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A MODIFIED STRONG SQUEEZING PROPERTY AND THE EXISTENCE OF INERTIAL MANIFOLDS OF SEMIFLOWS

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Abstract. Sometimes so-called cone invariance and squeezing properties are used to show the existence of inertial manifolds for evolution equations. We propose and motivate a modification of these properties for semiflows. We show that the cone invariance and modified squeezing properties together with a coercivity assumption are sufficient for a general, continuous semiflow to have an inertial manifold with exponential tracking property.

Keywords. semiflow; inertial manifold; asymptotic phase; cone invariance property; squeezing property

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1. Inertial Manifolds for Semiflows

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and let S be a semiflow on $\mathbb{X}$, i.e., let $S\colon \mathbb{R}_{\geq 0} \times \mathbb{X} \to \mathbb{X}$, $S^t x := S(t,x)$ satisfy

(S1) $(S^t)_{t\in\mathbb{R}_{\geq 0}}$ is a strongly continuous semigroup of (nonlinear) continuous operators, i.e.,

$$S^0 = I, \quad S^t S^\theta = S^{t+\theta} \text{ for all } t, \theta \geq 0,$$

and $S(\cdot, x)$ and $S^t = S(t,\cdot)$ are continuous for all $x \in \mathbb{X}$, $t \in \mathbb{R}_{\geq 0}$.

Our goal is to find a submanifold M of $\mathbb{X}$ with the following properties:

(M1) M is a finite-dimensional Lipschitz submanifold of $\mathbb{X}$;

(M2) M is *positively invariant* with respect to S, i.e.,

$$\forall u \in M\ \forall t \geq 0\colon\ S^t u \in M; \tag{1}$$

(M3) M has the *exponential tracking property*, i.e., there is an $\eta > 0$ such that, for every $x \in \mathbb{X}$, there are $x' \in M$, $c \geq 0$ with $S^t x' \in M$ and

$$\left\| S^t x - S^t x' \right\| \leq c\mathrm{e}^{-\eta t} \quad \text{for all } t \geq 0.$$

Obviously, such a manifold is a generalization of inertial manifolds for evolution equations which were first introduced and studied by P. Constantin, C. Foias, B. Nicoalenko, G.R. Sell and R. Temam [4,3,1], see also [14], and [6,12] for the exponential tracking property.

As usual, we look for M as a trivial submanifold of $\mathbb{X}$, i.e., we look for

$$M = \operatorname{graph}(m) := \{\xi + m^*(\xi)\colon \xi \in \pi_1\mathbb{X}\}$$

as the graph of function m over a finite-dimensional subspace $\mathbb{X}_1$ of $\mathbb{X}$, where π_1 is a continuous projector from $\mathbb{X}$ onto $\mathbb{X}_1$. Moreover, m shall belong to the Banach space $\mathbb{G} = \mathrm{C_b}(\pi_1\mathbb{X}, \pi_2\mathbb{X})$ of continuous, bounded functions and shall satisfy the Lipschitz inequality

$$\|m(\xi_1) - m(\xi_2)\| \leq \chi \|\xi_1 - \xi_2\| \quad \text{for all } \xi_1, \xi_2 \in \pi_1\mathbb{X}$$

with some fixed $\chi > 0$.

In Sect. 2, we introduce a modification of the cone invariance and squeezing properties (called *modified strong squeezing property*) as a natural geometric assumption on a semiflow to have an inertial manifold as graph of a bounded, globally Lipschitz function over a finite-dimensional subspace. In Sect. 3, we show that this property together with a coercivity property is actually sufficient for the existence of an inertial manifold. In the both last sections, we give a short application to evolution equations and we propose some extensions to more general results.

