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TOPOLOGICAL STRUCTURE OF SOLUTION SETS: CURRENT RESULTS

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Abstract. We shall present current results concerning Browder–Gupta-type theorems, Banach principle for multivalued mappings and the inverse limit method including its applications to ordinary differential equations and differential inclusions. Consequently, Aronszajn's-type topological characterization of the set of solutions for differential equations and inclusions is considered. Note that some new results of the above mentioned type will be discussed.

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1. Introduction

In 1923, H. Kneser proved that the Peano existence theorem can be formulated in this way that the set of all solutions is not only nonempty but also compact connected (comp. also [139], [140]). Later, in 1942 N. Aronszajn improved the Kneser theorem by showing that the set of all solutions is even R_δ-set. Evidently the characterization of the set of fixed points for some operators implies the respective result for solution sets. This paper is an attempt to give a systematic presentation of results and methods which concern the topological structure of fixed point sets and solution sets. In this subject there are three methods so called Browder–Gupta method, Banach method and inverse limit method. We survey most important results concerning the above three methods. Our considerations concentrate on simplest cases and main ideas. We included rich literature in which the reader can find further results.

Our paper is devoted for mathematicians and students interested in the topological fixed point theory or in the qualitative theory of differential equations and differential inclusions.

In what follows we shall assume that all topological spaces considered in our paper are metric.

2. Browder–Gupta type results

The famous Schauder Fixed Point Theorem or more generally the Lefschetz Fixed Point Theorem says that there exists a fixed point theorem for some classes of mappings. So, a natural question is to characterize the set of fixed points. The first result, which is still a main one, was proved in 1969 by F. Browder and C. Gupta (comp [21]). Below we shall present a slight generalization of the above mentioned result.

To do this we need some topological notions (for details see: [69]).

Definition 2.1. A space X is called *contractible* provided there exists a (continuous) homotopy $h : X \times [0,1] \to X$ such that:

$$h(x,0) = x \quad \text{for every } x \in X$$

and

$$h(x,1) = x_0 \quad \text{for every } x \in X \text{ and some fixed } x_0 \in X.$$

Definition 2.2. A space X is called an *absolute retract* (written $X \in \mathrm{AR}$) provided that for every space Y, its closed subset $B \subset Y$ and continuous map $f : B \to X$ there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e. $\widetilde{f}(x) = f(x)$ for every $x \in B$.

Definition 2.3. A space X is called an *R_δ-set* provided that there exists a sequence of compact nonempty contractible spaces $\{X_n\}$ such that:

$$X_{n+1} \subset X_n \quad \text{for every } n; \tag{2.1}$$

(2.2)
$$X = \bigcap_{n=1}^{\infty} X_n.$$

Let us remark (comp. [69]) that a space $X \in \mathrm{AR}$ if and only if X is a convex subset W of a normed space E or X is homeomorphic to a retract[1] of a convex subset $W \subset E$. So any absolute retract is contractible. If we restrict our considerations to compact spaces then we have:

$$\boxed{\mathrm{AR}} \subset \boxed{\mathrm{CONTRACTIBLE}} \subset \boxed{R_\delta}$$

Note that any R_δ-set is a compact nonempty connected space which is acyclic with respect to the Čech homology functor (comp. again [69]), i.e. it has the same homology as the one point space $\{x_0\}$.

Definition 2.4. Let $f : X \to Y$ be a continuous function and let $y \in Y$. We shall say that f is *proper at the point* y provided that there exists $\varepsilon > 0$ such that for any compact set $K \subset B(y, \varepsilon)$ the set $f^{-1}(K)$ is compact, where $B(y, \varepsilon)$ is the open ball in Y with the center at $y \in Y$ and radius ε.

Recall that $f : X \to Y$ is called *proper* provided that for any compact $K \subset Y$ the set $f^{-1}(K)$ is compact. Of course any proper map $f : X \to Y$ is proper at every point $y \in Y$.

Now we are able to formulate our reformulation of the Browder–Gupta theorem:

Theorem 2.1. *Let E be a Banach space and $f : X \to E$ be a continuous map such that the following conditions are satisfied:*

(2.1.1) f *is proper at* $0 \in E$,
(2.1.2) *for every $\varepsilon > 0$ there exists a continuous map $f_\varepsilon : X \to E$ for which we have:*
 (i) $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ *for every* $x \in X$,
 (ii) *the map* $\widetilde{f}_\varepsilon : f_\varepsilon^{-1}(B(0, \varepsilon)) \to B(0, \varepsilon)$, $\widetilde{f}_\varepsilon(x) = f_\varepsilon(x)$ *for every* $x \in f_\varepsilon^{-1}(B(0, \varepsilon))$, *is a homeomorphism.*

Then the set $f^{-1}(\{0\})$ is an R_δ-set.

Sketch of proof. First, we have to prove that $f^{-1}(\{0\})$ is nonempty. We take for every $\varepsilon = 1/n$, $n = 1, 2 \dots$ a map $f_n : X \to E$ which satisfies (2.1.2). In view of (2.1.2)(ii) for every n we can find a point $x_n \in X$ such that $f_n(x) = 0$. It follows that:
$$\|f(x_n)\| = \|f(x_n) - f_n(x_n)\| < \frac{1}{n}.$$
So the sequence $\{f(x_n)\}$ is convergent to the point $0 \in E$. Since f is proper at $0 \in E$, we can assume without loss of generality that the sequence $\{x_n\}$ is

[1] A space A is a retract of W if there exists a continuous function $r : W \to A$ such that $r(x) = x$ for every $x \in A$ (we have assumed that $A \subset W$).

convergent to a point $x \in E$. Now from the continuity of f it follows that $f(x) = 0$ and consequently $f^{-1}(\{0\}) \neq \emptyset$.

Now let us denote by S the set $f^{-1}(\{0\})$. It follows from (2.1.1) that S is compact. Moreover, we have proved that $S \neq \emptyset$. For every $\varepsilon = 1/n$, $n = 1, 2\ldots$ let $A_n = f_n(S)$ where f_n are chosen according to (2.1.2). Then from (2.1.2)(i) we deduce that $A_n \subset B(0, 1/n)$. Note that $\{A_n\}$ is a sequence of compact sets. We let:

$$C_n = \overline{\text{conv}}(A_n).$$

It follows from the Mazur's Lemma (comp. [69] or [90]) that C_n is a compact convex subset of $B(0, 1/n)$. Now by using (2.1.2)(ii) we deduce that set $D_n = f_n^{-1}(C_n)$ is an absolute retract (because it is homeomorphic to the convex set C_n). Therefore we can proceed in the same way as in the proof of Theorem 7 ([21]) and our theorem follows from Lemma 5 in [21].

Note that assumptions in 2.1 are analogous to Theorem 7 ([21]).

Let us remark also that Theorem 2.1 has exactly the same proof if we replace the Banach space E by an arbitrary Fréchet space and open balls by convex symmetric open neighbourhoods of the zero point $0 \in E$. We shall show it in the multivalued case.

Now, we are going to explain the scope of fixed point interpretation of Theorem 2.1.

