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ON CERTAIN THIRD ORDER EIGENVALUE PROBLEM

Michal Greguš

Dept. of Math. Anal., Faculty of Mathematics and Physics, Comenius University
Mlynská dolina, 842 15 Bratislava, Slovak Republic
Email: gregus@fmph.uniba.sk

Abstract. In this paper a singular third order eigenvalue problem is studied. The motivation was given by the paper [2] of Á. Elbert, T. Kusano and M. Naito for linear second order nonoscillatory differential equation.

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1. The aim of this paper is to study the following eigenvalue problem

$$y''' + 2A(t)y' + [A'(t) + \lambda b(t)]y = 0 \tag{a}$$

$$y(a,\lambda) = y(b,\lambda) = y(c,\lambda) = 0,\ a \le b < c < \infty \tag{1}$$

as well as the boundary condition at infinity

$$y(t,\lambda) = o\big(t[k_1 u_1(t) u_2(t) + k_2 u_2^2(t)]\big) \text{ for } t \to \infty \tag{2}$$

together with the requirement that

$$y(t,\lambda) \neq 0$$

in a certain neighborhood of infinity (t_0,∞), where $c \le t_0 < \infty$, and $u_1,\ u_2$ form a fundamental set of solutions of the second order differential equation

$$u'' + \frac{1}{2}A(t)u = 0 \tag{3}$$

with initial conditions $u_1(t_0) = 1,\ u_1'(t_0) = 0,\ u_2(t_0) = 0,\ u_2'(t_0) = 1,\ k_1,\ k_2$ are certain positive constants.

The basic suppositions on A and b in this paper are such that $A',\ b$ are continuous on $[a,\infty),\ b(t) > 0$ for (a,∞) and the differential equation (a) is strongly nonoscilatory for each real positive λ.

2. In this section we introduce certain auxiliary statements on the linear third order differential equation, given in monograph [1].

Consider equation (a) and the third order differential equation

$$y''' + 2A(t)y' + [A'(t) + b(t)]y = 0. \tag{a_1}$$

Lemma 1 (2, Theorem 2.1). *Let $A(t) < 0,\ b(t) > 0$ for $t \in [a,\infty)$ and let $|A(t)| \geq \int_a^t b(\tau)d\tau$ for $t \geq a$. Then the differential equation (a_1) is disconjugate in the interval $[a,\infty)$.*

Lemma 2. *Let the suppositions of Lemma 1 be fulfilled and let $\int_a^\infty b(\tau)d\tau < \infty$. Then to each $\bar\lambda \in [1,\infty)$ there exists $t_0 > a$ such that $|A(t)| > \bar\lambda \int_{t_0}^t b(\tau)d\tau$ holds for $t \geq t_0$ and the differential equation (a) is disconjugate for $\lambda = \bar\lambda$ on the interval $[t_0,\infty)$.*

The proof follows immediately from Lemma 1.

Lemma 3 (2, Theorem 2.14). *Let $A(t) < 0,\ b(t) > 0$ and $A'(t) + b(t) > 0$ for $t \in [a,\infty)$. If, moreover*

$$\int_T^\infty \left[A'(t) + b(t) - \frac{4}{3}\sqrt{\frac{2}{3}}\sqrt{-A^3(t)}\right]dt = +\infty,$$

$a < T < \infty$, then the differential equation (a_1) is oscillatory in $[a,\infty)$.

Lemma 4. *Let $A(t) < 0,\ b(t) > 0$ and $|A(t)| < K,\ |A'(t)| < K,\ b(t) > K,\ K > 0$, for $t \in [a,\infty)$. Then there exists $\tilde\lambda > 0$ such that the differential equation (a) is oscillatory in $[a,\infty)$ for all $\lambda \geq \tilde\lambda$.*

The proof of this lemma follows immediately from Lemma 3.

Consider, moreover, the second order differential equation

$$y'' + \frac{1}{2}A(t)y = 0 \tag{3}$$

Lemma 5. *Let $A(t) < 0$ for $t \in [a,\infty)$. Let $u_1,\ u_2$ be independent solutions of (3) and let $u_1(t_0) = 1,\ u_1'(t_0) = 0,\ u_2(t_0) = 0,\ u_2'(t_0) = 1,\ a < t_0 < \infty$. Then there is $u_1(t) > 0,\ u_2(t) > 0$ for $t > t_0$ and $u_1(t) \to \infty,\ u_2(t) \to \infty$ for $t \to \infty$.*

The proof follows from equation (3).