2. The Strong Squeezing Properties

Cone Invariance Property: If we look for $m \in \mathbb{G}$ satisfying the Lipschitz condition

$$\|m(\xi_1) - m(\xi_2)\| \leq \chi \|\xi_1 - \xi_2\| \quad \text{for all } \xi_1, \xi_2 \in \pi_1\mathbb{X}, \tag{2}$$

and if we don't have additional boundedness properties, we have to look for m in $\mathfrak{M}$, where $\mathfrak{M}$ is the set of all $m \in \mathbb{G}$ with (2). Introducing the cone

$$C_\chi := \{x \in X\colon\ \|\pi_2 x\| \leq \chi \|\pi_1 x\|\},$$

we have

$$m \in \mathfrak{M} \text{ if and only if } m \in \mathbb{G} \text{ and } \forall x \in \operatorname{graph}(m)\colon\ \operatorname{graph}(m) \in x + C_\chi. \tag{3}$$

The required positive invariance (1) and the equivalence (3) yield

$$x_1, x_2 \in \mathrm{graph}(m),\ t \geq 0 \text{ imply } S^t x_1 - S^t x_2 \in C_\chi.$$

Since we only know $m \in \mathfrak{M}$ and because of (3), we replace $x_i \in \mathrm{graph}(m)$ by $x_1 - x_2 \in C_\chi$ and get the following relation

$$\text{(CIP)} \qquad x_1 - x_2 \in C_\chi \text{ implies } S^t x_1 - S^t x_2 \in C_\chi \text{ for } t \geq 0$$

as a natural assumption for the existence of the manifold.

Since (CIP) means the invariance of the cone C_χ with respect to the difference of two positive trajectories, (CIP) is called ***cone invariance property***.

Squeezing Properties: In order to motivate the squeezing properties, we consider the following situation: We assume that S satisfies a cone invariance property (CIP) with parameter $\chi > 0$, and we assume that we have a positively invariant manifold $M = \mathrm{graph}(m)$, $m \in \mathfrak{M}$, with exponential tracking property. Concretely, we assume that for each $x_1 \in \mathbb{X} \setminus M$ there is a $\tilde{x}_1 \in M$ with

$$\text{(4)} \qquad \left\| S^t x_1 - S^t \tilde{x}_1 \right\| \leq c_1 \mathrm{dist}(x_1, M) \mathrm{e}^{-\eta t} \quad \text{for all } t \geq 0,$$

i.e., we assume that the exponential decays of the difference of the trajectory and its asymptotic phase is estimated by the initial distance of x_1 to the manifold.

We sharpen the assumptions on m by the additional assumption that m actually has a Lipschitz constant $L < \chi$.

Then there is a constant $c_2 > 0$ such that

$$\text{(5)} \qquad \forall x, y, z \in \mathbb{X} \text{ with } x - z \notin C_\chi, y - z \in C_L\colon \ \ \|x - z\| \leq c_2 \|x - y\|.$$

Let $x_1 \in \mathbb{X} \setminus M$ and $\tilde{x}_1 \in M$ with (4) and $x_1 - \tilde{x}_1 \notin C_\chi$, and let $\theta > 0$ and $x_2 \in M$ with $S^\theta x_1 - S^\theta x_2 \notin C_\chi$. Then (CIP) implies $S^t x_1 - S^t x_2 \notin C_\chi$ for $t \in [0, \theta]$. With $x = S^t x_1$, $y = S^t \tilde{x}_1$, $z = S^t x_2$ and (5), we obtain

$$\text{(6)} \qquad \left\| S^t x_1 - S^t x_2 \right\| \leq c_2 \left\| S^t x_1 - S^t \tilde{x}_1 \right\| \leq c_1 c_2 \mathrm{dist}(x_1, M) \mathrm{e}^{-\eta t}$$

for all $\theta > 0$, $t \in [0, \theta]$ and all $x_2 \in M$ with $S^\theta x_2 - S^\theta x_1 \notin C_\chi$. Since $\mathrm{dist}(x_1, M) \leq \|x_1 - x_2\|$ and $\|x_1 - x_2\| \leq \sqrt{1 + \chi^{-2}}\, \|\pi_2[x_1 - x_2]\|$, we obtain

$$\left\| S^t x_1 - S^t x_2 \right\| \leq c_3 \|\pi_2[x_1 - x_2]\| \,\mathrm{e}^{-\eta t}$$

with some $c_3 > 0$ and for all $\theta > 0$, $t \in [0, \theta]$ and all $x_1 \in \mathbb{X} \setminus M$, $x_2 \in M$ with $S^\theta x_1 - S^\theta x_2 \notin C_\chi$.