Assume that $X \subset E$ and $F : X \to E$ is a given mapping. We let $f : X \to E$, $f(x) = x - F(x)$. Then f is called the *field* associated with F. We have:

$$f^{-1}(\{0\}) = \text{Fix}(F) = \{x \in X \mid F(x) = x\}.$$

Observe that if $F_\varepsilon : X \to E$ is an ε-approximation of F then f_ε $(f_\varepsilon(x) = x - F_\varepsilon(x))$ is an ε-approximation of f $(f(x) = x - F(x))$.

It is well known that if F is a compact map or k-set contraction or condensing map which has ε-approximation of the same type then all assumptions of Theorem 5.2 are satisfied for the field f $f(x) = x - F(x))$ associated with F.

We would like to conclude that Theorem 5.2 contains as a special case many results, the called generalizations of the Browder–Gupta theorem (com. [21], [38], [39], [54], [55], [56], [101], [102], [141], [147], [158], [175], [176]).

There is a natural and essential problem to formulate an appropriate multivalued version of the Browder–Gupta Theorem. In this order see: [6], [12], [19], [34], [35], [61], [62], [60], [75], [84], [88], [101], [123], [124], [54]. The most general result was obtained in 1999 by G. Gabor (see [60]). We shall present below the Gabor result.

To do this recall some notation. In what follows the symbol $\varphi : X \multimap Y$ is reserved for multivalued mappings. In this Section we shall assume that for every $x \in X$ the set $\varphi(x)$ is compact nonempty.

A map $\varphi : X \multimap Y$ is called *upper semicontinuous* (u.s.c.) provided that for every open $U \subset Y$ the set $\{x \in X \mid \varphi(x) \subset U\}$ is open; φ is called lower semicontinuous (l.s.c.) provided that for every open $U \subset Y$ the set:

$$\{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$$

is open; φ is continuous, if φ is both u.s.c. and l.s.c.

A map $\varphi : X \multimap Y$ is proper provided that for every compact $K \subset Y$ the set

$$\{x \in X \mid \varphi(x) \cap K \neq \emptyset\}$$

is compact. In what follows for given $\varphi : X \multimap Y$ and $A \subset Y$ we let:

$$\varphi^{-1}(A) = \{x \in X \mid \varphi(x) \subset A\},$$
$$\varphi_+^{-1}(A) = \{x \in X \mid \varphi(x) \cap A \neq \emptyset\}.$$

Assume that $X \subset Y$ and $\varphi : X \multimap Y$ is a given multivalued map. We let

$$\mathrm{Fix}(\varphi) = \{x \in X \mid x \in \varphi(x)\}.$$

Now we are able to formulate the multivalued version of the Browder–Gupta Theorem (see: [60]).

Theorem 2.2. *Let X be a metric space, E a Fréchet space, $\{U_k\}$ a base of open convex symmetric neighbourhoods of the origin in E, and let $\varphi : X \multimap E$ be an u.s.c. proper map with compact values. Assume that there is a sequence of compact convex valued u.s.c. proper maps $\varphi_k : X \to E$ such that*

(i) $\varphi_k(x) \subset \varphi(N_{1/k}(x)) + U_k$, *for every* $x \in X$,
(ii) *if* $0 \in \varphi(x)$, *then* $\varphi_k(x) \cap \overline{U_k} \neq 0$,
(iii) *for every* $k \geq 1$ *and every* $u \in E$ *with* $u \in U_k$ *the inclusion* $u \in \varphi_k(x)$ *has an acyclic set of solutions.*

Then the set $\mathcal{S} = \varphi^{-1}(0)$ is compact and acyclic[2].

Proof. We show that $\mathcal{S}$ is nonempty. To this end, notice that for every $k \geq 1$ we can find $x_k \in X$ such that $0 \in \varphi_k(x_k)$. Assumption (i) implies that there are $z_k \in N_{1/k}(x_k)$, $y_k \in \varphi_k(z_k)$ and $u_k \in U_k$ such that $0 = y_k + u_k$. Thus $y_k \to 0$. Consider the compact set $K = \{y_k\} \cup \{0\}$. Since φ is proper, the set $\varphi_+^{-1}(K)$ is compact. Moreover, $\{z_k\} \subset \varphi_+^{-1}(K)$. Thus we can assume, without loss of generality, that $\{z_k\}$ converges to some point $x \in X$. By the upper semicontinuity of φ, we have $0 \in \varphi(x)$ and, what follows, $\mathcal{S} \neq \emptyset$.

Since φ is proper, the set $\mathcal{S}$ is compact. We show that it is acyclic. By assumption (ii), the set $A_k = \varphi_{k+}^{-1}(\overline{U_k})$ is nonempty. Consider the map $\psi : A_k \multimap \overline{U_k}$, $\psi_k(x) = \varphi_k(x) \cap \overline{U_k}$. Since $\overline{U_k}$ is contractible and ψ_k is u.s.c. convex valued surjection (see (iii)), we can apply Corollary 3.12 in [60] to obtain that A_k is acyclic.

Now we show that for every open neighbourhood U of $\mathcal{S}$ in X there exists $k \geq 1$ such that $A_k \subset U$. Indeed, assume on the contrary that there is an open neighbourhood U of $\mathcal{S}$ in X such that $A_k \not\subset U$ or every $k \geq 1$. It means that there are $x_k \in A_k$ with $x_k \notin U$ and, consequently, there are $y_k \in \varphi_k(x_k)$ such that

[2] i.e. the Čech homology of $\mathcal{S}$ are the same as a singleton $\{x_0\}$.

$y_k \in \overline{U_k}$. Assumption (i) implies that there are $z_k \in B(x_k, 1/k)$, $v_k \in \varphi(z_k)$ and $u_k \in U_k$ such that $y_k = v_k + u_k$. Therefore, $v_k = y_k - u_k \in 2U_k$ which implies that $v_k \to 0$. Consider the compact set $K_0 = \{v_k\} \cup \{0\}$. Since φ is proper, we can assume that $\{z_k\}$ and, consequently, $\{x_k\}$ converges to some point $x \in X$. Thus $x \in \mathcal{S}$. On the other hand, $x \notin U$, a contradiction and our theorem follows from Lemma 3.10 in [60].

Remark 2.1. It is easy to see that in the above result we can assume that X is a subset of a Fréchet space. Then, instead of neighbourhoods, we can consider sets $x + V_k$, where $\{V_k\}$ is the base of open convex symmetric neighbourhoods of the origin.

As a consequence of Theorem 3.6 and properties of a topological degree of u.s.c. compact convex valued maps (see e.g. [69] or [104]) one can obtain the following theorem generalizing the result of Czarnowski in [39].

Theorem 2.3. *Let Ω be an open subset of a Fréchet space E, $\{U_k\}$ the base of open convex symmetric neighbourhoods of the origin in E, and $\Phi : \overline{\Omega} \multimap E$ a compact u.s.c. map with compact convex values. Suppose that $x \notin \Phi(x)$ for every $x \in \partial\Omega$, and the topological degree* $\deg(j - \Phi, \Omega, 0)$ *of $(j - \Phi)$ is different from zero, where $j : \overline{\Omega} \to E$ is an inclusion. Assume that there exists a sequence $\{\Phi_k : \Omega \multimap E\}$ of compact u.s.c. maps with compact convex values such that*

(i) *$\Phi_k(x) \subset \Phi(x + U_k) + U_k$, for every $x \in \overline{\Omega}$,*
(ii) *if $x \in \Phi(x)$, then $x \in \Phi_k(x) + U_k$,*
(iii) *for every $u \in \overline{U_k}$ the set $\mathcal{S}_u^k$ of all solutions to the inclusion $x - \Phi_k(x) \ni u$ is acyclic or empty, for every $n > 0$.*

Then the fixed point set $\mathrm{Fix}(\Phi)$ *of Φ is compact and acyclic.*

Proof. Define the maps φ, $\varphi_k : \overline{\Omega} \multimap E$, $\varphi = j - \Phi$, $\varphi_k = j - \Phi_k$. One can check that φ, φ_k are proper maps. To apply Theorem 2.2 it is sufficient to show that, for sufficiently big k and for every $u \in \overline{U_k}$ the set $\mathcal{S}_u^k$ is nonempty.

