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THE USE OF LYAPUNOV FUNCTIONS IN UNIQUENESS AND NONUNIQUENESS THEOREMS

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Abstract. The contribution is devoted to using Lyapunov functions in uniqueness and nonuniqueness theorems. The survey of nonuniqueness results utilizing Lyapunov functions is given.

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1. Introduction

The question of the existence and uniqueness for the solutions of ordinary differential equations is an old problem of great importance. There is an enormous amount of literature offering various sufficient conditions for the uniqueness. We shall mention here only several mathematicians that have contributed to this problem.

The first result on the uniqueness of a scalar initial value problem

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{1}$$

where $f = (f_1, f_2, \ldots, f_n)$, $x = (x_1, x_2, \ldots, x_n)$, $x_0 = (x_{01}, x_{02}, \ldots, x_{0n})$, was given by A. Cauchy in 1820–1830. The result was improved by R. Lipschitz in 1876, who introduced so called Lipschitz condition of the form

$$|f(t,x) - f(t,y)| \le L|x-y|. \tag{2}$$

The Lipschitz condition was generalized by many authors such as W. F. Osgood (1898), P. Montel (1926), L. Tonelli (1925), M. Nagumo (1926). Very general is a condition of Perron's type

$$|f(t,x) - f(t,y)| \le g(t, |x-y|). \tag{3}$$

Perron's result (1926) was improved by E. Kamke (1930). His well-known theorem (see e. g. [1, pp. 56–57]) can be formulated for vector differential equations.

Theorem 1 (Kamke). *Assume that*

(i) $g \in C(R_0, \mathbb{R}^+)$, *where* $R_0 = \{(t,u) \in \mathbb{R}^2 : t_0 < t \le t_0 + a, 0 \le u \le 2b\}$, $\mathbb{R}^+ = [0, \infty)$ *and for every* $t_1 \in (t_0, t_0+a)$, *the function* $u(t) \equiv 0$ *is the only solution of* $u' = g(t,u)$ *defined on* (t_0, t_1) *and satisfying* $\lim_{t \to t_0} [u(t)/(t - t_0)] = 0$.

(ii) $f : R \to \mathbb{R}^n$, $R = \{(t,x) \in \mathbb{R}^{n+1} : t_0 \le t \le t_0 + a, |x - x_0| \le b\}$ *and*

$$|f(t,x) - f(t,y)| \le g(t, |x-y|) \quad \text{for} \quad (t,x),(t,y) \in R, \quad t \ne t_0. \tag{4}$$

Then the initial value problem (1) *has at most one solution in* $[t_0, t_0 + a]$.

2. The use of Lyapunov functions

Kamke's theorem was generalized in several manners. One of the fruitful ways is the use of Lyapunov functions method. This approach allows to obtain very general and flexible results. These results contain the most of previous results as special cases and, by special choices, new interesting criteria for the uniqueness can be obtained. There exists a lot of variants of criteria utilizing Lyapunov functions. We can mention here the results of H. Okamura (1934–42), T. Sato (1936), O. Borůvka (1956), J. Chrastina (1969), S. C. Chu and J. B. Diaz (1970), T. Roger (1972), F. Brauer and S. Sternberg (1958), R. D. Moyer (1966), S. R. Bernfeld - R. D. Driver - V. Lakshmikantham (1976), Z. Tesařová - O. Došlý (1980), H. A. Antosiewicz (1962), V. Lakshmikantham - M. Samimi (1983).

