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PROPERTY A OF THE $(n+1)^{th}$ ORDER DIFFERENTIAL EQUATION $$\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t)+p(t)x(t)\right)\ \right]' = f(t,x(t),\cdots,x^{(n)}(t))\,.$$

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Abstract. The aim of this contribution is to study properties of solutions of the $n+1^{th}$-order differential equation of the form

$$\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t)+p(t)x(t)\right)\ \right]' = f(t,x(t),\cdots,x^{(n)}(t))\,. \tag{1}$$

where $n \geq 2$ is a natural number. A new approach using "submersivity" of a solution of an equation is presented, by means of it a sufficient condition for the property A is proved. This approach can be also used to prove necessary condition for the property A.

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1. Preliminaries

The main goal of this paper is to study certain properties of solutions of the differential equations, which are very appropriate for exploring the Property A. In this paper we consider only the proper solutions of the equations.

A solution $u(.) \in \mathbb{C}^n[T_0,\infty)$ is called oscillatory at $+\infty$ if it is proper, and there exists a sequence of numbers $\{t_k\}, k \in N$ such that $t_k \in [T_0,\infty)$, $u(t_k)=0$, $k \in N$ and $\lim\limits_{t\to\infty} t_k = +\infty$ hold.

A solution $u(.)$ is called non-oscillatory proper (briefly non-oscillatory) if it is proper and there exists a number $\bar{t} \in R^+$ such that $u(t) \neq 0$ for $t \geq \bar{t}$.
Briefly, we can say that the proper solution is said to be oscillatory, if it has a sequence of zeros converging to $+\infty$, otherwise is said to be non-oscillatory.

We will say that an equation has the Property A, if each proper solution of this equation is oscillatory when n is even and is either oscillatory or satisfies the condition

$$\lim_{t\to\infty} u^{(i)}(t) = 0\,, \qquad \text{monotonically,} \qquad i = 0, 1, \dots, n-1\,,$$

when n is odd.

2. "Submersivity" of a solution of the equation $y^{(n)}(t) + \alpha_1(t)\ y^{(n-1)}(t) + ... + \alpha_n(t)\ y(t) + p(t)\ y(t) = r(t)$.

One can describe "submersivity" as the ability of the function not to overcome a certain level ε for a certain time interval $[t_0, t_0 + \delta]$. The function having these properties behaves as follows: from a certain $t > t_0$, it dives under a certain level of ε and keeps being under this level maximally during a time interval δ.

Let us find a criterion of "submersivity" as the simplest possible conditions to be imposed on the equations, usually to the left-hand side of the equations.

This property is of major importance for exploring the questions about the oscillating and non-oscillating properties of a solution, and it can be directly used to prove necessary and sufficient conditions for property A.

Similar problems were posed and solved by I. T. Kiguradze [5] for the equation of the form $u^{(n)} + u^{(n-2)} = f(t, u, u', \dots, u^{(n-1)})$. In his paper for the first time was considered the case of oscillatory left-hand side operator. The results given in it fill this gap to some extent.

Consequently the knowledge about situation in the oscillatory cases was studied in a few papers for the third order diff. equation. The result of this kind was presented e.g. by Cecchi, Došlá, Marini [1], Greguš, Graef [2], Greguš, Gera, Graef, [3,4].

Similar properties of solutions were investigated by several authors, namely for the equation of the form $u^{(n)} = f(t, u, u', \dots, u^{(n-1)})$. Many results also have been obtained for the equation of the type $u^{(n)} + \sum p_k(t) u^{(k)} = f(t, u, \dots, u^{(n-1)})$, with a disconjugated left-hand side operator.

Our aim is a little different: We consider here *the case on the left-hand side kernel operator of the equations (1) can be oscillatory.* The new what we bring to this problem was directly the assumption on the oscillatory left-hand side kernel operator.

"Submersivity" properties can help us to explore the questions of oscillation of solutions in the case of the oscillatory left-hand side operator. Similar theorem, as follows, appeared in [5], but only for the case $\alpha_i(t) \equiv 0$, $p(t) \equiv 1$.

