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ON CERTAIN THIRD ORDER BOUNDARY VALUE PROBLEMS ON INFINITE INTERVAL

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Abstract. In this paper certain third order eigenvalue problems are studied. The motivation was given by the theory of the third order linear eigenvalue problems on finite intervals and asymptotic properties of solutions of the third order linear differential equations [1].

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1. The aim of this paper is to study these two boundary value problems

$$y''' + 2A(t)y' + [A'(t) + \lambda b(t)]y = 0, \tag{a}$$

$$y(a,\lambda) = y(b,\lambda) = y(c,\lambda) = 0,\ a \le b < c < \infty \tag{1}$$

$$\lim_{t\to+\infty} y(t,\lambda) = 0 \tag{2}$$

and the problem (a), (1) and $(2')$ where

$$|y(t,\lambda)| < K,\ |y'(t,\lambda)| < K,\ K > 0,\ \lambda > 0, \tag{$2'$}$$

under certain suppositions on the functions A, A', b on $[a,\infty)$.

The result of this paper complete those which are given in monograph [1] in the case that equation (a) is oscillatory on $[a,\infty)$ for each $\lambda > 0$, i.e. every solution of (a) with one zero has infinite number of zeros on $[a,\infty)$.

2. In this section we introduce certain auxiliary statements on the third order differential equation given in [1].

In this and in the next section we will suppose that $A'(t)$ and $b(t)$ are continuous functions on $[a,\infty)$ and $b(t) > 0$ for all $t \in [a,\infty)$.

Have the linear third order differential equation

$$y''' + 2A(t)y' + [A'(t) + b(t)]y = 0, \tag{a_1}$$

(i.e. equation (a) with $\lambda = 1$).

Lemma 1 (1, Theorem 3.17). *Assume that every solution of the second order differential equation*

$$y'' + \frac{1}{2}A(t)y = 0 \tag{3}$$

converges to zero as $t \to +\infty$ and $\int_a^t b(\tau)d\tau$ converges. Then every solution of the differential equation (a_1) converges to zero as $t \to \infty$.

This Lemma was formulated and proved by M. Ráb in [2].

One of the sufficient conditions for the solutions of (3) to converge to zero as $t \to \infty$ is given in the following lemma [3].

Lemma 2. *Let $A(t) > 0$ be non decreasing on $[a,\infty)$ and let $A'(t) \geq l > 0$ and $\int_a^\infty \frac{dt}{A(t)} = +\infty$. Then every solution y of (3) has the property $\lim\limits_{t\to\infty} y(t) = 0$.*

Lemma 3 (1, Theorem 3.18). *Let the following assumptions in $[a,\infty)$ hold:*

1. *$A(t) > 0$, $\lim\limits_{t\to\infty} A(t) = \infty$.*
2. *The function $A^{-\frac{1}{4}}(t)$ is convex.*
3. *The integral $\int\limits_a^t \frac{b(\tau)}{A(\tau)}d\tau$ converges.*

Then every solution of (a_1) and its first derivative are bounded in $[a,\infty)$.

This lemma was formulated and proved by M.Zlámal [4].

Lemma 4 (1, Corollary 2.3). *Let the second order differential equation (3) be oscillatory in $[a,\infty)$.Then (a_1) is oscillatory in $[a,\infty)$ too, i.e. its every solution having a zero is oscillatory in $[a,\infty)$.*

Adaptation of the oscillation theorem [1, Theorem 4.5] to (a) yields the following lemma.

Lemma 5. *Suppose that $A \geq \frac{p}{2}$ for all $t \in [a,\infty)$, where p is a real constant and moreover $b(t) \geq k > 0$ on $[a,\infty)$ for some positive constant k.*
Let $\lambda \in (0,\infty)$ and let $y(t,\lambda)$ be a nontrivial solution of (a) with $y(a,\lambda) = 0$. Then, for any fixed $b > a$, the number of zeros of y on $[a,b]$ increases to infinity as $\lambda \to \infty$, in which case the distance between any consecutive zeros of y converges to zero.

The continuous dependence of zeros of solutions of (a) upon the parameters λ is given in following lemma.

Lemma 6 (1, Lemma 4.2). *Let y be a nontrivial solution of (a) on $[a,\infty)$ such that $y(a,\lambda) = 0$. Then, the zeros of y on $[a,\infty)$ (if they exist) are continuous functions of the parameter $\lambda \in (0,\infty)$.*

With the help of the results given in the preceding lemmas one can prove the following two theorems regarding the multipoint eigenvalue problems (a), (1), (2) and (a), (1), $(2')$.