For unknown M, this leads to the assumption

$$\text{(SP)} \qquad \begin{array}{l} \text{There are } \chi_2, \eta > 0 \text{ such that } \theta > 0,\ S^\theta x_1 - S^\theta x_2 \notin C_\chi \text{ imply} \\ \|S^t x_1 - S^t x_2\| \leq \chi_2 \|\pi_2[x_1 - x_2]\| \,\mathrm{e}^{-\eta t} \text{ for all } t \in [0, \theta] \end{array}$$

called ***squeezing property***.

Let us restart with (6). Estimating $\operatorname{dist}(x_1, M) \le \|\pi_2[x_1 - x_3]\|$ with $x_3 \in M$ and $\pi_1 x_3 = \pi_1 x_1$, replacing $x_3 \in M$ by $x_3 - x_2 \in C_\chi$, and replacing $S^\theta x_1 - S^\theta x_2 \notin C_\chi$ by the sharper assumption $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$, we find the following modification of squeezing property:

(modSP) There are $\chi_{21}, \chi_{22}, \eta > 0$ such that $\theta > 0$, $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$ imply $\|\pi_i[S^t x_1 - S^t x_2]\| \le \chi_{2i} \|\pi_2[x_1 - x_3]\| \mathrm{e}^{-\eta t}$ for all $t \in [0, \theta]$ and all x_3 with $\pi_1 x_3 = \pi_1 x_1$ and $x_3 - x_2 \in C_\chi$

called ***modified squeezing property***.

The combination of the cone invariance property (CIP) with the squeezing property (SP) is called ***strong squeezing property***, see [11]. Analogously, the combination of the cone invariance property (CIP) with the modified squeezing property (modSP) is called ***modified strong squeezing property***.

In the next section we will see the usefullness of the modified strong squeezing property for the existence proof of an inertial manifold. Before this, we compare the strong squeezing property with the modified strong squeezing property.

Checking the proofs of cone invariance properties found in [2,5,6,10,11,14], one can see that the number χ usually is a solution of an inequality $F(\chi) > 0$, where $F\colon]0, \infty[\to \mathbb{R}$ is a smooth function. Obviously, at least in these cases a second cone invariance property is satisfied. At least in [11, Proposition 3], such a second cone invariance property is explicitly used.

Lemma 1. *Let the cone invariance property* (CIP) *and the squeezing property* (SP) *be satisfied with the parameter $\chi > 0$. Suppose, there exists $\chi' > \chi$ such that we have a second cone invariance property with χ' instead of χ. Then the modified squeezing property* (modSP) *is satisfied with* $\chi_{21} := \frac{\chi_2 \chi'}{\chi(\chi' - \chi)}$, $\chi_{22} := \frac{\chi_2 \chi'}{\chi' - \chi}$.

Proof. Let $x_1, x_2 \in \mathbb{X}$ with $\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2$ and $\pi_2 S^\theta x_1 \neq \pi_2 S^\theta x_2$. Then $S^\theta x_1 - S^\theta x_2 \notin C_\chi$ and $S^\theta x_1 - S^\theta x_2 \notin C_{\chi'}$. The cone invariance property implies $S^t x_1 - S^t x_2 \notin C_\chi$ and $S^t x_1 - S^t x_2 \notin C_{\chi'}$ for all $t \in [0, \theta]$, i.e., we have

$$\chi \left\|\pi_1[S^t x_1 - S^t x_2]\right\| \le \chi' \left\|\pi_1[S^t x_1 - S^t x_2]\right\| < \left\|\pi_2[S^t x_1 - S^t x_2]\right\| \tag{7}$$

for all $t \in [0, \theta]$. Let $x_3 \in \mathbb{X}$ with $\pi_1 x_3 = \pi_1 x_1$ and $x_3 - x_2 \in C_\chi$, i.e.,

$$\|\pi_2[x_2 - x_3]\| \le \chi \|\pi_1[x_2 - x_1]\| . \tag{8}$$

Using (7) and (8), we find $\chi' \|\pi_1[x_1 - x_2]\| \le \|\pi_2[x_1 - x_3]\| + \chi \|\pi_1[x_1 - x_2]\|$ and, hence,