The interesting and powerful uniqueness criteria for the Cauchy problem were derived by I. Kiguradze (1965). We shall remind a criterion for a singular Cauchy problem formulated for $t_0 = a$:

Theorem 2 (Kiguradze [6]). *Let* f *be defined for* $a < t \le b$, $|x - x_0| < r$ *and a function* $V(t,x)$ *be continuous and positive definite in* $R_0 = \{(t,x) \in \mathbb{R}^n : a < t \le b, |x| \le 2r\}$. *Assume that* $g(t,u)$ *satisfies Carathéodory conditions on any set* $\{R_c = \{(t,u) \in \mathbb{R}^2 : a \le t \le b, |u| \le c\}$, $c \in (0, \infty)$. *Suppose that* $g(t, \cdot)$ *is nondecreasing,* $g(t,0) \equiv 0$ *and the problem*

$$\frac{du}{dt} = g(t,u), \quad u(a) = 0$$

has only the trivial solution. If the conditions

$$\lim_{t \to a} V(t, x(t) - y(t)) = 0, \tag{5}$$

$$V(t, x(t) - y(t)) \le \int_a^t g(s, V(s, x(s) - y(s))ds \tag{6}$$

hold for any two solutions $x(t)$, $y(t)$ *of* (1), *then* (1) *has at most one solution.*

3. Nonuniqueness theorems

In contradistinction with the problem of uniqueness criteria, there are only several papers dealing with problem of nonuniqueness. The necessary and sufficient conditions for the uniqueness in the scalar case was derived by T. Yosie in 1926 (see e. g. [1, pp. 81–91]). His main result is the following:

Theorem 3 (Yosie's criterion). *The scalar initial value problem has at most one solution in the interval* $[t_0, t_0 + a]$ *if and only if for every* $\varepsilon > 0$ *there exists a pair of lower- and upper- functions* $\varphi(t)$, $\psi(t)$ *with respect to the initial value problem* (1) *such that* $0 < \psi(t) - \varphi(t) < \varepsilon$ *in the interval* $(t_0, t_0 + a]$.

The first nonuniqueness criterion appeared in 1922 (see e. g. [1, p. 98]):

Theorem 4 (Tamarkine). *Let* $f(t, x)$ *be a scalar function continuous in* $R = \{(t, x) \in \mathbb{R}^2 : |t - t_0| \le a, |x - x_0| \le b\}$ *with* $(t_0, x_0) = (0, 0)$ *and for all* $(x, y) \in R$ *the condition*

$$|f(t, x) - f(t, x(t))| \ge g(|x - x(t)|)$$

holds, where $x(t)$ *is a solution of* (1), $g(u)$ *being an increasing continuous function for* $u \ge 0$, *such that* $g(0) = 0$ *and* $\int_{0+} \frac{du}{g(u)} < \infty$. *Then the initial value problem* (1) *has at least two solutions in* $[t_0 - a, t_0 + a]$.

The Tamarkine criterion was generalized by V. Lakshmikantham (1964). His nonuniqueness condition formulated for $t_0 = 0$ has a form

$$|f(t, x) - f(t, y)| \ge g(t, |x - y|), \tag{7}$$

where $g \in C(R, \mathbb{R}^+)$, $R = \{(t, u) \in \mathbb{R}^2 : 0 < t \le a, 0 \le u \le 2b\}$, $g(t, 0) \equiv 0$, $g(t, u) > 0$ for $u > 0$, and, there exists a differentiable function $u(t) \not\equiv 0$ for which

$$u'(t) = g(t, u(t)), \quad u(0) = u'_+(0) = 0.$$

Lakshmikantham's theorem was generalized by M. Samimi in 1982, however, as it was noticed by H. Stettner and Chr. Nowak, the condition (7) should be replaced by a stronger one: $f(t, x) - f(t, y) \ge g(t, x - y)$ for $x > y$. Unfortunately, the last condition cannot be fulfilled (see [9]).

The first mathematician who used Lyapunov functions to obtain nonuniqueness criterion was H. Stettner (1974). In our paper [2] a general nonuniqueness result employing Lyapunov functions for the nonsingular Cauchy problem was given. A modification of this result was presented by M. Samimi [10] in 1982. Samimi supposes the boundedness of f and uses a function $B(t)$ for the description of the behaviour of the solutions near the initial point t_0 in sense of the following Theorem 5.

