k+1}} G_k(x, z), \tag{P}$$

where

$$G_k(x, z) = \frac{z_{k+1}}{z_k}|\Delta x_k|^p - \frac{z_{k+1}\Phi(\Delta z_k)}{z_k\Phi(z_{k+1})}|x_{k+1}|^p + \frac{z_{k+1}\Phi(\Delta z_k)}{z_k\Phi(z_k)}|x_k|^p$$

holds. The function $G_k(\cdot,\cdot)$ satisfies

$$G_k(x, z) \geq 0 \tag{1}$$

with equality if and only if $\Delta x_k = x_k\frac{\Delta z_k}{z_k}$, i.e. if and only if $x_{k+1} = x_k\frac{z_{k+1}}{z_k}$.

Lemma 2. *If $x_{k+1} = x_k\frac{z_{k+1}}{z_k}$ for $k \in [0, n]$, then $x_k = \frac{x_0}{z_0}z_k$ for $k \in [0, n+1]$.*

Proof. By induction.

2. Main results

In connection with Eq. (E) we will study also the first order Riccati-type difference equation

$$\Delta w_k + C_k + w_k\Big(1 - \frac{R_k}{\Phi(\Phi^{-1}(R_k) + \Phi^{-1}(w_k))}\Big) = 0, \tag{R}$$

where $\Phi^{-1}(\cdot)$ denotes the inverse function to the function $\Phi(\cdot)$.

The relationship between functional (J) and Eqs. (E), (R) has been studied in [5]. Here it is proved the equivalence between disconjugacy of (E), existence of solution of (R) and positive definiteness of (J) on the class of functions satisfying $x_0 = 0 = x_{n+1}$.

The difference between these results and the results from this paper lies in another type of boundary conditions for the function x. The fact that we use another types of boundary conditions causes that we obtain information about solution of Eq. (E) given by another initial condition, than in [5].

First let us recall the definition of generalized zero, which is known to be the convenient substitution for zeros of the continuous function.

Definition 1. An interval $(m, m+1]$ is said to contain a *generalized zero* of a solution z_k of Eq. (E) if $z_m \neq 0$ and $R_m z_m z_{m+1} \leq 0$.

The following theorem establishes the relationship between the half–linear equation, Riccati equation and the $p-$degree functional. Results of this type are sometimes referred as Reid's Roundabout–type theorem.

Theorem 1. *The following statements are equivalent:*

(i) The solution z_k of Eq. (E) *given by $R_0\Phi\Big(\frac{\Delta z_0}{z_0}\Big) = A$ satisfies $R_k z_k z_{k+1} > 0$ for $k \in [0, n]$.*
(ii) Equation (R) *has a solution on $[0, n]$ such that $w_0 = A$ and $R_k + w_k > 0$ on $[0, n]$.*
(iii) Functional (J) *is positive definite on the class of functions defined on $[0, n+1]$ satisfying $x_{n+1} = 0$.*

Proof. "(i)$\Longleftrightarrow$(ii)" If z_k is the solution of (E) satisfying $R_k z_k z_{k+1} > 0$ for $k \in [0, n]$, then the function $w_k = R_k\Phi\Big(\frac{\Delta z_k}{z_k}\Big)$ is well-defined on $[0, n+1]$ and satisfies (R) and $R_k + w_k > 0$ on $[0, n]$, which follows from [5].

Conversely, if w_k is a solution of Eq. (R) satisfying $R_k + w_k > 0$, then $z_{k+1} = z_k\Big(1 + \Phi^{-1}\big(\frac{w_k}{r_k}\big)\Big)$ defines solution of Eq. (E) satisfying $R_k z_k z_{k+1} > 0$ for $k \in [0, n]$.

In addition, $R_0\Phi\Big(\frac{\Delta z_0}{z_0}\Big) = A$ is equivalent to $w_0 = A$.

"(i)$\Longrightarrow$(iii)" Let x be defined on $[0, n+1]$ and $x_{n+1} = 0$. Summation of Picone identity (P) for $k \in [0, n]$ gives

$$J(x) = A|x_0|^p + \sum_{k=0}^{n} \Big[\Delta\Big(|x_k|^p R_k \Phi\Big(\frac{\Delta z_k}{z_k}\Big)\Big) + \frac{R_k z_k z_{k+1}}{z_{k+1}^2} G_k(x,z) \Big] \geq$$
$$A|x_0|^p + |x_{n+1}|^p R_{n+1}\Phi\Big(\frac{\Delta z_{n+1}}{z_{n+1}}\Big) - |x_0|^p R_0 \Phi\Big(\frac{\Delta z_0}{z_0}\Big) = 0$$

and the functional is positive semidefinite.

The equality holds throughout only if $G_k(x,z) = 0$ for $k \in [0,n]$. From here it follows $x_{k+1} = x_k \frac{z_{k+1}}{z_k}$ for $k \in [0,n]$, or equivalently $x_k = z_k \frac{x_0}{z_0}$ for $k = [0, n+1]$. In view of the fact $x_{n+1} = 0 \neq z_{n+1}$, it holds $x_0 = 0$ and $x \equiv 0$. Hence $J(x) = 0$ only if $x \equiv 0$ and the functional is positive definite.