Theorem 1. *Let $n \geq 2$ and let the functions $\{\alpha_i(.)\}_{i=1}^n,\ p(.),\ r(.) \in \mathbb{C}[T_0,\infty)$ satisfy the assumptions*

$$(2)\qquad (i)\quad \text{for all } i \in \{1,\cdots,n\}, \qquad \lim_{t\to\infty} \alpha_i(t) = 0\,,$$

(ii) there exist constants $r_{max},\ r_{min} > 0$ such that

$$(3)\qquad |\, p(t)\,| < r_{max} \quad \text{and} \quad r_{min} \leq r(t) \leq r_{max} \qquad \text{for all } \ t \in [T_0,\infty).$$

Then for each $\delta_0 > 0$ and each $p_0 \in (0,1)$, there exist $T \geq T_0$ and $\varepsilon > 0$ with the following property:

If $y(.) \in \mathbb{C}^n[T_0,\infty)$ is a non-negative solution of the differential equation

$$(4)\qquad y^{(n)}(t) + \alpha_1(t)\, y^{(n-1)}(t) + ... + \alpha_n(t)\, y(t) + p(t)\, y(t) = r(t)\,,$$

then for all $t_0 > T$ and for all $\delta > \delta_0$

$$(5)\qquad \mu\left([t_0, t_0+\delta_0] \cap y^{-1}[0,\varepsilon]\right) \leq p_0\ \delta\,,$$

where μ denotes the Lebesgue measure of sets.

Proof. Let the functions $\{\alpha_i(.)\}_{i=1}^n,\ p(.), r(.)$ and constants $r_{max}, r_{min},\ \delta_0, p_0$ satisfied the conditions (2) and (3). Let m be the least natural number satisfying the inequality $r_{min} < 2^{2+n(n+1)}\,(2^m-1)$. Let

$$q_{min} = \frac{r_{min}}{2^m} \qquad \text{and} \qquad q_{max} = r_{max} + \frac{r_{min}}{2^m}\,(2^m-1)\,.$$

Moreover, put

$$(6)\qquad \alpha_0 = \frac{2.2^{n\,(n+1)}}{q_{min}}\,, \qquad \varepsilon_{max} = \left(\min\{p_0\,\delta_0\ ,\ \frac{r_{min}}{2\,r_{max}}\}\right)^n\,.$$

Let $p_0 \in (0,1)$ be an arbitrary, but fixed number. Lemma 5.1 from [6] ensures the existence of a constant $P_1 > 0$ such that: If $z(.) \in C^n[0,1]$ is a solution of the differential equation

$$(7)\qquad z^{(n)}(t) + p_1(t)\, z^{(n-1)}(t) + \cdots + p_n(t)\, z(t) = \alpha \cdot q(t)\,,$$

with the property

$$(8)\qquad 0 \leq z(t) \leq 1 \qquad \text{for all } \ t \in [0,1]\,,$$

where the functions $p_i(.),\ q(.)$ satisfy conditions

$$(9)\qquad 0 < q_{min} \leq q(t) \leq q_{max}\,, \qquad P(t) = \sum_{i=1}^n |\, p_i(t)\,| \leq P_1 \qquad \forall t \in [0,1]\,,$$

then for the constant α from the equation (7) we have $\alpha \leq \frac{2.2^{n(n+1)}}{q_{min}}$.

Theorem 5.2 from [6] guaranties that for arbitrary constants $q_{min}, q_{max}, \alpha_{max}$, such that $\alpha_{max} > 0$, $0 < q_{min} \le q_{max}$, and $p \in (0,1)$, there exist constants $P_2 > 0$ and $\varepsilon_2 \in (0,1)$, with the following property:

If $z(.) \in C^n[0,1]$ is a solution of the differential equation (7) such that (8) and $z(0) = 1$, where the constant α and functions $p_i(.),\ q(.) \in C[0,1]$ satisfy condition (9) and

$$0 < \alpha \le \alpha_{max}\,, \qquad P(t) = \sum_{i=1}^{n} \mid p_i(t) \mid \le P_2 \qquad \forall t \in [0,1]\,,$$

then $\mu(z^{-1}[0,\varepsilon_2]) \le p$.

First we will find the required $T > T_0$ and $\varepsilon > 0$. Set $\varepsilon = \varepsilon_2\ \frac{\varepsilon_{max}}{\alpha_0}$ and choose T_1 sufficiently large such that

$$\sum_{i=1}^{n} |\alpha_i(t)| \le \min\{P_1, P_2\}\,, \qquad \forall t > T_1\,. \tag{10}$$

Then the required T can be defined by $T = \max\{T_1, T_0\}$. It is enough to prove that ε and T chosen in this way ensure the validity of Theorem 1.