$$\|\pi_1[x_1 - x_2]\| \le \frac{1}{\chi' - \chi} \|\pi_2[x_1 - x_3]\| .$$

By the squeezing property (SP) and (7), we have

$$\begin{aligned}
\left\|\pi_2[S^t x_1 - S^t x_2]\right\| &\le \chi_2 \mathrm{e}^{\eta t} \left(\|\pi_2[x_1 - x_3]\| + \|\pi_2[x_3 - x_2]\|\right) \\
&\le \chi_2 \mathrm{e}^{\eta t} \left(\|\pi_2[x_1 - x_3]\| + \chi \|\pi_1[x_1 - x_2]\|\right) \\
&\le \frac{\chi_2 \chi'}{\chi' - \chi} \mathrm{e}^{\eta t} \|\pi_2[x_1 - x_3]\| ,
\end{aligned}$$

for all $t \in [0, \theta]$, i.e., (modSP) holds.

Thus, the strong squeezing property together with a second cone invariance property implies our modified strong squeezing property, i.e., in general, the modified strong squeezing property is the weaker assumption.

3. Construction of Inertial Manifolds

Let S be a semiflow on the Banach space $\mathbb{X}$. Let $\mathbb{X}_1$ be a finite-dimensional subspace of $\mathbb{X}$, π_1a continuous projector from $\mathbb{X}$ onto $\mathbb{X}_1$ and let $\pi_2 = I - \pi_1$. We assume that S satisfies the cone invariance property (CIP) and the modified squeezing property (modSP) with fixed $\chi > 0$. As technical assumptions we need

(S2) S satisfies the coercivity property $\|\pi_1 S^t x\| \to \infty$ as $\|\pi_1 x\| \to \infty$ in $\mathbb{X}$ for $t \geq 0$.

(S3) There is a positively invariant strip $\sum := \{x \in \mathbb{X} : \|\pi_2 x\| \leq \sigma\}$.

Theorem 1. *Under the above assumptions, there is an inertial manifold* $M =$ graph(m) *with bounded* $m\colon \pi_1\mathbb{X} \to \pi_2\mathbb{X}$ *satisfying a global Lipschitz condition with constant* χ*. Moreover, for each* $x_1 \in \mathbb{X}$*, there is a* $x_2 \in M$ *with*

$$\left\|\pi_i[S^t x_1 - S^t x_2]\right\| \leq \chi_{2i} \left\|\pi_2 x_1 - m^*(\pi_1 x_1)\right\| \mathrm{e}^{-\eta t} \quad \textit{for all } t > 0.$$

Proof. We devide the proof into the following three steps:

Step 1: The Graph Transformation Mapping. We wish to construct $M =$ graph(m^*) by an graph transformation mapping, i.e., m^* shall be the fixed point of suitable mappings $G^\theta\colon \mathfrak{M} \to \mathbb{G}$, $\theta > 0$, with

$$\mathrm{graph}(G^\theta m) = S^\theta \mathrm{graph}(m) \quad \text{for all } m \in \mathfrak{M}.$$

where $\mathfrak{M}$ is the set of all $m \in \mathbb{G}$ with (2) and graph$(m) \subset \sum$. Concretely, we wish to define G^θ by $(G^\theta m)(\xi) := \pi_2 S^\theta x$ if $\pi_1 S^\theta x = \xi$. For it, we have to show that, for any $\xi \in \pi_1\mathbb{X}$, $\theta > 0$, $m \in \mathfrak{M}$, the boundary value problem

$$x \in \mathrm{graph}(m), \quad \pi_1 S^\theta x = \xi \tag{9}$$

has a unique solution $x = X(\theta, \xi, m)$.