For each $k \geq 1$ define the map $\Psi : \overline{\Omega} \multimap E$, $\Psi(x) = \Phi_k(x) + u$, for every $x \in \overline{\Omega}$. We prove that, for sufficiently big k, $\deg(j - \Psi_k, \Omega, 0) \neq 0$ which implies, by the existence property of a degree, a nonemptiness of $\mathcal{S}_u^k$.

Since φ is a closed[3] map (see e.g. [69]), we can find, for sufficiently big k, a neighbourhood U_k of the origin such that $\varphi(\partial\Omega) \cap \overline{U_k} = \emptyset$.

Consider the following homotopy $H_k : \overline{\Omega} \times [0,1] \multimap E$, $H(x,t) = (1-t)\Phi(x) + t\Psi_k(x)$. We show that

$$Z_k = \{x \in \partial\Omega \mid x \in H_k(x,t) \text{ for some } t \in [0,1]\} = \emptyset$$

for sufficiently big k. Suppose, on the contrary, that there are a subsequence of $\{H_k\}$ (we denote it also by $\{H_k\}$), points $x_k \in \partial\Omega$, and numbers $t_k \in [0,1]$

[3] φ is closed provided for every closed $K \subset \overline{\Omega}$ the set $\Phi(K) = \bigcup_{x \in K} \Phi(x)$ is a closed subset of E.

such that $x_k \in H_k(x_k, t_k)$, that is $x_k = (1 - t_k)y_k + t_k s_k + t_k u$, for some $y_k \in \Phi(x_k)$ and $s_k \in \Phi(x_k)$. Assumption (i) implies that there are $z_k \in x_k + U_k$ and $v_k \in \Phi(z_k)$ such that $s_k \in v_k + U_k$. By the compactness of Φ, we can assume that $y_k \to y$ and $v_k \to v$. Therefore, $s_k \to v$. Moreover, we can assume that $t_k \to t \in [0,1]$. This implies that $x_k \to x_0 = (1-t)y + tv + tu$ or, equivalently, that $0 = (1-t)(x_0 - y) + t(x_0 - v) - tu$. But by the upper semicontinuity of φ, we obtain that $x_0 - y \in \varphi(x_0)$ and $x_0 - v \in \varphi(x_0)$. Since φ is convex valued, $0 \in (1-t)\varphi(x_0) + t\varphi(x_0) - tu \subset \varphi(x_0) - tu$. This implies that $\varphi(x_0) \cap \overline{U_k} \neq \emptyset$, a contradiction.

Now, by the homotopy property of a topological degree, one obtains

$$\deg(\Psi_k, \Omega, 0) = \deg(\Phi, \Omega, 0) \neq 0$$

which ends the proof of the theorem.

3. Aronszajn type results

In 1890 Peano [140] showed that the Cauchy problem

$$\left.\begin{aligned} &\dot{x}(t) = g(t, x(t)) \quad \text{for } t \in [0,a], \\ &x(0) = x_0, \end{aligned}\right\} \tag{3.1}$$

where $g : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, has local solutions although the uniqueness property does not hold in general.

This observation became a motivation for studying the structure of the set $\mathcal{S}$ of solutions to (3.1). Peano himself showed that, in the case $n = 1$, all sections $\mathcal{S}(t) = \{x(t) \mid x \in \mathcal{S}\}$ are nonempty, compact and connected (that is, a continuum) in the standard topology of the real line, for t in some neighbourhood of t_0. Kneser generalized this result in 1923 [89] into the case of arbitrary n. In 1928 Hukuhara [80] proved that $\mathcal{S}$ is a continuum in the Banach space of continuous functions with the sup norm.

A more precise characterization of $\mathcal{S}$ was found in 1942 by Aronszajn [10], who showed that $\mathcal{S}$ is an R_δ-set, i.e. it is homeomorphic to the intersection of a decreasing sequence of compact contractible spaces (or compact absolute retracts). This implies that $\mathcal{S}$ is acyclic which means that, without a lipschitzianity of the right hand side f of (3.1), the set $\mathcal{S}$ of solutions (3.1) may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point, in the sense that it has the same homology groups as one point space $\{x_0\}$.

Aronszajn's result was improved by several authors (see: [1], [3], [4]–[6], [9], [14], [15], [16], [17], [19], [23], [24], [32], [34], [13], [38]–[39], [40], [42]–[44], [46]–[47], [48]–[49], [53], [54], [60], [66], [68], [73], [75], [77]–[78], [97]–[98], [101], [121]–[138], [156]–[168], [171]–[173], [174]–[178]) but always a main tool to do it is a version of the Browder–Gupta theorem. We shall sketch it in the case of problem (3.1) first for the singlevalued case and later for the multivalued case.

The singlevalued case follows immediately from the Browder–Gupta Theorem and the Szufla's type lemma (see [164] or [68]) which we shall present below.

The following result is a slight reformulation of Lemma 1 in [164].

Theorem 3.1. *Let $E = C([0,a],\mathbb{R}^m)$ be the Banach space of continuous maps with the usual max-norm and let $X = K(0,r) = \{u \in E \mid \|u\| \leq r\}$ be the closed ball in E.*

If $F : X \to E$ is a compact map and $f : X \to E$ is a compact vector field associated with F, i.e. $f(u) = u - F(u)$, such that the following conditions are satisfied:

(3.1.1) *there exists an $x_0 \in \mathbb{R}^m$ such that $F(u)(0) = x_0$, for every $u \in K(0,r)$;*
(3.1.2) *for every $\varepsilon \in]0,a]$ and for every $u, v \in X$, if $u(t) = v(t)$ for each $t \in [0,\varepsilon]$, then $F(u)(t) = F(v)(t)$ for each $t \in [0,\varepsilon]$;*

then there exists a sequence $f_n : X \to E$ of continuous proper mappings satisfying conditions (2.1.1)–(2.1.2) *with respect to f.*

Sketch of proof. For the proof it is sufficient to define a sequence $F_n : X \to E$ of compact maps such that:

(i) $$F(x) = \lim_{n\to\infty} F_n(x), \quad \text{uniformly in } x \in X,$$

and

(ii) $$f_n : X \to E, \quad f_n(x) = x - F_n(x), \text{ is a one-to-one map.}$$

To do this we additionally define the mappings $r_n : [0,a] \to [0,a]$ by putting:

$$r_n(t) = \begin{cases} 0, & t \in \left[0, \dfrac{a}{n}\right], \\ t - \dfrac{a}{n}, & t \in \left(\dfrac{a}{n}, a\right]. \end{cases}$$