Let $y \in \mathbb{C}^n[T,\infty)$ be a solution of the equation (4), such that $y(t) \ge 0$ for all $t \in [T,\infty)$. For $t_0 > T$ and $\delta > \delta_0$ define the set M by $M = (t_0, t_0+\delta) \cap y^{-1}(-1, \frac{\varepsilon_{max}}{\alpha_0})$. The set M is open, so it can be expressed as at most countable union of the disjoint open intervals $(t_i, t_i+\delta_i)$, i. e. , $M = \bigcup_{i=1}^{l} (t_i, t_i+\delta_i)\,, 1 \le l \le \infty$. Note, if M is an empty set, the assertion of Theorem 1 holds.

Let us take one of these intervals $(t_j, t_j+\delta_j)$, $j \in \{1,\dots,l\}$ and set $s = t_j$ and $\tilde{\delta} = \min\{1, \delta_j\}$. Applying the transformation

$$z(t) = \frac{\alpha_0}{\varepsilon_{max}}\ y(s+\tilde{\delta}t)\,, \tag{11}$$

the equation (4) can be changed into the form

$$z^{(n)}(t) + p_1(t)\ z^{(n-1)}(t) + \cdots + p_n(t)\ z(t) = \alpha \cdot q(t)\,,$$

where

$$\begin{aligned} &z^{(k)}(t) = \tfrac{\alpha_0}{\varepsilon_{max}}\ \tilde{\delta}^k\ y^{(k)}(s+\tilde{\delta}t)\,, && p_k(t) = \tilde{\delta}^k\ \alpha_k(s+\tilde{\delta}t)\,,\\ &q(t) = r(s+\tilde{\delta}t) - p(s+\tilde{\delta}t)\ \ y(s+\tilde{\delta}t)\,, && \alpha = \frac{\alpha_0}{\varepsilon_{max}}\ \ \tilde{\delta}^n\,. \end{aligned} \tag{12}$$

It is easy to verify the fulfilling of the assumptions of Lemma 5.1 from [6] and Theorem 5.1 from [6] by the functions $z(.), p_i(.), q(.)$ and the constant α.

So, summarizing, we have $0 < q_{min} \le q(t) \le q_{max}$. Next, it is clear that $0 \le z(t) \le 1$, for all $t \in [0,\tilde{\delta}]$. According to (10), and the definitions of $\tilde{\delta}$ and $p_i(.)$ we have $\sum_{i=1}^{n} |p_i(t)| \le P_1$. Hence Lemma 5.1 from [6], whose assumptions are fulfilled, ensures that the constant $\alpha > 0$, given by (12) satisfies the inequality $\alpha \leq \frac{2.2^{n(n+1)}}{q_{min}}$. This inequality implies

$$\frac{\alpha_0}{\varepsilon_{max}}\,\tilde{\delta}^n \leq \frac{2.2^{n(n+1)}}{q_{min}}$$

which using (6) gives

$$\frac{\tilde{\delta}^n}{\varepsilon_{max}} \leq 1\,.$$

Due to (6) we have

$$\tilde{\delta}^n \leq \left(\min\{p_0\,\delta_0\,, \frac{r_{min}}{2\,r_{max}}\}\right)^n, \qquad \text{or} \qquad \tilde{\delta} \leq \left(\min\{p_0\,\delta_0\,, \frac{r_{min}}{2\,r_{max}}\}\right),$$

which means that

$$\tilde{\delta} \leq p_0\ \delta_0 \qquad \text{and} \qquad \tilde{\delta} \leq \frac{r_{min}}{2\,r_{max}} \leq \frac{1}{2}\,.$$

Clearly, for all intervals $[t_i, t_i+\delta_i],\ i \geq 2$, the assumptions of Theorem 5.1 from [6] are fulfilled. The property $y(t_i) = \frac{\varepsilon_{max}}{\alpha_0}$ implies $z(t_i) = 1$. Due to Theorem 5.1 from [6] there exists a positive constant ε_2 such that $\mu\left(z^{-1}[0,\varepsilon_2]\right) \leq p_0$.
From (11) we can see that

$$\frac{1}{\delta_i}\ \mu\left([t_i, t_i+\delta_i] \cap y^{-1}[0, \varepsilon_2\ \frac{\varepsilon_{max}}{\alpha_0}]\right) \leq p_0\,. \tag{13}$$

This clearly forces

$$\mu\left([t_i, t_i+\delta_i] \cap y^{-1}[0,\varepsilon]\right) \leq p_0\ \delta_i \tag{14}$$

on all intervals $[t_i, t_i+\delta_i]$, for $i \geq 2$.