Let $\theta > 0$, $\xi \in \pi_1\mathbb{X}$, $m \in \mathfrak{M}$, and x_1, x_2 with

$$\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2 = \xi \quad \text{and} \quad x_2 \in \mathrm{graph}(m).$$

If we choose $x_3 := \pi_1 x_1 + m(\pi_1 x_1)$, then the modified squeezing property (modSP) implies

$$\left\|\pi_i[S^t x_1 - S^t x_2]\right\| \leq \chi_{2i} \left\|\pi_2 x_1 - m(\pi_1 x_1)\right\| \mathrm{e}^{-\eta t} \quad \text{for all } t \in [0, \theta]. \tag{10}$$

In particular, for $x_1 \in \mathrm{graph}(m)$, we have $\pi_2 x_1 = m(\pi_1 x_1)$ and (10) implies

$$\forall\theta > 0 \forall x_1, x_2 \in \mathrm{graph}(m)\colon\ \pi_1 S^\theta x_1 = \pi_1 S^\theta x_2 \Longrightarrow x_1 = x_2, \tag{11}$$

i.e., for each $\theta > 0$, $m \in \mathfrak{M}$, $\xi \in \pi_1\mathbb{X}$ there is at most one $x \in \mathrm{graph}(m)$ with $\pi_1 S^\theta x = \xi$.

Let $\theta > 0$, $m \in \mathfrak{M}$ be fixed and let $H\colon \pi_1\mathbb{X} \to\colon \pi_1\mathbb{X}$ be defined by $H(\zeta) := \pi_1 S^\theta(\zeta + m(\zeta))$.

By the continuity of S^θ, H is continuous with inverse H^{-1} given by $H^{-1}(\xi) = \pi_1 X(\theta, \xi, m)$ on $H\pi_1\mathbb{X}$. In order to show $H\pi_1\mathbb{X} = \pi_1\mathbb{X}$, we wish to show the continuity of H^{-1}. Suppose, there is a $\xi \in \pi_1\mathbb{X}$ such that H^{-1} is not continuous at ξ. Then there are $\varepsilon > 0$ and a sequence $(\xi_k)_{k\in\mathbb{N}}$ in $\pi_1\mathbb{X}$ such that $\xi_k \to \xi$ as $k \to \infty$ and

$$\|\zeta - \zeta_k\| \geq \varepsilon \quad \text{for all } k \in \mathbb{N} \tag{12}$$

where $\zeta := X(\theta, \xi, m)$, $\zeta_k := X(\theta, \xi_k, m)$.

First we suppose that there is a subsequence of $(\zeta_k)_{k\in\mathbb{N}}$, denoted for shortness again by $(\zeta_k)_{k\in\mathbb{N}}$, with $\|\zeta_k\| \to \infty$ as $k \to \infty$. Then the coercivity property (S2) implies $\left\|\pi_1 S^\theta(\zeta_k + m(\zeta_k))\right\| \to \infty$ in contradiction to $S^\theta(\zeta_k + m(\zeta_k)) \to S^\theta(\zeta + m(\zeta))$.

Remains the boundedness of $(\zeta_k)_{k\in\mathbb{N}}$. Since $\pi_1\mathbb{X}$ is finite-dimensional space $\pi_1\mathbb{X}$, there is a convergent subsequence, denoted for shortness again by $(\zeta_k)_{k\in\mathbb{N}}$, with a limit $\zeta_\infty \in \pi_1\mathbb{X}$. By the continuity of S^θ, we have $S^\theta(\zeta_\infty + m(\zeta_\infty)) = S^\theta(\zeta + m(\zeta))$, and hence $\zeta = \zeta_\infty$ in contrast to (12) and (11).

Therefore, H and H^{-1} are continuous. Because of (S2), we have $\|H(\xi)\| \to \infty$ for $\|\xi\| \to \infty$. Thus, H is a homeomorphism from $\pi_1\mathbb{X}$ onto $\pi_1\mathbb{X}$ and hence we have $H\pi_1\mathbb{X} = \pi_1\mathbb{X}$. Therefore, for each $\theta > 0$, $m \in \mathfrak{M}$, $\xi \in \pi_1\mathbb{X}$, we have a unique solution $X(\theta, \xi, m)$ of (9), and we can define the graph transformation mappings G^θ by

$$\left(G^\theta m\right)(\xi) = \pi_2 S^\theta X(\theta, \xi, m) \quad \text{for } \theta > 0, m \in \mathfrak{M}, \xi \in \pi_1\mathbb{X}.$$