Now we are able to define the sequence $\{F_n\}$ as follows:

(iii) $$F_n(x)(t) = F(x)(r_n(t)), \quad \text{for } x \in X,\ n = 1,2,\ldots .$$

It is easily seen that F_n is a continuous and compact mapping, $n = 1,2,\ldots$. Since $|r_n(t) - t| \leq a/n$ we deduce from compactness of F and (iii) that

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \text{uniformly in } x \in X.$$

Now we shall prove that f_n is a one-to-one map. Assume that for some $x, y \in X$ we have

$$f_n(x) = f_n(y).$$

This implies that

$$x - y = F_n(x) - F_n(y).$$

If $t \in [0, a/n]$ then we have

$$x(t) - y(t) = F(x)(r_n(t)) - F(y)(r_n(t)) = F(x)(0) - F(y)(0).$$

Thus, in view of (3.1.1), we obtain

$$x(t) = y(t), \quad \text{for every } t \in [0, a/n].$$

Finally, by successively repeating the above procedure n times we infer that

$$x(t) = y(t), \quad \text{for every } t \in [0, a].$$

Therefore f_n is a one-to-one map and the proof is complete.

Now from Theorems 2.1 and 3.1 we get:

Corollary 3.1. *Assume that f and F are as in Theorem (3.1). Then $f^{-1}(0) = \operatorname{Fix}(F)$ is an R_δ-set.*

Now we come back to problem (3.1). We shall denote by $\mathcal{S}(g, 0, x_0)$ the set of all solutions of the Cauchy problem (3.1).

Theorem 3.2 (Aronszjan). [4] *Let $g : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping such that:*

(3.2.1) *$g(\,\cdot\,, x)$ is a measurable function for every $x \in \mathbb{R}^n$,*
(3.2.2) *$g(t, \,\cdot\,)$ is a continuous function for every $t \in [0, a]$,*
(3.2.3) *there exists a Lebesgue integrable function $\alpha : [0, a] \to [0, +\infty)$ such that:*

$$\|g(t, x)\| \le \alpha(t) \quad \textit{for every } (t, x) \in [0, a] \times \mathbb{R}^n.$$

Then $\mathcal{S}(g, 0, x_0)$ is an R_σ-set.

Sketch of proof. We define the integral operator:

$$F : C([0, a], \mathbb{R}^n) \to C([0, a], \mathbb{R}^n)$$

by putting

$$F(u)(t) = x_0 + \int_0^t g(\tau, u(\tau))\, d\tau \quad \text{for every } u \text{ and } t. \tag{3.2}$$

Then $\operatorname{Fix}(F) = \mathcal{S}(g, 0, x_0)$. It is easy to see that F satisfies all the assumptions of Theorem 2.1. Consequently we deduce Theorem 3.2 from 3.1 and the proof is complete.

Now, let g be a Carathéodory map with linear growth. Assume further that $u \in \mathcal{S}(g, 0, x_0)$. Then we have (cf. (3.2.1))

$$u(t) = F(u)(t) = x_0 + \int_0^t g(\tau, u(\tau))\, d\tau,$$

[4] A mapping $g : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying conditions (3.2.1) and (3.2.2) will be called a Carathéodory function; if g satisfies (3.2.3) then it is called integrably bounded.

and consequently

$$\|u(t)\| \leq \|x_0\| + \int_0^a \mu(\tau)\,d\tau + \int_0^t \mu(\tau)\|u(\tau)\|\,d\tau.$$

Therefore from the well-known Gronwall inequality we get

$$\|u(t)\| \leq (\|x_0\||\gamma)\exp(\gamma) \quad \text{for every } t,$$

where $\gamma = \int_0^a \mu(\tau)\,d\tau$. We let

$$g_0 : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$$

by putting

$$g_0(t,x) = \begin{cases} g(t,x), & \text{if } \|x\| \leq M \text{ and } t \in [0,a], \\ g(t, Mx/\|x\|), & \text{if } \|x\| \geq M \text{ and } t \in [0,a], \end{cases}$$

where $M = (\|x_0\| + \gamma)\exp(\gamma)$.

Proposition 3.1. *If g is a Carathéodory map with linear growth, then*

(3.1.a) *g_0 is Carathéodory and integrably bounded; and*
(3.1.b) $\mathcal{S}(g_0,0,x_0) = \mathcal{S}(g,0,x_0)$.

The proof of Proposition 3.1 is straightforward (cf. [68], [69], [91]).
Now from Theorem 3.2 and Proposition 3.1 we obtain immediately:

Corollary 3.2. *If $g : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map and has linear growth, then $\mathcal{S}(g,0,x_0)$ is an R_σ-set.*

We recall the following classical result:

Theorem 3.3. *If $g : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping which is integrably bounded and satisfies condition* (3.41) *and it is locally Lipschitz with respect to the second variable*[5]*, then $\mathcal{S}(g,0,x_0)$ is an R_δ-set.*

In 1986 F. S. De Blasi and J. Myjak (see [47]) generalized Aronszajn's result for differential inclusions with u.s.c. convex valued right hand sides. Below we shall show the method presented in [68] (comp. also [101], [102]). For the simplicity we shall consider the following Cauchy problem:

$$\left.\begin{aligned} &x'(t) \in \varphi(t,x(t)), \\ &x(0) = x_0, \end{aligned}\right\} \tag{3.3}$$

where $\varphi : [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an u.s.c. bounded map with compact convex values.

We shall denote by $\mathcal{S}(\varphi;0,x_0)$ the set of all solutions of (3.3). In what follows we keep all assumptions on φ contained in (3.3).

First we have:

[5] Such a mapping g is called integrably bounded measurable-locally Lipschitz.

Proposition 3.2. *If φ possesses a measurable-locally Lipschitz selector $f : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$, (written $f \subset \varphi$), i.e. $f(t,x) \in \varphi(t,x)$ for every $(t,x) \in [0,a] \times \mathbb{R}^n$, then $\mathcal{S}(\varphi; 0, x_0)$ is contractible.*

Sketch of proof. Let $f \subset \varphi$ be measurable-locally Lipschitz selector. By Theorem 3.3 the following Cauchy problem:

$$\left.\begin{aligned} x'(t) &= f(t,x(t)), \\ x(t_0) &= u_0, \end{aligned}\right\} \tag{3.4}$$

has exactly one solution for every $t_0 \in [0,a]$ and $u_0 \in \mathbb{R}^n$. For the proof it is sufficient to define a homotopy $h : \mathcal{S}(\varphi, 0, x_0) \times [0,1] \to \mathcal{S}(\varphi, 0, x_0)$ such that

$$h(x,s) = \begin{cases} x & \text{for } s = 1 \text{ and } x \in \mathcal{S}(\varphi, 0, x_0), \\ \overline{x} & \text{for } s = 0, \end{cases}$$

where $x = \mathcal{S}(\varphi, 0, x_0)$ is exactly one solution given for the Cauchy problem (3.4).

We put

$$h(x,s)(t) = \begin{cases} x(t), & 0 \le t \le sa, \\ \mathcal{S}(f, sa, x(sa))(t), & sa \le t \le a. \end{cases}$$

Then h is a continuous homotopy contracting $\mathcal{S}(\varphi, 0, x_0)$ to the point $\mathcal{S}(\varphi, 0, x_0)$.