It remains to prove the validity of the estimation (14) on the interval $[t_1, t_1+\delta_1]$.

If, $y(t_1) = y(t_0) = \frac{\varepsilon_{max}}{\alpha_0}$, then the estimation (14) is evidently true.

If, $y(t_1) < \frac{\varepsilon_{max}}{\alpha_0}$, then using the backward transformation $x(t) = y(t_1+\delta_1-t)$ we obtain $x(t_1) = y(\delta_1) = \frac{\varepsilon_{max}}{\alpha_0}$. Let $z(t) = \frac{\alpha_0}{\varepsilon_{max}}\ x(t_1+\delta_1 t)$. Then by similar arguments we can obtain the estimation $\mu\left(z^{-1}[0,\varepsilon_2]\right) \leq p_0$ also on the interval $[t_1, t_1+\delta_1]$. Thus (compare with (13))

$$\frac{1}{\delta_1}\mu\left([t_1, t_1+\delta_1] \cap x^{-1}[0, \varepsilon_2\ \frac{\varepsilon_{max}}{\alpha_0}]\right) \leq p_0$$

and further

$$\mu\left([t_1, t_1+\delta_1] \cap x^{-1}[0,\varepsilon]\right) = \mu\left([t_1, t_1+\delta_1] \cap y^{-1}[0,\varepsilon]\right) \leq p_0\delta_1\,. \tag{15}$$

Recall that $\varepsilon = \varepsilon_2 \frac{\varepsilon_{max}}{\alpha_0}$, and $\varepsilon_2 \in (0,1)$. From (14) and (15) we can conclude that

$$\mu\left([t_0, t_0+\delta_0] \cap y^{-1}[0,\varepsilon]\right) \leq \mu\left(M \cap y^{-1}[0,\varepsilon]\right) =$$
$$= \mu\left([\bigcup_{i=1}^{l}(t_i, t_i+\delta_i)] \cap y^{-1}[0,\varepsilon]\right) \leq \sum_{\substack{1<i\leq l\\ t_i>t_0}} \mu\left([t_i, t_i+\delta_i] \cap y^{-1}[0,\varepsilon]\right) + p_0\,\delta_1 \leq$$
$$\leq p_0\,\delta_1 + \sum_{\substack{1<i\leq l\\ t_i>t_0}} p_0\ \delta_i \leq p_0\left[\delta_1 + \sum_{1<i\leq l} \mu\left(t_i, t_i+\delta_i\right)\right] = p_0\ \mu(M) \leq p_0\ \delta\,,$$

which is the required conclusion. □

3. Formulation of the Problem

The aim of this paper is to study properties of solutions of the nonlinear $n+1^{th}$ - order differential equation of the form

$$\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t) + p(t)x(t)\right)\ \right]' = f(t, x(t), \cdots, x^{(n)}(t))\,. \tag{16}$$

where $n \geq 2$ is a natural number.

Let M_1 and M_2 be constants such that $0 < M_1 \leq M_2$.

Let the functions $p(.)$, $r_1(.)$ and $f = f(t, x_0, \ldots, x_n)$ satisfy conditions:

- Let the function $r_1(.) \in \mathbb{C}^1[0,\infty)$ and have the property

$$0 < M_1 \leq r_1(t) \leq M_2\,, \qquad \forall t \geq \tilde{T} > 0\,, \tag{17}$$

- Let the function $p(.) \in \mathbb{C}^1[0,\infty)$ have the property

$$0 < M_1 \leq p(t) \leq M_2\,, \qquad \forall t \geq \tilde{T} > 0\,, \tag{18}$$