Step 2: Fixed-Points of the Graph Transformation Mapping. Let $\theta > 0$, $m \in \mathfrak{M}$, $\xi_1, \xi_2 \in \pi_1\mathbb{X}$ be arbitrary. By (S3) we have $\mathrm{graph}(G^\theta m) \subset \sum$. By the cone invariance property (CIP), we have

$$\begin{aligned}\left\|\left(G^\theta m\right)(\xi_1) - \left(G^\theta m\right)(\xi_2)\right\| &\leq \chi \left\|\pi_1[S^\theta X(\theta, \xi_1, m) - S^\theta X(\theta, \xi_2, m)]\right\| \\ &= \chi \left\|\xi_1 - \xi_2\right\|,\end{aligned}$$

i.e., G^θ maps $\mathfrak{M}$ into itself for each $\theta > 0$.

Now let $\theta > 0$, $\xi \in \pi_1\mathbb{X}$, $m_1, m_2 \in \mathfrak{M}$, and x_1, x_2 with

$$\pi_1 S^\theta x_1 = \pi_1 S^\theta x_2 = \xi \quad \text{and} \quad x_i \in \mathrm{graph}(m_i).$$

If we choose $x_3 := \pi_1 x_1 + m_2(\pi_1 x_1)$, then $x_3 - x_2 \in C_\chi$ and (modSP) imply

$$\left\|\pi_2[S^\theta x_1 - S^\theta x_2]\right\| \leq \chi_{22} \left\|m_1(\pi_1 x_1) - m_2(\pi_1 x_1)\right\| \mathrm{e}^{-\eta\theta} \quad \text{for } \theta \geq 0.$$

Thus

$$\left\| \left(G^\theta m_1\right)(\xi) - \left(G^\theta m_2\right)(\xi) \right\| \leq \chi_{22} \mathrm{e}^{-\eta\theta} \left\| m_1(\pi_1 X(\theta, \xi, m_1)) - m_2(\pi_1 X(\theta, \xi, m_1)) \right\|,$$

i.e.,

$$\left\| G^\theta m_1 - G^\theta m_2 \right\|_{\mathbb{G}} \leq \kappa(\theta) \left\| m_1 - m_2 \right\|_{\mathbb{G}} \quad \text{for all } \theta > 0 \text{ and } m_1, m_2 \in \mathfrak{M},$$

where $\kappa(\theta) := \chi_{22}\mathrm{e}^{-\eta\theta}$. Since $\eta > 0$, there is a $\theta_0 > 0$ with $\kappa(\theta) < 1$ for $\theta \geq \theta_0$. Thus, for $\theta \geq \theta_0$, G^θ is a contractive self-mapping on the closed subset $\mathfrak{M}$ of the Banach space $\mathbb{G}$. Hence, for each $\theta \geq \theta_0$, there is a unique fixed-point $m^{(\theta)}$ of G^θ in $\mathfrak{M}$.

Let $p \in \mathbb{N}_{>0}$. Then $m^{(\theta)}$ is a fixed-point of $G^{p\theta}$ and hence $m^{(p\theta)} = m^{(\theta)}$ for $\theta \geq \theta_0$ and $p \in \mathbb{N}_{>0}$. Let $q \in \mathbb{N}_{>0}$. Because of

$$G^\theta \left(G^{\frac{1}{q}\theta} m^{(\theta)} \right) = G^{\frac{1}{q}\theta} \left(G^\theta m^{(\theta)} \right) = G^{\frac{1}{q}\theta} m^{(\theta)}$$

and the uniqueness of the fixed-point $m^{(\theta)}$ of G^θ, $m^{(\theta)}$ is the unique fixed-point of $G^{\frac{1}{q}\theta}$ for $\theta \geq \theta_0$ and each $q \in \mathbb{N}_{>0}$. Thus, for each $\theta > 0$, there is a unique fixed-point $m^{(\theta)}$ of G^θ and we have $m^{(\frac{p}{q}\theta)} = m^{(\theta)}$ for $\theta > 0$ and all $p, q \in \mathbb{N}_{>0}$. Hence,