In 1992, Chr. Nowak [8] attempted to remove the condition on the boundedness of f in Samimi's theorem. In the paper [3] a general nonuniqueness criterion was derived, which contains as a consequence a revised form of Nowak's nonuniqueness criterion and the most of previous known nonuniqueness criteria. The notation

$$\mathrm{D}^+V(t,x) := \limsup_{h\to 0+} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h}$$

is used and the criterion is given here in a simplified form formulated for $t_0 = a$, where $-\infty \le a < \infty$:

Theorem 5 (Kalas [3]). *Let $t_1 \in (a, A)$. Assume that $f \in C[R, \mathbb{R}^n]$, where $R = \{(t,x) \in \mathbb{R}^{n+1} : a < t < A, |x - x_0| \le b\}$, and*
(i) there exists a function $g \in C[(a, t_1] \times \mathbb{R}^+, \mathbb{R}]$ nondecreasing in the second variable and such that a certain solution $\varphi(t)$, $t \in (a, t_1]$ of

$$u' = g(t,u)$$

satisfies conditions

$$\varphi(t_1) > 0, \qquad \lim_{t\to a+} \frac{\varphi(t)}{B(t)} = 0,$$

where $B \in C[(a, t_1], \mathbb{R}]$ is positive;
(ii) $V \in C[R, \mathbb{R}^+]$ is such that

$$V(t_1, y_0) < \varphi(t_1) \qquad \text{for some } \ y_0 \in \mathbb{R}^n, \ |y_0 - x_0| < b, \tag{8}$$

$$V(t,x) > \varphi(t) \qquad \text{for } \ a < t < t_1, \ |x - x_0| = b, \tag{9}$$

$$V(t,x) \ge \Phi(t)\,\Psi(|x - z(t)|) \qquad \text{for } \ a < t \le t_1, \ |x - x_0| < b, \tag{10}$$

where $\Phi \in C[(a, t_1], \mathbb{R}^+]$, $\Psi \in C[[0, 2b), \mathbb{R}^+]$, $z \in C[(a, t_1], \mathbb{R}^n]$ satisfy

$$\liminf_{t\to a+} \frac{\Phi(t)}{B(t)} > 0, \quad \Psi(0) = 0, \quad \Psi(u) > 0 \ \text{ for } \ u \in (0, 2b) \tag{11}$$

and

$$\lim_{t\to a+} z(t) = x_0, \quad |z(t) - x_0| < b \ \text{ for } \ t \in (a, t_1];$$

(iii) there exists a positive function $\varepsilon \in C[(a, t_1), \mathbb{R}^+]$ such that $V(t,x)$ satisfies locally the Lipschitz condition with respect to x for $(t,x) \in \Omega_\varphi$ and

$$\mathrm{D}^+V(t,x) \ge g(t, V(t,x)) \qquad \text{on } \ \Omega_\varphi$$

holds, Ω_φ being defined by

$$\Omega_\varphi = \{(t,x) \in \mathbb{R}^{n+1} : \varphi(t) < V(t,x) < \varphi(t) + \varepsilon(t), a < t < t_1, |x - x_0| < b\}. \tag{12}$$

Then the problem (1) *has at least two different solutions $x(t)$ on $(a, t_1]$ such that*

$$\lim_{t\to a+} \frac{V(t, x(t))}{B(t)} = 0$$

is valid.

All the mentioned nonuniqueness criteria have the disadvantage that they cannot be applied for the n-th order differential equations. In the following result formulated for $t_0 = a$, where $-\infty \le a < \infty$, we use Lyapunov functions that need not be positive definite in x (in sense of the condition (10)), but only in some components of x and thus we need the estimations only of several components of f. Such a result is applicable to the n-order differential equation. In the result we use the projection Pr defined by $\operatorname{Pr} x = (x_{i_1}, \dots, x_{i_l})$, where i_j $(j = 1, \dots, l)$ are integers such that $1 \le i_1 < \cdots < i_l \le n$.