- Let the function $f = f(t, x_0, x_1, \ldots, x_n)$ be continuous on $R^+ \times R^{n+1}$ and has the sign property

$$f(t, x_0, x_1, \ldots, x_n)\, x_0 \leq 0\,. \tag{19}$$

  Moreover there exist functions $p_0(.)$, and $\omega(.) \in \mathbb{C}[0,\infty)$ such that

$$f(t, x_0, x_1, \ldots, x_n)\ sign\ (x_0) \leq -p_0(t)\,\omega(|x_0|)\,, \tag{20}$$

where the functions $p_0(.)$, and $\omega(.)$ have the properties

(21) $\omega: \ [0,\infty) \to [0,\infty)$ is non-decreasing function,

(22) $\omega(0) = 0\,,$

(23) $\omega(s) > 0\,, \qquad \forall s > 0\,,$

(24) $p_0: \ [0,\infty) \to [0,\infty)$

$p_0(t) \not\equiv 0$ on any subinterval of $[0,\infty)\,,$

and the function $p_0(t)$ is strongly non-integrable

(25) on the interval $[1,\infty)\,.$

Definition 1. We call a function $f(.)$ **strongly non-integrable**, if $f(.)$ is non-negative and locally integrable function on an interval $[T,\infty)$ and if there exist the constants $\delta_0 > 0$ and $p_0 \in (0,1)$ with the property:

For each set $M \in \mathcal{B}(\mathcal{R})$ such that $M \subset [T,\infty)$ and

(26) $$\mu\left(M \cap [t_0, t_0+\delta]\right) \geq (1-p_0)\,\delta \qquad \forall t_0 \geq T\,,\ \forall \delta \geq \delta_0$$

the function $f(.)$ satisfies

(27) $$\int\limits_M f\ d\mu = +\infty\,.$$

Remark 1. It is easy find a function, which is strongly non-integrable:
e.g. let $f(.) \in \mathbb{C}[T,\infty)\,, f(t) > s > 0\,, \forall t \in [T,\infty)$, where s be an positive constant. $f(.)$ is a non-integrable function and in the sense of the Definition 1 is strongly non-integrable, too.

Remark 2. Non all non-integrable function are strongly non-integrable:
e.g. let $f(.) \in \mathbb{C}[T,\infty)$, be defined by

$$f(t) = \begin{cases} 1/t^2\,, & t \in [2n, 2n+1] \\ 1\,, & t \in (2n+1, 2n+2) \quad \text{for all } n \in N\,. \end{cases}$$

If we take $M = \cup_{n\in N}\ [2n, 2n+1]$ then M satisfy (26) e.g. for $p_0 = 1/4\,, \delta_0 = 2$ we get $\int\limits_M f\ d\mu < +\infty\,.$

4. "Submersivity " of a solution of the equation $\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t) + p(t)x(t)\right)\ \right]' = f(t, x(t), \cdots, x^{(n)}(t))\,.$

In this section we will examine "submersivity " of a solution of the equation

$$\left[\frac{1}{r_1(t)}\ \left(x^{(n)}(t) + p(t)x(t)\right)\ \right]' = f(t, x(t), \cdots, x^{(n)}(t))\,.$$

We will compare the properties of the solution of (4) with the properties of solutions of our equation. Our aim was to find conditions of " submersivity" for the solution of the equation $y^{(n)}(t)+\alpha_1(t)\,y^{(n-1)}(t)+...+\alpha_n(t)\,y(t)+p(t)\,y(t)=r(t)$.

We will see that the similar conditions of "submersivity" can be found for the solutions of the equation (16). The conclusions obtained in the next theorem, we will use for proving the main theorem of this contribution.

Let us define on some interval $[\tilde{T},\infty)$, for a function $x(.)\in\mathbb{C}^{n+1}\,[T,\infty)$ the function

$$\alpha_1(t)=\frac{1}{r_1(t)}\,\left[x^{(n)}(t)+p(t)\,x(t)\right]\,, \tag{28}$$

Theorem 2. *Let functions* $p(.)$, $r_1(.)\in\mathbb{C}^1[T_0,\infty)$ *satisfy the assumptions* (17) *and* (18) *on the interval* $[T_0,\infty)$. *Let the function* $f=f(t,x_0,x_1,\dots,x_n)$ *be continuous on* $R^+\times R^{n+1}$ *and satisfy properties* (19)–(25). *Moreover let* $0<\lim\limits_{t\to\infty}\alpha_1(t)$.