Lemma 6. *Let* $A(t) < 0,\ b(t) > 0$ *for* $t \in [a,\infty)$ *and let* $\lambda > 0$. *Let* y *be a solution of* (a) *and let for* $\lambda = \bar\lambda$ *be* $y(t_0,\bar\lambda) = 0,\ y'(t_0,\bar\lambda) \neq 0,\ y''(t_0,\bar\lambda) \neq 0$ *and let* $y(t,\bar\lambda) \neq 0$ *for* $t > t_0$. *Then*

$$y(t,\bar\lambda) = u_2(t)\left[\frac{y''(t_0,\bar\lambda)}{2}u_2(t) + y'(t_0,\bar\lambda)u_1(t)\right] - \frac{1}{2}\bar\lambda\int_{t_0}^{t} b(\tau)\begin{vmatrix} u_1(t) & u_2(t) \\ u_1(\tau) & u_2(\tau)\end{vmatrix}^2 y(\tau,\bar\lambda)d\tau. \tag{4}$$

where $u_1,\ u_2$ *form a fundamental set of solutions of* (3) *with the properties as in the formulation of Lemma 5.*

The proof of Lemma 6 is given in [2], Chap. I, §3 at the beginning of section 3 by method of variation of constants for

$$y''' + 2A(t)y' + A'(t) = -\bar\lambda b(t)y.$$

Remark 1. If in (4) $y(t,\bar\lambda) > 0$ $[y(t,\bar\lambda) < 0]$ for $t > t_0$, then $y'(t_0,\bar\lambda) > 0$ $[y'(t_0,\bar\lambda) < 0]$ and $u_2(t) > 0,\ u(t) = y'(t_0,\bar\lambda)u_1(t) + \frac{y''(t_0,\bar\lambda)}{2}u_2(t) > 0$ $[u(t) = y'(t_0,\bar\lambda)u_1(t) + \frac{y''(t_0,\bar\lambda)}{2}u_2(t) < 0]$ for $t > t_0$.

Corollary 1. *Let the supposition of Lemma 6 be fulfilled. Then there exist constants* $k_1 > 0,\ k_2 > 0$ *such that* $|y(t,\bar\lambda)| \le u_2(t)[k_1u_1(t) + k_2u_2(t)]$ *for* $t > t_0$ *where* $k_1 = |y'(t_0,\bar\lambda)|,\ k_2 = \frac{|y''(t_0,\bar\lambda)|}{2}$, *or*

$$y(t,\bar\lambda) = o\big(tu_2(t)[k_1u_1(t) + k_2u_2(t)]\big) \quad \text{for} \quad t \to \infty. \tag{2}$$

Adaptation of oscillation theorem [2, Theorem B, or Theorem 4.5 in the same section] to (a) in our case yields the following lemma.

Lemma 7. *Suppose that* $|A(t)| \le K,\ |A'(t)| \le K,\ K > 0$ *and* $b(t) \ge k > 0$ *for* $t \in [a,\infty)$. *Let* $\lambda \in (0,\infty)$ *and let* $y(t,\lambda)$ *be a nontrivial solution of* (a) *with* $y(a,\lambda) = 0$. *Then for any fixed* $b > a$, *the number of zeros of* y *on* $[a,b]$ *increases to infinity as* $\lambda \to \infty$, *and the distance between any consecutive zeros of* y *converges to zero.*

The continuous dependence of zeros of solutions of (a) upon the parameter λ is given in following lemma.

Lemma 8 (2, Lemma 4.2). *Let* y *be a nontrivial solution of* (a) *on* $[a,\infty)$ *such that* $y(a,\lambda) = 0$. *Then, the zeros of* y *on* (a,∞) *(if they exist) are continuous functions of the parameter* $\lambda \in (0,\infty)$.

With the help of results given in the preceding lemmas and Corollary 1 one can prove the following theorem regarding the singular eigenvalue problem (a), (1), (2).