Observe that if $\varphi : [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an intersection of the decreasing sequence $\varphi_k : [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ i.e. $\varphi(t,x) = \bigcap_{k=1}^{\infty} \varphi_k(t,x)$ and $\varphi_{k+1}(t,x) \subset \varphi_k(t,x)$ for almost all $t \in [0,a]$ and for all $x \in \mathbb{R}^n$, then

$$\mathcal{S}(\varphi, 0, x_0) = \bigcap_{k=1}^{\infty} \mathcal{S}(\varphi_k, 0, x_0).$$

We have (see: [102] or [69]):

Theorem 3.4. *Assume that φ is as in* (3.3)*. Then there exists a decreasing sequence $\varphi_k : [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ of compact convex valued and bounded u.s.c. mappings such that:*

(3.4.1) *$\varphi(t,x) = \bigcap_{k=1}^{\infty} \varphi_k(t,x)$ for every $t,x) \in [0,a] \times \mathbb{R}^n$,*
(3.4.2) *every φ_k possesses a measurable locally Lipschitz selector $f_k \subset \varphi_k$.*

Now we are in the position to prove the following Aronszajn-type result:

Theorem 3.5. *Under assumptions of* (3.3) *the set $\mathcal{S}(\varphi; 0, x_0)$ is R_δ.*

Sketch of proof. Consider the sequence $\{\varphi_k\}$ according to (3.4). Then:

$$\mathcal{S}(\varphi; 0, x_0) = \bigcap_{k=1}^{\infty} \mathcal{S}(\varphi_k, 0, x_0).$$

In view of Proposition 3.2 the set $\mathcal{S}(\varphi_k; 0, x_0)$ is contractible. Since φ_k is u.s.c. bounded with convex compact values if follows that $\mathcal{S}(\varphi_k; 0, x_0)$ is compact nonempty (see for example [69]). Therefore $\mathcal{S}(\varphi; 0, x_0)$ is an intersection of compact nonempty and contractible spaces and hence $\mathcal{S}(\varphi; 0, x_0)$ is R_δ.

Remark 3.1. Theorem (3.5) remains true for φ a Carathéodory map with sublinear growth (see: [69] or [47]).

Above we have showed only an application of Browder-Gupta Theorem to the Cauchy problem for the first order ordinary differential equations (inclusions) in the Euclidean space $\mathbb{R}^n$. We would like to point out that another applications are possible, namely:

(A) to the Cauchy problem in Banach spaces on compact or noncompact intervals (see: [2], [4], [9], [18], [32], [34], [35], [38], [39], [40], [49], [50], [53], [55], [56], [59], [61], [62], [66]–[68], [77], [78], [88], [97], [98], [94]–[96], [121]–[138], [152], [174], [175], [171]–[173]);
(B) to higher order differential equations or inclusions (see: [14], [15], [22], [16], [29], [34], [47], [48], [107], [108], [156]–[168], [169], [170]);
(C) to more general boundary value problems both ordinary differential equations and inclusions (see: [5], [6], [12], [17], [72], [13], [93], [149]–[151], [110]–[117]);
(D) to integral equations and inclusions (see: [1], [23]–[25], [87], [171], [172]).

We shall end this section by showing you another possibility. We mean differential equations (inclusions) on compact subsets of $\mathbb{R}^n$ or more generally of Banach spaces. There are only few papers devoted this problem (see: [17], [13], [54], [72], [66], [121]–[123] [143]). For simplicity we shall restrict our considerations to subsets of $\mathbb{R}^n$ (for the Banach case see: [13], [72] and [54]).

Let K be a compact subset of $\mathbb{R}^n$. For a point $x \in K$ by T_xK we shall denote the Bouligand tangent cone to K at x.

We have (see: [66] or [69]):

$$T_xK = \left\{ y \in \mathbb{R}^n \;\middle|\; \liminf_{t\to 0^+} \frac{\operatorname{dist}(x+ty, K)}{t} = 0 \right\}.$$

A compact subset $K \subset \mathbb{R}^n$ is called a *proximate retract* provided there exists an open neighbourhood U of K in $\mathbb{R}^n$ and a retraction $r : U \to K$ such that:

$$\|x - r(x)\| = \operatorname{dist}(x, K), \quad \text{for every } x \in U.$$

It is well known that the class of all proximate retracts is quite rich, in particular it contains convex sets and C^2-manifolds.

Now, let $\varphi : [0,a] \times K \multimap \mathbb{R}^n$ be an u.s.c. map which is bounded and compact convex valued. We shall assume also the following:

$$\varphi(t,x) \cap T_xK \neq \emptyset, \quad \text{for every } (t,x) \in [0,a] \times K. \tag{3.5}$$

For such a map φ we consider the following Cauchy problem:

$$\left.\begin{aligned} &x'(t) \in \varphi(t, x(t)), \\ &x(0) = x_0, \qquad\quad x_0 \in K, \end{aligned}\right\} \tag{3.6}$$

where solutions are considered as absolutely continuous functions $x : [0,a] \to \mathbb{R}^n$ such that $x(t) \in K$ for every $t \in [0,a]$.

Let $\mathcal{S}_K(\varphi; 0, x_0)$ denote the set of all solutions of (3.6).

In 1992 S. Plaskacz proved (see: [143])

Theorem 3.6. *Under all of the above assumptions the set $\mathcal{S}_K(\varphi; 0, x_0)$ is R_δ.*

For the proof of Theorem 3.6 we recommend [143] or [66] or [69].

Remark 3.2. There exists a recent result of R. Bader and W. Kryszewski ([13]) where Theorem 3.6 is taken up for regular sets in Hilbert spaces and Carathéodory-type mappings.

4. Fixed points of multivalued contractions and applications

The Banach contraction principle is one of few fixed point theorems, where, besides the existence, some further information is included, namely how the unique fixed point can be successively approximated with arbitrary accuracy. In the case of a multivalued contraction we have the set of fixed points. So a natural question of its topological characterization arises. In this section we shall review most important results of this type. For more details we recommend: [20], [7], [37], [54], [61], [70], [71], [105], [144], [145].

For a metric space (X, d), by $C(X)$ we shall denote the family of all closed nonempty subsets of X For $A \in C(X)$ and $\varepsilon > 0$, we let

$$0_\varepsilon(A) = \{x \in X \mid \exists y \in A, d(x, y) < \varepsilon\}.$$

Let $A, B \in C(X)$. We define the Hausdorff distance $d_H(A, B)$ between A and B as follows:

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B) \text{ and } B \subset O_\varepsilon(A)\}.$$

It is well known that $d_H(A, B)$ can be equal to infinity. If we restrict our considerations to the family $BC(X)$ of all bounded closed and nonempty subsets of X, then d_H is a metric $BC(X)$, the so called Hausdorff metric.

Let E be a Banach space and A, B, C, $D \in BC(E)$. It is easy to see that:

$$\begin{array}{llr} d_H(A + B, C + D) \leq & d_H(A, C) + d_H(B, D), & (i) \\ d_H(\{x + A\}, \{y\}) = & d_H(\{x\}, \{y - A\}), & (ii) \\ d_H(tA, tB) & \leq d_H(A, B), \quad \text{for } t \in [0, 1], & (iii) \end{array}$$

where $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$ is the algebraic sum of A and B and $tA = \{tx \mid x \in A\}$.