Then for each $\delta_0>0$ *and each* $p_0\in(0,1)$, *there exist* $T>T_0$ *and* $\varepsilon>0$ *with the following property:*

If $x(.)\in\mathbb{C}^{n+1}[T_0,\infty)$ *is a non-negative solution of the differential equation* (16), *then for all* $t_0>T$ *and for all* $\delta>\delta_0$ *we have*

$$\mu\left([t_0,t_0+\delta]\cap\left[\frac{x}{\alpha_1}\right]^{-1}[0,\varepsilon]\right)\le p_0\,\delta\,, \tag{29}$$

where μ *denotes the Lebesgue measure of sets.*

Proof. Let the function $\alpha_1(t)$ be defined by (28). Due to sign property (19) we get

$$\alpha_1{}'(t)=f(t,x(t),x'(t),\dots,x^{(n)}(t))\le 0\,.$$

Since $0<\lim\limits_{t\to\infty}\alpha_1(t)$ and $\alpha_1{}'(t)\le 0$, it follows that $0<\lim\limits_{t\to\infty}\alpha_1(t)<\infty$. Further according to (28) we have

$$x^{(n)}(t)+p(t)\,x(t)=\alpha_1(t)\,r_1(t)\,. \tag{30}$$

The equation (30) formally can be written in the form (4), where $\alpha_i(t)\equiv 0$ for all $i\in\{1,\dots,k\}$ and $r(.)=\alpha_1(.)\,r_1(.)$. Hence, by Theorem 1 whose assumptions are satisfied on some $[T',\infty)\,,T'\ge T_0$, for all constants $\delta_0>0$ and $p_0\in(0,1)$ there exist $T\ge T'\ge T_0$ and $\varepsilon_1>0$ such that for all $t_0>T$ and for all $\delta>\delta_0$ we get $\mu\left([t_0,t_0+\delta_0]\cap x^{-1}[0,\varepsilon_1]\right)\le p_0\ \delta$. If we take $\varepsilon=\varepsilon_1/\sup\limits_{t\ge T}\alpha_1(t)$, then it holds

$$\mu\left([t_0,t_0+\delta]\cap\left[\frac{x}{\alpha_1}\right]^{-1}[0,\varepsilon]\right)\le p_0\,\delta\,,$$

which proves the claim of our theorem. □

5. The Main Theorem

Theorem 3. *Let $x(.) \in \mathbb{C}^{n+1}[0,\infty)$ be a solution of the differential equation of the form* (16)*, where $n \geq 2$ is natural s number.*

Let the functions $p(.)$ and $r_1(.)$ satisfy the conditions (18) *and* (17) *respectively.*

Let further the right-hand side of (16) *satisfy* (19) - (25)*.*

Then $x(.)$ is either oscillatory solution of the diff. equation (16)*, or there exists some function $\alpha(t)$, with the property*

$$\alpha(t) \geq 0\,, \qquad \forall t \in [T_0,\infty), T_0 \geq 0\,, \qquad \text{and } \lim_{t\to\infty} \alpha(t) = 0\,,$$

and $x(.)$ solves the diff. equation

$$x^{(n)}(t) + p(t)\; x(t) = \alpha(t)\; sign\; x(t)\,,$$

on some neighbourhood of infinity.

Proof. Let $x(.)$ be a proper non-oscillatory solution of the diff. eq. (16).

It is sufficient to study a non-negative non-oscillatory solution $x(t)$ on an interval $[\tilde{T},\infty)$. Otherwise if $x(t) \leq 0$ in $[\tilde{T},\infty)$, then the function

$$y(t) = -x(t)\,, \qquad t \in [\tilde{T},\infty)\,,$$

satisfies the differential equation

$$\left[\frac{1}{r_1(t)}\;\left(y^{(n)}(t) + p(t)y(t)\right)\;\right]' = f_1(t, y(t), y'(t), \cdots, y^{(n)}(t))\,, \tag{31}$$

where the function $f_1(t, x_0, x_1, \ldots, x_n) = -f(t, -x_0, -x_1, \ldots, -x_n)$ has all properties of the function f, i.e. f_1 is continuous on $R^+ \times R^{n+k}$ and satisfies the conditions (19) – (25).

Thus all properties of non-negative solutions of the equation (31) can be transformed to the similar ones of the non-positive solution of (16).

Hence in this theorem we will consider only non-negative solution and the statement will be true also for the non-positive solutions.

