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SADDLE CONNECTIONS IN PLANAR SYSTEMS

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Abstract. The class of planar autonomous systems with a small parameter-dependent perturbation is considered. We derive a sufficient condition for existence of a saddle connection in such system.

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1. Introduction

We consider systems of the form

$$\dot{x} = f(x) + \varepsilon g(x, \alpha), \quad x \in R^2, \quad \varepsilon, \alpha \in R, \tag{1}$$

where f, g are C^r, $r \geq 2$, and bounded on bounded sets, ε is a small parameter. Such systems are viewed as planar systems with a small perturbation which depends on a real parameter α. If we assume that unperturbed system (for $\varepsilon = 0$) possesses a saddle connection, then a natural question arises whether there are values of a parameter α for which a perturbed system possesses a saddle connection. There are many results related to similar questions, see for instance [1] for the problem of existence of periodic orbits in a perturbed system, or [4, §4.4], where the impact of a small time-dependent periodic perturbation on homoclinic orbit in Hamiltonian systems is studied. The paper [3] explores existence and number of periodic and homoclinic orbits, but only for a particular Hamiltonian system (whirling pendulum equation) with a special perturbation (a friction). None of the results in mentioned (and other) works has been directly applicable to our problem. To solve it, we follow a geometrical point of view as it is presented in [2].

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2. Assumptions and background material

We will assume that for $\varepsilon = 0$ (1) has two saddle points p_1 and p_2, which are connected by heteroclinic trajectory Γ. (The reasoning in the case of a saddle connected to itself by a homoclinic loop is very similar). More precisely, one branch, say Γ^u, of the global unstable manifold W^u of p_1 coincides with one branch, say Γ^s of the global stable manifold W^s of p_2, and they form a saddle connection Γ (see Fig. 1a).

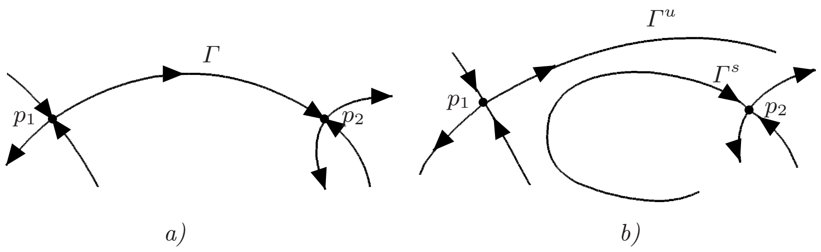

Fig. 1. The phase portrait of $\dot{x} = f(x) + \varepsilon g(x, \alpha)$ for a) $\varepsilon = 0$, b) $\varepsilon \neq 0$.

This situation is not resistent to perturbations – in general, any perturbation will break the saddle connection, although the local phase portraits will not change under a small perturbation (see Fig. 1b). Particularly, the following facts are well-known for (1) with $\varepsilon \neq 0$ (for details we refer the reader to [2, §4.5] and the references given there):

F1 For each ε sufficiently small, (1) has two unique saddles $p_1^\varepsilon = p_1^\varepsilon + \mathcal{O}(\varepsilon)$, $p_2^\varepsilon = p_1^\varepsilon + \mathcal{O}(\varepsilon)$. This is a straightforward application of the implicit function theorem, since Jacobi matrices $Df(p_1)$, $Df(p_2)$ are invertible (they have nonzero real eigenvalues).

F2 Perturbed local stable and unstable manifolds of the saddles p_1^ε, p_2^ε‘are C^r-close to unperturbed local stable and unstable manifolds of the saddles p_1, p_2. This fact follows from invariant manifold theory.

F3 If we denote by $\gamma(t)$ a solution of the unperturbed system lying in Γ, by $\gamma^u(t)$ and $\gamma^s(t)$ solutions of the perturbed system lying in Γ_ε^u and Γ_ε^s (branches of W_ε^u and W_ε^s corresponding to Γ^u and Γ^s), the following expressions holds, with uniform validity in the indicated intervals:

$$\begin{aligned} \gamma^s(t) &= \gamma(t) + \varepsilon\gamma_1^s(t) + \mathcal{O}(\varepsilon^2), \quad t \in [0, \infty), \\ \gamma^u(t) &= \gamma(t) + \varepsilon\gamma_1^u(t) + \mathcal{O}(\varepsilon^2), \quad t \in (-\infty, 0]. \end{aligned} \tag{2}$$

Here $\gamma_1^s(t)$ and $\gamma_1^u(t)$ are solutions of the first variational equations

$$\dot{\gamma}_1^{s,u}(t) = Df(\gamma(t))\gamma_1^{s,u}(t) + g(\gamma(t), \alpha). \tag{3}$$

This fact represents both local and global dynamics – near a saddle point (infinite time interval) it is governed by exponential attraction and repulsion, while away from a saddle (finite time interval) the closeness of solutions may be derived thanks to Gronwall's inequality.