"(iii)$\Longrightarrow$(i)" Suppose, by contradiction, that the functional is positive definite and for the solution z of Eq. (E) given (uniquely up to the constant multiple) by the condition $R_0 \Phi\Big(\frac{\Delta z_0}{z_0}\Big) = A$ there exists $N \in [0,n]$ such that

$$\begin{aligned} R_k z_k z_{k+1} &> 0 \quad \text{for } 0 \leq k < N \\ R_N z_N z_{N+1} &\leq 0. \end{aligned}$$

Denote

$$x_k = \begin{cases} z_k & k \in [0,N] \\ 0 & k \in [N+1, n+1]. \end{cases}$$

Since $z_0 \neq 0$, clearly $x \not\equiv 0$. Suppose $N \geq 1$. From the definition of the function x it follows $L[x_k] = 0$ for $k \in [0, N-2]$. Summation by parts gives

$$\begin{aligned} J(x) &= A|x_0|^p + \sum_{k=0}^{n} \Big[R_k |\Delta x_k|^p - C_k |x_{k+1}|^p \Big] \\ &= A|x_0|^p + \big[x_k R_k \Phi(\Delta x_k)\big]_{k=0}^{n+1} - \sum_{k=0}^{n} x_{k+1} L[x_k] \\ &= -\sum_{k=0}^{N} x_{k+1} L[x_k] = -x_N L[x_{N-1}] \\ &= -z_N \Big[\Delta(R_{N-1}\Phi(\Delta x_{N-1})) + C_{N-1}\Phi(z_N) \Big] \\ &= z_N \Big[R_{N-1}\Phi(\Delta z_{N-1}) - R_N \Phi(\Delta x_N) + \Delta(R_{N-1}\Phi(\Delta z_{N-1})) \Big] \\ &= z_N R_N \Phi(\Delta z_N) + z_N R_N \Phi(z_N), \end{aligned}$$

since $\Delta x_N = -z_N$ and $\Delta x_{N-1} = \Delta z_{N-1}$. Hence $J(x) = R_N |z_N|^p \Big[\Phi\big(\frac{\Delta z_N}{z_N}\big) + 1 \Big]$. Now $z_{N+1} \neq 0$. Really, if $z_{N+1} = 0$ would hold, then $J(x) = 0$, a contradiction. Hence

$$J(x) = \frac{R_N z_N z_{N+1}}{z_{N+1}^2} |z_N|^p \Big[\frac{z_{N+1}}{z_N} \Phi\Big(\frac{z_{N+1}}{z_N} - 1\Big) + \frac{z_{N+1}}{z_N} \Big].$$

In view of the fact $R_N z_N z_{N+1} \leq 0$ and with respect to inequality $\alpha\Phi(\alpha - 1) + \alpha \geq 0$ we obtain $J(x) \leq 0$, a contradiction. This contradiction ends the proof.

Corollary 1 (Leighton type comparison theorem). *Let y_k be solution of Eq.* (e), *such that $y_{n+1} = 0 \neq y_0$. Denote $a = r_0\Phi\big(\frac{\Delta y_0}{y_0}\big)$. Let A be such that*

$$V(y) := (A-a)|y_0|^p + \sum_{k=0}^{n}\Big[(R_k - r_k)|\Delta y_k|^p - (C_k - c_k)|y_{k+1}|^p\Big] \leq 0.$$

Then the solution of Eq. (E) *given by $R_0\Phi\big(\frac{\Delta z_0}{z_0}\big) = A$ has a generalized zero on $[0, n+1]$, i.e., there exists $i \in [0, n]$ such that $R_i z_i z_{i+1} \leq 0$ holds.*

Proof. Define the functional $j(x) = \alpha|x_0|^p + \sum_{k=0}^{n} r_k|\Delta x_k|^p - c_k|x_{k+1}|^p$. Using summation by parts we obtain $j(y) = 0$ and hence $J(y) = J(y) - j(y) = V(y) \leq 0$. Since $y \not\equiv 0$, the statement follows from Theorem 1.

An immediate consequence is the following

Corollary 2. *Let y_k be solution of Eq.* (e), *such that $y_{n+1} = 0 \neq y_0$. Denote $a = r_0\Phi\big(\frac{\Delta y_0}{y_0}\big)$. Let $A < a$, $R_k \leq r_k$ on $[0, n]$ and $c_k \leq C_k$ on $[0, n-1]$. Then the solution of Eq.* (E) *given by $R_0\Phi\big(\frac{\Delta z_0}{z_0}\big) = A$ has a generalized zero on $[0, n+1]$, i.e., there exists $i \in [0, n]$ such that $R_i z_i z_{i+1} \leq 0$ holds.*