Let the function $\alpha_1(t)$ be define by (28). We conclude from the sign property (19) of the equation (16) on interval $[\tilde{T},\infty)$ that

$$\alpha'_1(t) = f(t, x(t), x'(t), \ldots, x^{(n)}(t)) \leq 0 \qquad \text{for } \; x(t) \geq 0\,,$$

hence that

$$\alpha'_1(t) \leq 0\,, \qquad \forall t \in [\tilde{T},\infty)\,. \tag{32}$$

By (32), it is obvious that $\alpha_1(t)$ is non-increasing function on interval $[\tilde{T},\infty)$. Thus there exists T', $T' \geq \tilde{T}$ such that the function $\alpha_1(t)$ does not change its sign on interval $[T',\infty)$.

We can certainly assume the existence of such T_0, $T_0 \geq T'$ with the property

$$sign\ (\alpha_1(t)) = \text{constant}, \qquad \forall t \in [T_0,\infty). \tag{33}$$

The proof will be divided into three cases.

(A) $\lim\limits_{t\to\infty} \alpha_1(t) < 0$,

(B) $\lim\limits_{t\to\infty} \alpha_1(t) > 0$,

(C) $\lim\limits_{t\to\infty} \alpha_1(t) = 0$.

5.1. (A) $\lim\limits_{t\to\infty} \alpha_1(t) < 0$

If $\lim\limits_{t\to\infty} \alpha_1(t) < 0$, then there exists T_1, $T_1 \geq T_0$ such that for all $t \geq T_1$, $\alpha_1(t) \leq -\varepsilon < 0$, where $\varepsilon > 0$. By the definition of $\alpha_1(t)$ and the constants M_1, M_2 we get $x^{(n)}(t) + p(t)x(t) = \alpha_1(t)\ r_1(t) \leq -\varepsilon\ M_1 < 0$.

As $x(t) \geq 0$, and $p(t) > 0$ on $[T_1,\infty)$ we have $x^{(n)}(t) \leq -\varepsilon\ M_1 < 0$. Let $T_1 \leq t_1 \leq t$ and $\tau \in [t_1, t]$. By integration we come to the inequality

$$x^{(n-1)}(t) \leq x^{(n-1)}(t_1) - \varepsilon\ M_1\ (t - t_1).$$

In the limit case, if $t \to \infty$ we obtain

$$\lim_{t\to\infty} x^{(n-1)}(t) \leq x^{(n-1)}(t_1) - \varepsilon\ M_1 \lim_{t\to\infty} (t - t_1) = -\infty,$$

which contradicts $x(t) \geq 0$.

5.2. (B) $\lim\limits_{t\to\infty} \alpha_1(t) > 0$

Consider the functions $p_0(.)$, and $\omega(.) \in \mathbb{C}[0,\infty)$, for which (20) – (25) hold.
By Theorem 2 for all $\delta_0 > 0$, $p_0 \in (0,1)$, there exist $T \geq T_0$ and $\varepsilon > 0$ with the property: If $x(.) \in \mathbb{C}^{n+1}[T_0,\infty)$ is a non-negative solution of the differential equation (16), then for all $t_0 > T$ and for all $\delta > \delta_0$ we have

$$\mu\left([t_0, t_0+\delta] \cap \left[\frac{x}{\alpha_1}\right]^{-1} [0,\varepsilon]\right) \leq p_0\ \delta, \tag{34}$$

where μ denotes the Lebesgue measure of sets.

Let the constants ε and T be given by Theorem 2. Let us denote by $\mathcal{M}$ the set

$$\mathcal{M} = \{t : t \geq T,\ x(t) \geq \varepsilon\ \alpha_1(t)\}. \tag{35}$$

The set $\mathcal{M}$ has the property (26) from Definition 1, Theorem 2, Definition 1 and the assumptions (24), (25) yield $\int\limits_{\mathcal{M}} p_0(t)\ d\mu = +\infty$.

Due to (33) we have $\alpha_1(t) \geq 0$ for all $t \geq T \geq T_0$. Choose arbitrary t_0, t such that $T \leq t_0 \leq t$. We have

$$0 \leq \alpha_1(t) = \alpha_1(t_0) + \int\limits_{t_0}^{t} {\alpha_1}'(s)\ ds = \alpha_1(t_0) - \int\limits_{t_0}^{t} |f(s, x(s), \cdots, x^{(n)}(s))|\ ds\,.$$

and hence $|\alpha_1(t_0)| \geq \int\limits_{t_0}^{t} |f(s, x(s), \cdots, x^{(n)}(s))| ds$, which for $t \to \infty$ implies, $|\alpha_1(t_0)| \geq \int\limits_{t_0}^{\infty} |f(s, x(s), \cdots, x^{(n)}(s))| ds$. Putting $t = t_0$ we obtain

$$|\alpha_1(t)| \geq \int\limits_{t}^{\infty} |f(s, x(s), \cdots, x^{(n)}(s))|\ ds\,, \qquad \forall t \geq t_0 \geq T\,.$$