Theorem 6 (Kalas [4]). *Let $f \in C(R, \mathbb{R}^n)$, where $R = \{(t,x) \in \mathbb{R}^{n+1} : a < t < A, |x - x_0| \le b\}$. Put $\mu(t) := \max_{|x-x_0|\le b} |f(t,x)|$. Suppose that*

$$\int_{a+} \mu(t)dt < \infty$$

holds and choose $t_1 \in (a, A)$ such that

$$\int_a^{t_1} \mu(t)dt \le b/2$$

is valid. Assume that
(i) there exists a function $g \in C[(a,t_1] \times \mathbb{R}^+, \mathbb{R}]$ nondecreasing in the second variable and such that a certain solution $\varphi(t)$, $t \in (a, t_1]$ of

$$u' = g(t,u)$$

satisfies conditions

$$\varphi(t_1) > 0, \qquad \lim_{t \to a+} \frac{\varphi(t)}{B(t)} = 0,$$

where $B \in C[(a,t_1], \mathbb{R}^+]$ is positive;
(ii) $V(t,x) \in C[R, \mathbb{R}^+]$ and there exists $y_0 \in \mathbb{R}^l$, $|y_0 - \operatorname{Pr} x_0| < b/2$, such that

$$V(t_1, y) < \varphi(t_1) \qquad \text{for} \;\; y \in \mathbb{R}^n,\; |y - x_0| \le b,\; \operatorname{Pr} y = y_0,$$

and

$$V(t,x) \ge \Phi(t)\,\Psi(|\operatorname{Pr} x - z(t)|) \qquad \text{for} \;\; a < t \le t_1,\; |x - x_0| < b,$$

where $\Phi \in C[(a,t_1], \mathbb{R}^+]$, $\Psi \in C[[0,2b), \mathbb{R}^+]$, $z \in C[(a,t_1], \mathbb{R}^l]$ satisfy (11) and

$$\lim_{t \to a+} z(t) = \operatorname{Pr} x_0, \quad |z(t) - \operatorname{Pr} x_0| < b \;\; \text{for } t \in (a, t_1];$$

(iii) there exists a positive function $\varepsilon \in C[(a,t_1), \mathbb{R}^+]$ such that $V(t,x)$ satisfies locally the Lipschitz condition with respect to x for $(t,x) \in \Omega_\varphi$ and

$$\mathrm{D}^+ V(t,x) \ge g(t, V(t,x)) \qquad \text{on} \;\; \Omega_\varphi$$

holds, Ω_φ being defined by (12). Then the problem (1) has at least two different *solutions* $x(t)$ *on* $(a, t_1]$ *such that*

$$\lim_{t \to a+} \frac{V(t, x(t))}{B(t)} = 0$$

is valid.

Proof. For the proof see [4].

Theorem 6 is formulated for the nonsingular Cauchy problem. Recently a result which attemps to extend the last result to a singular case was published in [5]. Moreover a vector Lyapunov function instead of a scalar one is used, which allows to apply achieved results to a wider class of differential equations. For the formulation of the result we need the following notation:

$\lvert \cdot \rvert$	Hölder's 1-norm (sum of the absolute values of components);
l	fixed number from the set $\{1, \cdots, n\}$;
$i_1, i_2, \cdots, i_l$	integers $1 \le i_1 < i_2 < \cdots < i_l \le n$;
I	$:= \{i_1, i_2, \cdots, i_l\}$;
N	$:= \{1, 2, \cdots, n\}$;
$\tilde{R}^k_{a,A}$	$:=\{(t, x) \in \mathbb{R}^{k+1} : a < t < A,\ x \in \mathbb{R}^k\}$;
$\hat{R}^n_{a,A}$	$:=\{(t, x) \in \mathbb{R}^{n+1} : a < t \le A,\ x \in \mathbb{R}^n\}$;
$R^k_{\alpha,A;\varrho}$	$:=\{(t, x) \in \mathbb{R}^{k+1} : \alpha < t < A, \lvert x\rvert \le \varrho\}$;
$\mathcal{L}[\hat{R}^n_{a,A}, {\mathbb{R}^+}^k]$	class of all functions $V(t, x) : \hat{R}^n_{a,A} \to {\mathbb{R}^+}^k$ with following property: $V(t, \cdot)$ is uniformly continuous and if $a < \alpha < \beta \le A$, then $V(t, x(t))$ is absolutely continuous on $[\alpha, \beta]$ for any absolutely continuous function $x : [\alpha, \beta] \to \mathbb{R}^n$;
$K[\tilde{R}^k_{a,A}, \mathbb{R}^n]$	class of all mappings $\tilde{R}^k_{a,A} \to \mathbb{R}^n$ which satisfy Caratheodory conditions on $R^k_{\alpha,A;\varrho}$ for any $\alpha \in (a, A)$, $\varrho \in (0, \infty)$;
$N_0(a, A; \tau_1, \cdots, \tau_n)$	$:=\{\Lambda = (\lambda_{ij}(t))^n_{i,j=1} : \lambda_{ij} \in L[[a, A], \mathbb{R}^+]\}$ such that the system of differential inequalities $\lvert x_i'(t)\rvert \le \sum_{j=1}^n \lambda_{ij}(t)\lvert x_j(t)\rvert$, $t \in [a, A]$, $i \in N$ possesses no nontrivial solution $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in AC[[a, A], \mathbb{R}^n]$ satisfying $x_i(\tau_i) = 0$ $(i = 1, 2, \cdots, n)$;
$N_I(a, A)$	$:=N_0(a, A; \tau_1, \cdots, \tau_n)$, where $\tau_i = A$ for $i \in I$ and $\tau_i = a$ for $i \in N \setminus I$.

In the theorem, the initial value problem (1) with $t_0 = a$, where $-\infty \le a < \infty$, will be considered. We shall assume, that the vector function $f = (f_1, \cdots, f_n) \in K[\tilde{R}^n_{a,A}, \mathbb{R}^n]$ is such that there are $c_i \in \mathbb{R}$ $(i \in I)$, $\Lambda = (\lambda_{ij})^n_{i,j=1} \in N_I(a, A)$, $\mu_i \in L[[a, A], \mathbb{R}^+]$ $(i \in N)$ for which

$$-f_i(t, x)\,\mathrm{sgn}(x_i - c_i) \le \sum_{j=1}^n \lambda_{ij}(t)\lvert x_j\rvert + \mu_i(t) \qquad (i \in I)$$

and

$$f_i(t,x)\operatorname{sgn}(x_i - x_{0i}) \le \sum_{j=1}^{n} \lambda_{ij}(t)|x_j - x_{0j}| + \mu_i(t) \qquad (i \in N \setminus I)$$

hold for $(t,x) = (t, x_1, \ldots, x_n) \in \tilde{R}^n_{a,A}$.

Theorem 7 (Kalas [5]). *Assume that*

(i) there exists a function $g = (g_1, \ldots, g_k) \in K[\tilde{R}^k_{a,A}, \mathbb{R}^k]$ *such that any component* $g_j(t, u_1, \ldots, u_{j-1}, \cdot, u_{j+1}, \ldots, u_k)$ *is nondecreasing for* $j = 1, \ldots, k$ *and there is a solution* $\varphi(t) = (\varphi_1(t), \ldots, \varphi_k(t))$, $t \in (a, A)$ *of*

$$u' = g(t,u)$$

satisfying

$$\varphi(t) > 0, \qquad \lim_{t\to a+} \varphi(t) = 0, \qquad \liminf_{t\to A-} \varphi(t) > 0;$$

(ii) $V(t,x) = (V_1(t,x), \ldots, V_k(t,x)) \in \mathcal{L}[\hat{R}^n_{a,A}, {\mathbb{R}^+}^k]$ *and there exists* $y_0 \in \mathbb{R}^l$ *such that*

$$\sup\{V_j(A,y) : y \in \mathbb{R}^n, \operatorname{Pr} y = y_0\} < \liminf_{t\to A-} \varphi_j(t) \ (j = 1, \ldots, k)$$

and,

$$|V(t,x)| \ge \Psi(|\operatorname{Pr} x - z(t)|) \quad \text{for } a < t < A,$$

where $\Psi \in C[\mathbb{R}^+, \mathbb{R}^+]$, $z \in C[(a,A), \mathbb{R}^l]$ *are such that*

$$\Psi(0) = 0, \qquad \Psi(u) > 0 \quad \text{for } u > 0, \qquad \lim_{t\to a+} z(t) = \operatorname{Pr} x_0;$$