Recall that a mapping $F : Y \to BC(X)$ is called Hausdorff-continuous if it is continuous w.r.t. the metric d in Y and d_H in $BC(X)$.

F is called *measurable* if, for every closed $U \subset X$, the set $F_+^{-1}(U)$ is measurable.

Proposition 4.1 ([69]). *A map $F : Y \to BC(X)$ is Hausdorff-continuous with compact values if and only if F is both u.s.c. and l.s.c.*

Note that, for $F : Y \to BC(X)$, the Hausdorff continuity implies only l.s.c. (see again [69]). It is easy to see that the following proposition is true.

Proposition 4.2. *If $F : Y \to BC(X)$ is l.s.c. with connected values and $F(Y) = \bigcup_{y \in Y} F(y) = X$, then X is connected, provided Y is connected.*

If what follows we need some additional topological notions. A metric space (X, d) is C^n (i.e. n-connected) if, for every $k \leq n$, every continuous map from the k-sphere S^k into X is null homotopic (i.e. homotopic to a constant map). Namely,

it means that every continuous map $f : S^k \to X$ has a continuous extension over the closed ball K^{n+1}, where S^n and K^{n+1} stand for the unit sphere and the unit closed ball in the Euclidean $(n+1)$-space $\mathbb{R}^n$, respectively.

A space X is C^∞ (i.e. infinitely connected), if it is C^n, for every n. A collection $\mathcal{E} \subset 2^X$ is equi-LC^n if, for every $y \in \bigcup\{B \mid B \in \mathcal{E}\}$, every neighbourhood V of y in X contains a neighbourhood W of y in X such that, for all $B \in \mathcal{E}$ and $k \leq n$, every map from S^k into $W \cap B$ is null-homotopic over $V \cap E$ (i.e. a homotopy taking values in $V \cap E$). We shall also make use of the following (comp. [69]).

Theorem 4.1 (Michael's Selection Theorem). *Let X be a metric space and Y be a complete metric space. Let $F : X \to BC(Y)$ be a l.s.c. map such that the topological dimension* $\dim X \leq n+1$ *and $F(x)$ is C^n and for all $x \in X$ with the collection $\{F(x) \mid x \in X\}$ equi-LC^n. Then F has a continuous selection.*

A mapping $F : X \to C(X)$ is called a *multivalued contraction* if there exists $\alpha < 1$ such that:

$$d_H(F(x), F(y)) < \alpha d(x, y), \quad \text{for every } x, y \in X.$$

In 1970, H. Covitz and S. B. Nadler proved:

Theorem 4.2 ([37]). *If X is a complete metric space and $F : X \to C(X)$ is a contraction, then* $\mathrm{Fix}(F) = \{x \in X \mid x \in F(x)\} \neq \emptyset$.

Let $F : X \to C(X)$ be a contraction. Obviously, the set $\mathrm{Fix}(F)$ is not a singleton, in general. For example, let $F(x) = A$, for every $x \in A$, be a constant map. Evidently, F is a contraction and $\mathrm{Fix}(F) = A$.

The following theorem is due to B. Ricceri ([145]).

Theorem 4.3. *Let E be a Banach space and let $F : E \to C(E)$ be a contraction such that $F(x)$ is convex, for every $x \in E$. Then* $\mathrm{Fix}(F)$ *is a retract of E.*

In 1991, A. Bressan, A. Cellina and A. Fryszkowski proved:

Theorem 4.4 ([20]). *If $E = L^1(T)$ is the space of integrable functions on a measure space T and $F : E \to BC(E)$ is a contraction with decomposable*[6] *values, then* $\mathrm{Fix}(F)$ *is a compact* AR*-space.*

[6] $A \subset L^1(T)$ is decomposable if, for every $\gamma, \mu \in A$ and a measurable subset $J \subset T$, we have:

$$(\gamma \cdot \chi_J + \mu \chi_{T \setminus J}) \in A,$$

where χ_S is the characteristic function of the subset $S \subset T$.

In view of [70] and [71], we would like to generalize both 4.4 and 4.5.

A simple argument shows that the following proposition [71], Proposition 1.1 is true.

Proposition 4.3. *Let X be a separable metric space and let X_0 be a nonempty closed subset of X. If $X \in \mathrm{AR}$ and, for any separable space Y and any nonempty closed set $Y_0 \subset Y$, every continuous function $f_0 : Y_0 \to X_0$ admits a continuous extension over Y, then $X_0 \in \mathrm{AR}$.*

Let (T, F, μ) be a finite, positive, nonatomic measure space and let $(E, \|\cdot\|)$ be a Banach space. We denote by $L^1(T, E)$ the Banach space of all (equivalent classes of) μ-measurable functions $u : T \to E$ such that the function $t \to \|u(t)\|$ is μ-integrable, equipped with the norm

$$\|u\|_{L^1(T,E)} = \int_T \|u(t)\|\, d\mu.$$

We always assume that the space $L^1(T, E)$ is separable. The multifunction $F : X \to C(X)$ is called *Lipschitzean* if there exists a real number $L \geq 0$ such that $d_H(F(x'), F(x'')) \leq L d(x', x'')$, for all $x', x'' \in X$. If $L < 1$, we say that F is a multivalued contraction. It can be easily checked that any Lipschitzean multifunction is l.s.c. The following property of Lipschitz multifunctions will play an important role in proving the main result of this section.

Proposition 4.4. *Let (X, d) be a metric space and let $F : X \to C(X)$ be a Lipschitzean multifunction. Set, for every $x \in X$, $\varphi(x) = d(x, F(x))$. Then the function $\varphi : X \to [0, +\infty)$ is Lipschitzean.*

Proof. Let $L \leq 0$ be such that $d_H(F(x'), F(x'')) \leq L d(x', x'')$, for all $x', x'' \in X$. Pick $x', x'' \in X$ and choose $\varepsilon > 0$. Owing to the definition of φ, there exists $z' \in F(x')$ fulfilling

$$-\varphi < -d(x', z') + \varepsilon.$$

Using the inequality $d(z', F(x'')) \leq L d(x', x'')$, we can find $z'' \in F(x'')$ such that,

$$d(z', z'') < L d(x', x'') + \varepsilon.$$

Therefore,

$$\begin{aligned} e(x'') - e(x') &< d(x'', F(x'')) - d(x', z') + \varepsilon \\ &\leq d(x'', z'') - d(x', z') + \varepsilon < (L+1) d(x', x'') + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, we actually have

$$\varphi(x'') - \varphi(x') \leq (L+1) d(x', x'')$$

and, interchanging x' with x'',

$$\varphi(x') - \varphi(x'') \leq (L+1) d(x', x'').$$

This completes the proof.

We now recall the notion of the Michael family of subsets of a metric space [71], Definition 1.4.