If we use the previous result, for all $t \geq T$ we get

$$|\alpha_1(t)| \geq \int\limits_{t}^{\infty} |f(s, x(s), \cdots, x^{(n)}(s))|\ ds \overset{(20)}{\geq} \int\limits_{t}^{\infty} p_0(s)\ \omega(|x(s)|)\ ds\,.$$

Define the function $n : [T, \infty) \to \{0, 1\}$ as follows

$$n(s) = \begin{cases} 1 & s \in \mathcal{M} \\ 0 & s \notin \mathcal{M}\,, \end{cases}$$

where the set $\mathcal{M}$ is given by (35).

Since the functions $\alpha_1(t)$ do not change their signs on the interval $[T_0, \infty)$ and $\alpha'_1(t) \leq 0$, the function $\alpha_1(t)$ is non-increasing function on interval $[T_0, \infty)$ with the property $\lim\limits_{t\to\infty} \alpha_1(t) = \tilde{\alpha}_1 > 0$. Put $\varepsilon_1 = \frac{\tilde{\alpha_1}}{\alpha_1(T_0)}$. We thus get

$$\alpha_1(\tau) \geq \tilde{\alpha}_1 = \varepsilon_1\ \alpha_1(T_0) \geq \varepsilon_1\ \alpha_1(t)\ , \qquad \forall \tau \geq t \geq T_0\,.$$

Hence

$$\alpha_1(\tau) \geq \varepsilon_1\ \alpha_1(t)\,, \qquad \forall \tau \geq t \geq T\,. \tag{36}$$

Since the function $\omega(.)$ is non-decreasing non-negative function and using the estimation (36), we obtain on the set $\mathcal{M}$

$$\omega(|x(s)|) = \omega(x(s)) \geq \omega(\varepsilon\ \alpha_1(s))\,.$$

According to the above definition of the function $n(.)$

$$\omega(x(s)) \geq n(s)\ \omega(\varepsilon\ \alpha_1(s)) \geq n(s)\ \omega(\varepsilon\ \varepsilon_1 \alpha_1(t))\,, \qquad \forall s \geq t \geq T\,,$$

be valid. Further, the function $\omega(.)$ satisfies (22), (23) and due to (36) we get

$$\alpha_1(t) \geq \int_t^\infty p_0(s)\ n(s)\ \omega(\varepsilon\ \varepsilon_1\ \alpha_1(t_0))\ ds\,, \qquad \forall\ t \geq t_0 \geq T\,,$$

and hence

$$\infty > \frac{\alpha_1(t_0)}{\omega(\varepsilon\ \varepsilon_1\ \alpha_1(t_0))} \geq \int_{t_0}^\infty p_0(s)\ n(s)\ ds = \int_{\mathcal{M}\cap[t_0,\infty)} p_0(s)\ ds = +\infty\,.$$

which is impossible.

The cases (A) and (B) led to contradiction. Therefore the case (C), holds.

5.3. (C) $\lim_{t\to\infty} \alpha_1(t) = 0$

Due to sign property (32) we get $\alpha'_1(t) \leq 0$ for all $t \geq T_0$ and $\lim_{t\to\infty} \alpha_1(t) = 0$ and hence

$$\alpha_1(t) \geq 0\,, \qquad \forall t \geq T_2 \geq T_0\,.$$

If we take $\alpha(t) = r_1(t)\ \alpha_1(t)$, then we have $\lim_{t\to\infty} \alpha(t) = 0$ and $\alpha(t) \geq 0$, for all $t \geq T_2 \geq T_0$.

From the above it follows that, either $x(.)$ is oscillatory solution, or $x(.)$ is proper non-oscillatory solution on some interval $[T_0, \infty)$ and then there exists the function $\alpha(t)\,,\ \alpha(t) \geq 0$ for all $t \in [T_2, \infty)$, with the property $\lim_{t\to\infty} \alpha(t) = 0$, such that $x(.)$ will be a solution of equation

$$x^{(n)}(t) + p(t)x(t) = \alpha(t)\ sgn\ x(t)\,,$$

and the proof is complete. □

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