(iii) there exist positive functions $\varepsilon_j \in C[(a,A), \mathbb{R}^+]$ *such that*

$$V_j'(t,x(t)) \ge g_j(t, \varphi_1(t), \ldots, \varphi_{j-1}(t), V_j(t,x(t)), \varphi_{j+1}(t), \ldots, \varphi_k(t))$$

holds for $j = 1, 2, \ldots, k$ *and for any solution* $x(t)$ *of* (1) *a. e. on any interval* $(\alpha_1, \alpha_2) \subseteq (a, A)$ *for which*

$$V_i(t,x(t)) < \varphi_i(t) + \varepsilon_i(t) \qquad \text{on } (\alpha_1,\alpha_2), \qquad (i = 1, \ldots, k), \tag{13}$$

$$V_j(t,x(t)) > \varphi_j(t) \qquad \text{on } (\alpha_1,\alpha_2). \tag{14}$$

Then the initial value problem (1) *possesses at least two different solutions* $x(t)$ *on* $[a,A]$, *either of which satisfies* $V(t,x(t)) \le \varphi(t)$ *for* $t \in (a,A)$.

As a consequence we easily obtain the result for the nonuniqueness for the n-th order differential equation (for details see [5]).

Corollary 1. *Let* $\tilde{f} \in K[\tilde{R}^n_{a,A}, \mathbb{R}]$. *Suppose* $c \in \mathbb{R}$, $\lambda, \mu \in L[[a,A], \mathbb{R}^+]$ *are such that*

$$-\tilde{f}(t, x_1, \ldots, x_n)\operatorname{sgn}(x_n - c) \le \lambda(t)|x_n| + \mu(t)$$

for $(t,x) \in \tilde{\mathbb{R}}^n_{a,A}$. *Assume that*

(i) there exists a function $g \in K[\tilde{R}^1_{a,A}, \mathbb{R}]$ *such that* $g(t,\cdot)$ *is nondecreasing and there is a solution* $\varphi(t)$, $t \in (a,A)$ *of* $u' = g(t,u)$ *satisfying*

$$\varphi(t) > 0, \qquad \lim_{t \to a+} \varphi(t) = 0;$$

(ii) there are $z \in C[[a,A],\mathbb{R}]$, $\varepsilon \in C[(a,A),\mathbb{R}^+]$ *such that* z *is absolutely continuous on* $[\alpha, A]$ *for any* $\alpha \in (a,A)$, $z(a) = x_{0n}$ *and*

$$(\tilde{f}(t,x_1,\dots,x_n) - z'(t))\,\mathrm{sgn}(x_n - z(t)) \geq g(t, |x_n - z(t)|)$$

holds on $\hat{\Omega} = \{(t,x_1,\dots,x_n) \in \mathbb{R}^{n+1} : \varphi(t) < |x_n - z(t)| < \varphi(t) + \varepsilon(t), a < t < A\}$ *for almost all* $t \in (a,A)$. *Then the initial value problem*

$$v^{(n)} = \tilde{f}(t,v,v',\cdots,v^{(n-1)}),$$
$$v(a) = x_{01},\ v'(a) = x_{02},\ \cdots,\ v^{(n-1)}(a) = x_{0n}$$

is nonunique.

Finally, notice that very interesting results for nonuniqueness of a singular Cauchy-Nicolletti problem were achieved by I. Kiguradze [7]. The sufficient conditions are given in the form of one-sided inequalities for the components of the right-hand side f. The estimating expression for the j-th component f_j of f depends on t and x_j and is linear in $|x_j|$. The proofs of Theorem 6 and Theorem 7 are based on the combination of the Lyapunov function method with the modified method of I. Kiguradze [7]. We mention also the paper [9], where the differences between the nonsingular and the singular initial value problem are analyzed.

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