Theorem 1. *Let $A(t) < 0$, $b(t) > 0$ and $|A(t)| < K$, $|A'(t)| < K$, $K > 0$ for $t \in [a, \infty)$. Let, further, $\int_a^\infty b(t)dt < \infty$ and $|A(t)| \geq \int_a^t b(\tau)d\tau$ for $t \in [a, \infty)$ and let $a \leq b < c < \infty$ be arbitrary, but fixed. Then there exists a natural number ν, a sequence of the values of the parameter λ, $\{\lambda_{\nu+p}\}_{p=0}^\infty$ (eigenvalues) such that $\lambda_{\nu+p} < \lambda_{\nu+p+1}$, $p = 0, 1, 2, \dots$ and $\lim_{p\to\infty} \lambda_{\nu+p} = \infty$ and a corresponding sequence of functions $\{y_{\nu+p}\}_{p=0}^\infty$ (eigenfunctions) such that $y_{\nu+p} = y(t, \lambda_{\nu+p})$ is a solution of (a) for $\lambda = \lambda_{\nu+p}$, has a finite number of zeros on (a, ∞) with the last zero at $t_0^{\nu+p}$, fulfills the boundary conditions (1), (2) and has exactly $\nu + p$ zeros in (b, c).*

Proof. Let $a < b < c < \infty$. Let $y = y(t, \lambda)$, $\lambda > 0$ be a nontrivial solution of (a) such that $y(a, \lambda) = y(b, \lambda) = 0$ for all $\lambda > 0$. Construct, now, on $[a, \infty)$ differential equation

$$Y''' + 2A(t)Y' + [A'(t) + \lambda B(t)]Y = 0, \tag{A}$$

where

$$B(t) = \begin{cases} b(t) & \text{for } t \in [a, c] \\ b(c) & \text{for } t \geq c. \end{cases}$$

Let $Y = Y(t, \lambda)$ be a solution of (A) on $[a, \infty)$ such that $Y(a, \lambda) = Y(b, \lambda) = 0$ and $Y(t, \lambda) = y(t, \lambda)$ for $t \in [a, c]$ and $\lambda \in (0, \infty)$.
By Lemma 4, there exists $\bar{\lambda}$ such that the differential equation (A) is oscillatory in $[a, \infty)$ for all $\lambda > \bar{\lambda}$. Let $Y(t, \lambda^*)$, $\lambda^* \geq \bar{\lambda}$ have exactly ν zeros in (b, c). Let $t_\nu(\lambda)$ be the ν-th zero of $Y(t, \lambda)$. Then there is $t_\nu(\lambda^*) < c \leq t_{\nu+1}(\lambda^*)$. By Lemma 7 there exists $\bar{\lambda^*}$ such that $t_{\nu+1}(\bar{\lambda^*}) < c$ and by Lemma 8 (continuous dependence of zeros) there exists λ_ν, $\lambda^* \leq \lambda_\nu < \bar{\lambda^*}$ such that $t_{\nu+1}(\lambda_\nu) = c$ and $Y(t, \lambda_\nu)$ has exactly ν zeros in (b, c). But, we know that $Y(t, \lambda_\nu) = y(t, \lambda_\nu)$ on $[a, c]$. By Lemma 2 to λ_ν there exists $t_0^\nu \geq c$ such that $y(t, \lambda_\nu)$ has finite numbers of zeros to the right of c. Let t_0^ν be its last zero on $[c, \infty)$. Then by Corollary 1 the inequality (2) holds.

Continuing in the same manner we can find a sequence of values

$$\lambda_\nu, \lambda_{\nu+1}, \dots, \lambda_{\nu+p}, \dots$$

and the corresponding sequence of functions $\{y_{\nu+p}\}_{p=0}^\infty$ (eigenfunctions) with the prescribed properties and the theorem is proved.

Remark 2. If we take in consideration the fact, that equation (a) is for $\lambda = 1$ disconjugate on $[a, \infty)$, the oscillation Lemma 7 and Lemma 8 (continuous dependence of zeros on λ) then it is possible to prove Theorem 1 for $\nu = 0$.

References

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