Theorem 1. *Let the suppositions on A, A', b given in Lemma 1, be fulfilled and let $A(t) > 0$ and equation (3) be oscillatory on $[a,\infty)$. Let $a \le b < c < \infty$ be arbitrary, but fixed. Then there exists a natural number ν, a sequence of the values of parameter λ $\{\lambda_{\nu+p}\}_{p=0}^{\infty}$ (eigenvalues) such that $\lambda_{\nu+p} < \lambda_{\nu+p+1}$ and $\lim_{p\to\infty} \lambda_{\nu+p} = \infty$, and a sequence of functions $\{y(t,\lambda_{\nu+p})\}_{p=0}^{\infty}$ (eigenfunctions) such that $y(t,\lambda_{\nu+p})$, ${}_{p=0,1,\dots}$ is a solution of (a) with $\lambda = \lambda_{\nu+p}$, which fulfills the conditions (1), (2) and has exactly $\nu + p - 1$ zeros on (b,c).*

Proof. Let $a < b < c < \infty$. Let $y(t,\lambda)$, $\lambda > 0$ be a nontrivial solution of (a) such that $y(a,\lambda) = y(b,\lambda) = 0$. Such a solution of (a) evidently exists (see e.g. properties of bands of solutions of (a), [1]). Solution $y(t,\lambda)$ is oscillatory on $[b,\infty)$. Construct now on $[a,\infty)$ the differential equation

$$Y''' + 2A(t)Y' + [A'(t) + \lambda B(t)]Y = 0, \tag{A}$$

where

$$B(t) = \begin{cases} b(t) & \text{for } t \in [a,c] \\ b(c) & \text{for } t > c. \end{cases}$$

On account of Lemma 4 equation (A) is oscillatory on $[a,\infty)$ for all $\lambda > 0$. Let $Y(t,\lambda)$ be a solution of (A) on $[a,\infty)$ with the property $Y(a,\lambda) = Y(b,\lambda) = 0$. If we denote $Y'(b,\lambda) = y'(b,\lambda)$, $Y''(b,\lambda) = y''(b,\lambda)$ for all $\lambda > 0$, then clearly $Y(t,\lambda) = y(t,\lambda)$ is the solution of (a) for $t \in [a,b]$ and $\lambda > 0$, too.

The function $Y(t,\lambda)$ as a solution of (A) is oscillatory on $[b,\infty)$ for all $\lambda > 0$.

Let for $\lambda = \bar\lambda > 0$ the solution $Y(t,\bar\lambda)$ have exactly ν zeros in (b,c). Then clearly, for the ν-th zero $t_\nu(\bar\lambda)$ and the $(\nu+1)$-st zero $t_{\nu+1}(\bar\lambda)$ of $Y(t,\bar\lambda)$ we have $t_\nu(\bar\lambda) < c \le t_{\nu+1}(\bar\lambda)$.

It follows from Lemma 5 (oscillation lemma), that for some $\bar{\bar\lambda} > \bar\lambda$ we have $t_{\nu+1}(\bar{\bar\lambda} < c$. Since $t_{\nu+1}(\lambda)$ is a continuous function of the parameter λ (Lemma 6), there exist $\lambda_\nu \in [\bar\lambda, \bar{\bar\lambda})$ such that for $\lambda = \lambda_\nu$ we have $t_{\nu+1}(\lambda_\nu) = c$, i.e $Y(c,\lambda_\nu) = y(c,\lambda_\nu) = 0$ and $Y(t,\lambda_\nu)$ has exactly ν zeros in (b,c).

Continuing in this manner, we can find a sequence of values of the parameters $\lambda > 0$

$$\lambda_\nu < \lambda_{\nu+1} < \cdots < \lambda_{\nu+p} < \cdots ,$$

to which there corresponds a sequence of functions

$$Y_\nu,\ Y_{\nu+1},\dots,\ Y_{\nu+p},\dots$$

such that $Y_{\nu+p} = y(t,\lambda_{\nu+p})$ is a solution of (A) satisfying conditions (1) and $Y(t,\lambda_{\nu+p})$ has exactly $\nu+p-1$ zeros in (b,c).

But $Y(t,\lambda_{\nu+p}) = y(t,\lambda_{\nu+p})$ on $[a,c]$ with the same initial conditions in c. Therefore the solution $y(t,\lambda_{\nu+p})$ of (a) fulfills the boundary condition (1) and by Lemma 1 (where instead of $\int_a^\infty b(\tau)d\tau < \infty$ we have $\lambda_{\nu+p}\int_a^\infty b(\tau)d\tau < \infty$) the solution $y(t,\lambda_{\nu+p})$ has the property (2), too and Theorem 1 is proved in the case $a < b < c < \infty$.

If $a = b < c < \infty$, the proof is similar, but for the solution y with the double zero at a for $\lambda > 0$.

By the same argument we can prove the following

Theorem 2. *Let the suppositions on A, A', b given in Lemma 3 be fulfilled on $[a,\infty)$. Then the assertion of Theorem 1 holds with the exception that $y(t,\lambda_{\nu+p})$ fulfills the condition $(2')$, (not the condition (2)).*

References

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