In what follows, we will look for values of parameter α for which the saddle connection persists. The main idea is to measure, in some sense, the distance between perturbed branches Γ_ε^u and Γ_ε^s of the global manifolds W_ε^u and W_ε^s.

3. The distance function

Let $p \in \Gamma$ be a nonsingular point ($f(p) \neq 0$), and $p^u \in \Gamma_\varepsilon^u$, $p^s \in \Gamma_\varepsilon^s$ are lying on the normal $f^{\perp(p)}$ to Γ at p (Fig. 2). Then we define the oriented distance between Γ_ε^u and Γ_ε^s at the point p as

$$d(\varepsilon, \alpha) = \frac{f(p) \wedge (p^u - p^s)}{|f(p)|},$$

where $a \wedge b = a^\perp \cdot b$ is the wedge product.

We denote $\gamma(t)$, $\gamma^s(t)$ and $\gamma^u(t)$ solutions lying in Γ, Γ_ε^s and Γ_ε^u for which

$$\gamma(0) = p, \quad \gamma^s(0) = p^s, \quad \gamma^u(0) = p^u. \tag{4}$$

Using (2) and (4), we can write

$$d(\varepsilon, \alpha) = \varepsilon \frac{f(\gamma(0)) \wedge (\gamma_1^u(0) - \gamma_1^s(0))}{|f(\gamma(0))|} + \mathcal{O}(\varepsilon^2).$$

Now we define the time dependent distance function

$$\Delta(t) = f(\gamma(t)) \wedge (\gamma_1^u(t) - \gamma_1^s(t))$$

which may be written as $\Delta(t) = \Delta^u(t) - \Delta^s(t)$ with $\Delta^{s,u}(t) = f(\gamma(t)) \wedge \gamma_1^{s,u}(t)$. Note that

$$d(\varepsilon, \alpha) = \varepsilon \frac{\Delta(0)}{|f(\gamma(0))|} + \mathcal{O}(\varepsilon^2).$$

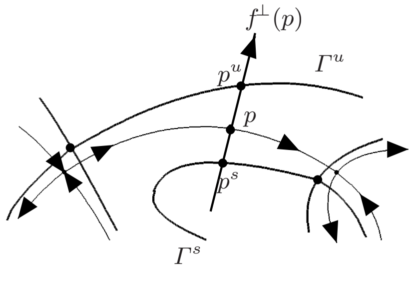

Fig. 2. Definition of the distance function.

The derivative of $\Delta^{s,u}(t)$ with respect to time is

$$\dot{\Delta}^{s,u}(t) = Df(\gamma(t))\dot{\gamma}(t) \wedge \gamma_1^{s,u}(t) + f(\gamma(t)) \wedge \dot{\gamma}_1^{s,u}(t).$$

Using (3) and the fact that $\dot{\gamma}(t) = f(\gamma(t))$, we obtain, after some matrix calculations,

$$\dot{\Delta}^{s,u}(t) = \mathrm{Tr}\ (Df(\gamma(t)))\, \Delta^{s,u}(t) + f(\gamma(t)) \wedge g(\gamma(t), \alpha).$$

Integrating the last equation from 0 to ∞ for Δ^s and from $-\infty$ to 0 for Δ^u yields

$$\Delta^s(\infty) - \Delta^s(0) = \int\limits_0^\infty f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{-\int_0^t \mathrm{Tr}\ (Df(\gamma(s)))\,\mathrm{d}s} \mathrm{d}t,$$

$$\Delta^u(0) - \Delta^u(-\infty) = \int\limits_{-\infty}^0 f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{\int_t^0 \mathrm{Tr}\ (Df(\gamma(s)))\,\mathrm{d}s} \mathrm{d}t.$$

Since

$$\Delta^s(\infty) = \lim_{t\to\infty} f(\gamma(t)) \wedge \gamma_1^s(t),$$

where $\gamma_1^s(t)$ is bounded and $\lim_{t\to\infty} f(\gamma(t)) = f(p_2) = 0$, we have $\Delta^s(\infty) = 0$. Similarly $\Delta^u(-\infty) = 0$. Then

$$\Delta^s(0) = \int\limits_0^\infty f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{-\int_0^t \mathrm{Tr}\ (Df(\gamma(s)))\,\mathrm{d}s} \mathrm{d}t.$$

In the case when the unperturbed system is Hamiltonian, i.e. $f = \left(\frac{\partial H}{\partial x_2}, -\frac{\partial H}{\partial x_1}\right)$ for some differentiable function $H(x_1, x_2)$, we have $\mathrm{Tr}\,(Df) \equiv 0$, and

$$\Delta(0) = \int_{-\infty}^{\infty} f(\gamma(t)) \wedge g(\gamma(t), \alpha) dt,$$

which is the homoclinic Melnikov function [2, p. 187].