$$S^{\frac{p}{q}\theta_0} x \in \operatorname{graph}(m^{(\theta_0)}) \quad \text{for } u \in \operatorname{graph}(m^{(\theta_0)}) \text{ and } m, n \in \mathbb{N}_{>0}$$

and the continuity of $t \mapsto S^t u$ yields

$$S^\theta x \in \operatorname{graph}(m^{(\theta_0)}) \quad \text{for } \theta > 0 \text{ and } x \in \operatorname{graph}(m^{(\theta_0)}).$$

Thus, $m^* := m^{(\theta_0)} = m^{(\theta)}$ for all $\theta > 0$, and $M = \operatorname{graph}(m^*)$ is positively invariant with respect to S.

Step 3: Existence of Asymptotic Phases. Let $x_1 \in \mathbb{X}$ and let $(t_k)_{k \in \mathbb{N}}$ be a monotonously increasing sequence of real numbers t_k with $t_k \to \infty$ for $k \to \infty$. Further, let $\zeta_k := \pi_1 X(t_k, S^{t_k} x_1, m^*)$. By (10), we have

$$\left\| \pi_i [S^t x_1 - S^t X(t_k, S^{t_k} x_1, m^*)] \right\| \leq \chi_{2i} \left\| \pi_2 x_1 - m^*(\pi_1 x_1) \right\| \mathrm{e}^{-\eta t} \quad \text{for all } t \in [0, t_k].$$

In particular, we find $\|\pi_1 x_1 - \zeta_k\| \leq \chi_{21} \|\pi_2 x_1 - m^*(\pi_1 x_1)\|$. If $\pi_1 \mathbb{X}$ is finite dimensional, then the bounded and closed set

$$\{\zeta \in \pi_1 \mathbb{X} \colon \|\pi_1 x_1 - \zeta\| \leq \chi_{21} \|\pi_2 x_1 - m^*(\pi_1 x_1)\|\}$$

is compact. Thus, there is subsequence of $(t_k)_{k \in \mathbb{N}}$, denoted again by $(t_k)_{k \in \mathbb{N}}$, such that $(\zeta_k)_{k \in \mathbb{N}}$ is converging to some $\zeta^* \in \pi_1 \mathbb{X}$. Let $x_2 := \zeta^* + m^*(\zeta^*)$. Then

$$\begin{aligned}
&\|\pi_i [S^t x_1 - S^t x_2]\| \\
&\quad \leq \|\pi_i [S^t x_1 - S^t X(t_k, S^{t_k} x_1, m^*)]\| + \|\pi_i [S^t X(t_k, S^{t_k} x_1, m^*) - S^t x_2]\| \\
&\quad \leq \chi_{2i} \|\pi_2 x_1 - m^*(\pi_1 x_1)\| \mathrm{e}^{-\eta t} + \|\pi_i [S^t (\zeta_k + m^*(\zeta_k) - S^t (\zeta^* + m^*(\zeta^*)]\|
\end{aligned}$$

for all $\theta > 0$, $t \in [0, \theta]$ and all $k \in \mathbb{N}_{>0}$ with $t_k \geq \theta$. By the continuity of m^* and S, and because of $\zeta_k \to \zeta^*$, the second term can be made arbitrary small on $[0, \theta]$ choosing k large enough. Therefore,

$$\left\|\pi_i[S^t x_1 - S^t x_2]\right\| \leq \chi_{2i} \left\|\pi_2 x_1 - m^*(\pi_1 x_1)\right\| \mathrm{e}^{-\eta t} \quad \text{for all } \theta > 0, t \in [0, \theta],$$

i.e., $t \mapsto S^t x_2$ is an asymptotic phase of $t \mapsto S^t x_1$ in M.