Definition 4.1. Let X be a metric space and let $M(X)$ be a family of a closed subsets of X, satisfying the following conditions:

(4.1.1) $X \in M(X)$, $\{x\} \in M(X)$, for all $x \in X$, and, if $\{A_i\}_{i\in I}$ is any sub-class of $M(X)$, then $\bigcap_{i\in I} A_i \in M(X)$;
(4.1.2) for every $k \in \mathbb{N}$ and every $x_1, x_2, \ldots, x_k \in X$, the set

$$A(x_1, x_2, \ldots, x_k) = \bigcap\{A \mid A \in M(X), x_1, x_2, \ldots, x_k \in A\}$$

is infinitely connected;
(4.1.3) to each $\varepsilon > 0$, there corresponds $\delta > 0$ such that, for any $A \in M(X)$, any $k \in \mathbb{N}$, and any $x_1, x_2, \ldots, x_k \in O_\delta(A)$, one has $A(x_1, x_2, \ldots, x_k) \subseteq O_\varepsilon(A)$;
(4.1.4) $\overline{A \cap B(x, r)} \in M(X)$, for all $A \in M(X)$, $x \in X$, and $r > 0$;

then we say that $M(X)$ is the *Michael family* of subsets of X.

This concept is closely related to the existence of continuous selections. Indeed, we have the following (see [69] or GMS):

Proposition 4.5. *Let X, Y be two metric spaces and let $F : X \to C(Y)$ be a l.s.c. multifunction. If Y is complete and there exists a Michael family $M(Y)$ of subsets of Y such that $F(x) \in M(Y)$, for each $x \in X$, then, for any nonempty closed set $X_0 \subseteq X$, every continuous selection f_0 from $F|_{X_0}$ admits a continuous extension f over X such that $f(x) \in F(x)$, for all $x \in X$.*

The proceeding result gains interest if we realize that significant classes of sets are the examples of the Michael families.

Example 4.1. Let X be a convex subset of a normed space and let $M(X)$ be the class of all sets $A \subseteq X$ such that $A = \emptyset$ or A is convex and closed in X. Then $M(X)$ is a Michael family of subsets of X.

Example 4.2 (comp. [70]). Let X be a metric space and let $M(X)$ be a simplicial convexity on X, whose elements are closed in X. Then $M(X)$ is a Michael family of subsets of X.

Definition 4.2. Let X be a metric space, let $F : X \to C(X)$ be l.s.c., and let $\mathcal{D}$ be a family of metric spaces. We say that F has the *selection property* w.r.t. $\mathcal{D}$ if, for any $Y \in \mathcal{D}$, any pair of continuous functions $f : Y \to X$ and $h : Y \to (0, +\infty)$ such that

$$G(y) = \overline{F((y)) \cap B(f(y), h(y))} \neq \emptyset, \quad y \in Y,$$

and any nonempty closed set $Y_0 \subseteq Y$, every continuous selection g_0 from $G|_{Y_0}$ admits a continuous extension g over Y fulfilling $g(y) \in G(y)$, for all $y \in Y$. If $\mathcal{D}$ is a family of all metric spaces, then we say that F has *selection property* (in symbols, $F \in SP(X)$).

Such notion has some meaningful features, as the remarks below point out.

Remark 4.1. Let X be a metric space and let $F : X \to C(X)$ be a l.s.c. m ultifunction. If X is complete and there exists a Michael family $M(X)$ of subsets of X such that $F(x) \in M(X)$, for all $x \in X$, then $F \in SP(X)$. This is an immediate consequence of Proposition 3.6.

Remark 4.2. Let X be a nonempty closed subset of $L^1(T, E)$ and $F : X \to C(X)$ be a l.s.c. multifunction with decomposable values. Then, arguing as in [71], it is possible to see that F has the selection property w.r.t. the family of all separable metric spaces.

We are now in a position to prove the main result of this section (see: [70] or [71]).

Theorem 4.5. *Let X be a complete absolute retract and let $F : X \to C(X)$ be a multivalued contraction. Suppose $F \in SP(X)$. Then the set* $\mathrm{Fix}(F)$ *is a complete absolute retract.*

Proof. Since $\mathrm{Fix}(F)$ is nonempty and closed in X, we only have to show that if Y is a metric space, $Y^\star$ is a nonempty closed subset of Y, and $f^\star : Y^\star \to \mathrm{Fix}(F)$ is a continuous function, then there exists a continuous extension $f : Y \to \mathrm{Fix}(F)$ of $f^\star$ over Y. Let d be the metric of Y, let $L \in (0, 1)$ be such that $d_H(F(x'), F(x'')) \leq Ld(x', x'')$, for all $x', x'' \in X$, and let $M \in (1, L^{-1})$. The assumption $X \in \mathrm{AR}$ yields a continuous function $f_0 : Y \to X$ fulfilling $f_0(y) = f^\star(y)$ in Y. We claim that there is a sequence $\{f_n\}$ of continuous functions from Y into X with the following properties:

(i) $f_n|Y^\star = f^\star$, for every $n \in \mathbb{N}$,
(ii) $f_n(y) \in F(f_{n-1}(y))$, for all $y \in Y$, $n \in \mathbb{N}$,
(iii) $d(f_n(y), f_{n-1}(y)) \leq L^{n-1}d(f_1(y), f_0(y) + M^{1-n}$, for every $y \in Y$, $n \in \mathbb{N}$.

To see this, we proceed by induction on n. It follows from Proposition 3.4 that the function $h_0 : Y \to (0, +\infty)$, defined by

$$h_0(y) = d(f_0(y), F(f_0(y))) + 1, \quad y \in Y,$$

is continuous; moreover, one clearly has $F(f_0(y)) \cap B(f_0(y), h_0(y)) \neq \emptyset$, for all $y \in Y$. Having in mind that $F \in SP(X)$, we obtain a continuous function $f_1 : Y \to X$ satisfying $f_1(y) = f^\star(y)$ in $Y^\star$ and $f_1(y) \in F(f_0(y))$ in Y. Hence, conditions (i), (ii), and (iii) are true for f_1. Now, suppose that we have constructed p continuous functions $f_1, f_2, \ldots, f_p$ from Y into X in such way that (i), (ii), and (iii) hold, whenever $n = 1, 2, \ldots, p$. Since F is Lipschitzean with the constant L, (ii) and (iii) apply for $n = p$, and $LM < 1$, for every $y \in Y$, we achieve

$$\begin{aligned} d(f_p(y), F(f_p(y))) &\leq d_H(F(f_{p-1}(y)), F(f_p(y))) \leq Ld(f_{p-1}(y), f_p(y)) \\ &\leq L^p d(f_1(y), f_0(y)) + LM^{1-p} \end{aligned}$$

$$\leq L^p d(f_1(y), f_0(y)) + M^{-p},$$

and subsequently

$$F(f_p(y)) \cap B(f_p(y), L^p d(f_1(y), f_0(y)) + M^{-p}) \neq \emptyset.$$

Because of the assumption $F \in SP(X)$, this produces a continuous function $f_{p+1} : Y \to X$ with the properties:

$$\begin{array}{lll} f_{p+1}|Y^\star = f^\star; & f_{p+1}(y) \in F(f_p(y)), & \text{for every } y \in Y; \\ d(f_{p+1}(y), f_p(y)) \leq L^p d(f_1(y), f_0(y)) + M^{-p}, & & \text{for all } y \in Y. \end{array}$$

Thus, the existence of the sequence $\{f_n\}$ is established. We next define, for any $a > 0$, $Y_a = \{y \in Y \mid d(f_1(y), f_0(y)) < a\}$. Obviously, the family of sets $\{Y_a \mid a > 0\}$ is an open convering of Y. Moreover, due to (iii) and the completeness of X, the sequence $\{f_n\}$ converges uniformly on each Y_a. Let $f : Y \to X$ be the point-wise limit of $\{f_n\}$. It can be easily seen that the function f is continuous. Furthermore, owing to (i), one has $f|_{Y^\star} = f^\star$. Finally, the range of f is a subset of $\mathrm{Fix}(F)$, because, by (ii), $f(y) \in F(f(y))$, for all $y \in Y$. This completes the proof.