Corollary 3. *Let y_k be solution of Eq.* (e), *such that $y_{n+1} = 0 \neq y_0$. Denote $a = r_0\Phi\big(\frac{\Delta y_0}{y_0}\big)$. Let A be such that*

$$\mathcal{V}(y) := (A - \frac{R_0}{r_0}a)|y_0|^p - \sum_{k=0}^{n}\Big\{\Delta\Big(\frac{R_k}{r_k}\Big) r_k\Phi(\Delta y_k)y_{k+1} + (C_k - \frac{R_{k+1}}{r_{k+1}}c_{k+1})|y_{k+1}|^p\Big\} \leq 0.$$

Then the solution of Eq. (E) *given by $R_0\Phi\big(\frac{\Delta z_0}{z_0}\big) = A$ has a generalized zero on $[0, n+1]$, i.e., there exists $i \in [0, n]$ such that $R_i z_i z_{i+1} \leq 0$ holds.*

Proof. Let y_k be solution of (e) on $[0, n]$ satisfying $y_{n+1} = 0 \neq y_0$. Then

$$\begin{aligned} L[y_k] &= \Delta\big(R_k\Phi(\Delta y_k)\big) + C_k\Phi\big(y_{k+1}\big) = \Delta\Big(\frac{R_k}{r_k} r_k\Phi(\Delta y_k)\Big) + C_k\Phi(y_{k+1}) \\ &= \Delta\Big(\frac{R_k}{r_k}\Big) r_k\Phi(\Delta y_k) + \frac{R_{k+1}}{r_{k+1}}\Delta\Big(r_k\Phi(\Delta y_k)\Big) + C_k\Phi(y_{k+1}) \\ &= \Delta\Big(\frac{R_k}{r_k}\Big) r_k\Phi(\Delta y_k) + \Phi(y_{k+1})\Big[C_k - \frac{R_{k+1}}{r_{k+1}}c_k\Big]. \end{aligned} \tag{2}$$

Since the integration by parts shows that

$$\begin{aligned} \sum_{k=0}^{n}\Delta\Big(\frac{R_k}{r_k}\Big) r_k\Phi(\Delta y_k)y_{k+1} &= \Big[R_k\Phi(\Delta y_k)y_{k+1}\Big]_{n=0}^{n+1} - \sum_{k=0}^{n}\frac{R_{k+1}}{r_{k+1}}\Delta\Big(r_k\Phi(\Delta y_k)y_{k+1}\Big) \\ &= R_{n+1}y_{n+2}\Phi(\Delta y_{n+1}) - R_0 y_1\Phi(\Delta y_0) \\ &\quad - \sum_{k=0}^{n}\frac{R_{k+1}}{r_{k+1}}\Big[\Delta\Big(r_k\Phi(\Delta y_k)\Big)y_{k+1} + \Delta y_{k+1}r_{k+1}\Phi(\Delta y_{k+1})\Big] \end{aligned}$$

$$
\begin{aligned}
&= R_{n+1}y_{n+2}\Phi(\Delta y_{n+1}) - R_0 y_1 \Phi(\Delta y_0)\\
&\quad - \sum_{k=0}^{n}\Big[-\frac{R_{k+1}}{r_{k+1}}c_k|y_{k+1}|^p + R_{k+1}|\Delta y_{k+1}|^p\Big]\\
&= R_{n+1}y_{n+2}\Phi(\Delta y_{n+1}) - R_0 y_1 \Phi(\Delta y_0)\\
&\quad - \sum_{k=0}^{n}\Big[-\frac{R_{k+1}}{r_{k+1}}c_k|y_{k+1}|^p + R_k|\Delta y_k|^p\Big] - R_{n+1}|\Delta y_{n+1}|^p + R_0|\Delta y_0|^p\\
&= R_{n+1}y_{n+1}\Phi(\Delta y_{n+1}) - R_0 y_0 \Phi(\Delta y_0)\\
&\quad - \sum_{k=0}^{n}\Big[R_k|\Delta y_k|^p - \frac{R_{k+1}}{r_{k+1}}c_k|y_{k+1}|^p\Big]
\end{aligned}
$$

the following relation holds

$$
\sum_{k=0}^{n} y_{k+1}L[y_k] = R_{n+1}\Phi(\Delta y_{n+1})y_{n+1} - R_0\Phi(\Delta y_0)y_0 - \sum_{k=0}^{m}\Big[R_k|\Delta y_k|^p - C_k|y_{k+1}|^p\Big].
$$

Then in view of (2) and $y_{n+1} = 0$, clearly

$$
J(y) = |y_0|^p\Big[A - R_0\Phi\Big(\frac{\Delta y_0}{y_0}\Big)\Big] - \sum_{n=0}^{n} y_{k+1}L[y_k] = \mathcal{V}(y)
$$

and the statement follows from Theorem 1.

3. Open problems

In the oscillation theory of discrete differential equations the concept of generalized zeros is used. This is caused by the fact that the sequence R_k is allowed to attain also negative values. However in the boundary conditions of the functional (J) "exact" zeros are used. It could be interesting to remove this disharmonicity and to find out, whether the concept of generalized zeros in boundary conditions would produce some fruitful extension of discrete variational technique.

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