In the next, we will use more suitable notation $\Delta(0) = M(\alpha)$, which takes into account the fact that $\Delta(0)$ depends on α. Thus

$$d(\varepsilon, \alpha) = \varepsilon \frac{M(\alpha)}{|f(p)|} + \mathcal{O}(\varepsilon^2). \tag{5}$$

Now we are ready to state and prove the main result:

Theorem 1. *Let there exist α_0 such that $M(\alpha_0) = 0$, $M'(\alpha_0) \neq 0$. Then for each ε sufficiently small there exists $\alpha(\varepsilon) = \alpha_0 + \mathcal{O}(\varepsilon)$ such that the perturbed system*

$$\dot{x} = f(x) + \varepsilon g(x, \alpha(\varepsilon))$$

possesses a saddle connection, which is C^r-close to the saddle connection of the unperturbed system.

Proof. We rewrite (5) in the form $d(\varepsilon, \alpha) = \varepsilon \overline{d}(\varepsilon, \alpha)$, where

$$\overline{d}(\varepsilon, \alpha) = \frac{M(\alpha)}{|f(p)|} + \mathcal{O}(\varepsilon).$$

Then, for $\varepsilon \neq 0$, d vanishes if and only if $\overline{d}$ vanishes. For α_0 with indicated properties we obtain

$$\overline{d}(0, \alpha_0) = 0, \qquad \frac{\partial \overline{d}}{\partial \alpha}(0, \alpha_0) \neq 0.$$

The implicit function theorem ensures the existence of a smooth curve of points $(\varepsilon, \alpha(\varepsilon))$ passing throw $(0, \alpha(0))$, $\alpha(0) = \alpha_0$, with a property

$$\overline{d}(\varepsilon, \alpha(\varepsilon)) = 0.$$

It means that the oriented distance between Γ_ε^u and Γ_ε^s at the point p is zero, which implies, thanks to the uniqueness theorem, that they coincide, forming a saddle connection. The C^r-closeness is ensured by F3.

4. Example

We will seek parameter α_0 for which there exists a smooth curve of parameters $\alpha(\varepsilon)$ with the property: the planar system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x + \varepsilon y(\cos x + \alpha(\varepsilon)) \end{aligned} \tag{6}$$

has a saddle connection that is C^r-close to the upper saddle connection of the planar pendulum equation, i.e. the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x. \end{aligned} \tag{7}$$

To obtain the value of α_0, we will compute $M(\alpha)$ for (6). First, we recall that the planar pendulum equation (7) is a Hamiltonian system with the energy

$$H(x, y) = \frac{y^2}{2} - \cos x + 1.$$

Saddles $-\pi$, π are connected by two heteroclinic orbits

$$y = \pm\sqrt{2(\cos x + 1)}$$

(upper and lower saddle connections) corresponding to the energy level $h = 2$. Then

$$M(\alpha) = \int_{-\infty}^{\infty} y^2(t)(\cos x(t) + \alpha)\mathrm{d}t.$$

Using the fact that $y\mathrm{d}t = \mathrm{d}x$, and trigonometrical identity $\cos x + 1 = 2\cos^2 \dfrac{x}{2}$, we obtain that along the upper saddle connection

$$M(\alpha) = \int_{-\pi}^{\pi} y(\cos x + \alpha)\mathrm{d}x = 8(\alpha + \frac{1}{3}).$$

Consequently, if we denote $\alpha_0 = -\dfrac{1}{3}$, then

$$M(\alpha_0) = 0, \qquad M'(\alpha_0) \neq 0.$$

By Theorem 1, for each ε sufficiently small there exists $\alpha(\varepsilon) = -\frac{1}{3} + \mathcal{O}(\varepsilon)$ such that (6) has an upper saddle connection. Moreover, from the definition of $d(\varepsilon, \alpha)$ we can deduce that for $\alpha > \alpha(\varepsilon)$ the unstable manifold of $[-\pi, 0]$ is lying above the stable manifold of $[\pi, 0]$, and reversely for $\alpha < \alpha(\varepsilon)$ (see Fig. 3, where the situation is depicted for two values of ε). The similar result may be obtained for the lower saddle connection.

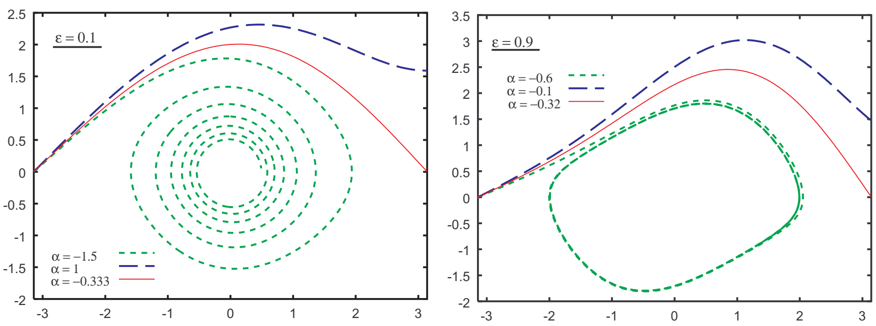

Fig. 3. Phase portraits of (6).

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