4. Application to Evolution Equations

We consider the evolution equation

$$\dot{u} + Au = f(u) \tag{13}$$

with selfadjoint, positive definite densly defined linear operator A in the separable Hilbert space $(\mathbb{H}, |\cdot|_0)$. Further let $f \in \mathrm{C_b}(D(A^\alpha), \mathbb{H})$ satisfy the Lipschitz inequality

$$|f(u) - f(u')|_0 \leq L|u - u'|_\alpha \quad \text{for all } u, u' \in D(A),$$

where $\alpha \in [0, \frac{1}{2}]$. Let π_1 be the orthogonal projection from $\mathbb{H}$ onto the N-dimensional subspace of $\mathbb{H}$ spanned by the N eigenvectors belonging to the first N eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ of A, counted with their multiplicity.

Then (13) generates a semiflow S on $\mathbb{X} = D(A^\alpha)$ satisfying (S1), cf. [7,9,8] for (S1). The coercivity property (S2) follows from the variation of constant formula, the boundedness of f and $|\pi_1 \mathrm{e}^{-At} u| \geq C\mathrm{e}^{-\lambda_N t}|\pi_1 u|$. Studying the quadratic form $Q_\chi(u) = |\pi_2 u|_\alpha^2 - \chi^2|\pi_1 u|_\alpha^2$ along the difference of solutions of (13), in [8] was shown, that there is a $\chi > 0$ with

$$\frac{d}{dt}Q_\chi(S^t u_1 - S^t u_2) \leq \Lambda(\chi) Q_\chi(S^t u_1 - S^t u_2) \text{ and } \Lambda(\chi) < 0 \tag{14}$$

if the spectral gap condition

$$\lambda_{N+1} - \lambda_N > cL\left(\lambda_N^\alpha + \lambda_{N+1}^\alpha\right) \tag{15}$$

holds with $c = 1$. Romanov showed in [13], that the spectral gap condition (15) is sharp in the following sense: For each $c \in [0, 1[$, there are two-dimensional evolution equations (13) in $\mathbb{X} = \mathbb{R}^2$ without inertial manifold (i.e., here instable manifolds) but satisfying (15).

In particular, we may choose $\chi = \chi_0 := \sqrt{\lambda_N^\alpha \lambda_{N+1}^{-\alpha}}$. Moreover, in [8] was shown that (14) implies the modified strong squeezing property (CIP), (modSP) of S. In particular, for $\chi = \chi_0$ we may choose $\eta := -\lambda_{N+1} + L\lambda_{N+1}^\alpha$, $\chi_{21} := \frac{1}{\chi_0 - \underline{\chi}}$, $\chi_{22} := \frac{\chi_0}{\chi_0 - \underline{\chi}}$, where $\underline{\chi} < \overline{\chi}$ are the positive solutions of

$$\left(\lambda_{N+1} - \lambda_N\right)^2 \chi^2 = L^2\left(\chi^2 + 1\right)\left(\lambda_N^{2\alpha} + \chi^2 \lambda_{N+1}^{2\alpha}\right).$$

5. Extensions

Let $\mathbb{X}$ be densely imbedded in the Banach space $\mathbb{Y}$. If the cone invariance and modified squeezing property are required only with respect to the weaker norm $\|\cdot\|_{\mathbb{Y}}$, we need an additional smoothing property of S in the form that there is a function $c_0\colon\ ]0,\infty[\ \to\ ]0,\infty[$ with $\|S^t u\|_{\mathbb{X}} \leq c_0(t)\,\|u\|_{\mathbb{Y}}$ for $u \in \mathbb{X}$ and $t > 0$. This approach allows $\alpha \in [0,1[$ for the evolution equation (13) if $\mathbb{Y} = D(A^{\nu})$ with $\nu \in [0, \min\{\alpha, \frac{1}{2}\}]$, see [8].

Another approach consists in the construction of a manifold $M = \mathrm{graph}(m)$ with bounded domain $D(m) \subset \mathbb{X}_1$ as an overflowing invariant manifold, see [8]. For it we need some overflowing and inflowing properties of the semiflow on the boundary of a subset V of $\mathbb{X}$ in which the manifold shall be constructed. Then the technical assumption (S2) can be removed, since the needed bijectivity of the corresponding mapping H can be shown by the homotopy theorem. For the evolution equation (13), this allows to replace the global boundedness and Lipschitz assumptions on f by corresponding assumptions on V.

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