The same arguments as in the proof of Theorem 4.5 actually lead to the following more general result.

Theorem 4.6. *Let $\mathcal{D}$ be a family of metric spaces, let X be a complete absolute retract, and let $F : X \to C(X)$ be a multivalued contraction having the selection property w.r.t. $\mathcal{D}$. Then, for any $Y \in \mathcal{D}$ and any nonempty closed set $Y_0 \subseteq Y$, every continuous function $f_0 : Y_0 \to \mathrm{Fix}(F)$ admits a continuous extension over Y.*

Theorem 4.6 has a variety of special cases of a particular interest. As an example, Remark 4.10 combined with Theorem 4.6 lead to

Theorem 4.7. *Let X be a complete absolute retract and let $F : X \to C(X)$ be a multivalued contraction. If there exists a Michael family $M(X)$ of subsets of X such that $F(x) \in M(X)$, for all $x \in X$, then the set $\mathrm{Fix}(F)$ is an absolute retract.*

Evidently, Theorem 4.6 generalizes earlier results formulated in (4.5) and (4.6). For details concerning 4.6 see: [70] and [71]. Now, we would like to study the topological dimension of the set $\mathrm{Fix}(F)$ for some multivalued contractions. Note that the above mentioned problem was initiated by J. Saint Raymond [144]. At first, we recall the following result (see: [144] or [7]).

Proposition 4.6. *If $F : X \to BC(X)$ is a contraction with compact values, then $\mathrm{Fix}(F)$ is compact.*

The following result due to Z. Dzedzej and B. Gelman ([58]) is a generalization of the result obtained by J. Saint Raymond ([144]).

Theorem 4.8. *Let E be a Banach space and $F : E \to BC(E)$ be a contraction with convex values and a constant $\alpha < 1/2$. Assume, furthermore, that the topological dimension $\dim F(x)$ of $F(x)$ is greater or equal to n, for some n and every $x \in E$. If $\mathrm{Fix}(F)$ is compact, then $\dim \mathrm{Fix}(F) > n$.*

Problem 1. Is it possible to prove 4.8, for $E = X$, to be a complete AR-space and $F : X \to CB(X)$ with values belonging to a Michael family $M(X)$?

Following D. Miklaszewski, we would like to discuss some generalizations of 4.8.

Theorem 4.9. *Let X be a retract of a Banach space E, and $F : X \to BC(X)$ be a compact continuous multivalued map with values being such elements of the Michael family $M(X)$ that $F(x) \setminus \{x\} \in C^{k-2}$, for every $x \in \mathrm{Fix}(F)$. Then the set $\mathrm{Fix}(F)$ has the dimension greater or equal to k.*

Proof. Suppose on the contrary that $\dim(\mathrm{Fix}(F)) < k$. Let us consider the maps $\psi : \mathrm{Fix}(F) \to BC(E)$ and $\varphi : \mathrm{Fix}(F) \to E \setminus \{0\}$ defined by the formulae: $\psi(x) = F(x) - x = \{y - x \mid y \in F(x)\}$ and $\varphi = \psi(x) \setminus \{0\} = (F(x) \setminus \{x\}) - x$. We are going to prove that the family $\{\varphi(x) \mid x \in \mathrm{Fix}(F)\}$ is equi-LC^∞. Let $y \in \varphi(x_0)$ and r be a positive number such that $0 \notin B_E(y, 3r)$. Suppose that the set $B_E(y, r) \cap \varphi(x)$ is non-empty, for a fixed point x of F. Then $B_E(y, r) \cap \varphi(x) = [(B_E(y + x, r) \cap F(x)) - x]$. Let $z \in B_E(y + x, r) \cap F(x)$. It is easy to show that $B_E(y + x, r) \cap F(x) \subset B_E(y + x, 3r) \cap F(x)$. But the second set of these three sets being in the Michael family $M(X)$ is C^∞ as well as its translation, so the inclusion of $B_E(y, r) \cap \varphi(x)$ into the set $B_E(y, 3r) \cap \varphi(x)$ is homotopically trivial, and the family $\{\varphi(x) \mid x \in \mathrm{Fix}(F)\}$ is equi-LC^∞. It follows from Theorem 1.8 that φ has a selection f. Then the map $g : \mathrm{Fix}(F) \to X$ defined by the formula: $g(x) = f(x) + x$ is a selection of F. We conclude that, in view of Theorem 4.10, there exists a selection h of F being an extension of g. But h has a fixed point $x' \in \mathrm{Fix}(F)$, $h(x') = g(x') = f(x') + x' = x'$, $f(x') = 0 \in \varphi(x)$, which is a contradiction.

In the case when $\dim X < +\infty$, by analogous considerations as in the proof of 4.9 we obtain:

Theorem 4.10. *Let X be a retract of a Banach space E and $F : X \to BC(X)$ be a continuous (i.e. both l.s.c. and u.s.c.) map such that $\overline{F(X)} = \overline{\bigcup\{F(x) \mid x \in X\}}$ is a compact set. Assume that the values of F satisfy the following conditions:*

(i) *$F(x) \setminus \{x\}$ is C^{k-2}, for every $x \in \mathrm{Fix}(F)$,*
(ii) *$F(x)$ is C^k for every $x \in X$,*
(iii) *$\{F(x) \mid x \in \mathrm{Fix}(F)\}$ is equi-LC^{k-2} in E,*
(iv) *$\{F(x) \mid x \in X\}$ is equi-LC^k in X.*

Them $\dim(\mathrm{Fix}(F)) \geq k$.

The proof of 4.10 is quite analogous to that of 4.9. Finally, note that one can show an example of a continuous (i.e. both l.s.c. and u.s.c.) map with contractible values of the local dimension 2 such that (iii) and (iv) are satisfied, but the dimension of the set of fixed points equals 1.

It is evident that the above results can be applied directly to differential inclusions where the right hand side is a measurable-Lipschitz multivalued map $f : [0,a] \times \mathbb{R}^n \to CB(\mathbb{R})$. A very general application to the so called almost-periodicity problem for differential inclusions in Banach spaces is presented in Section 5 of [7].

Namely, we shall give a topological characterization of the set of solutions of some boundary value problems for differential inclusions of order k.

Let E be a separable Banach space and let $\phi : [0,a] \times E^k \multimap E$ be a multivalued mapping, where $E^k = E \times \ldots \times E$ (k-times).

We shall consider the following problem

$$\left.\begin{aligned} x^{(k)}(t) &\in \phi(t, x(t), x'(t), \ldots, x^{(k-1)}(t)) \\ x(0) &= x_0 \\ x'(0) &= x_1 \\ &\vdots \\ x^{(k-1)}(0) &= x_{k-1}, \end{aligned}\right\} \tag{4.1}$$

where the solution $x : [0,a] \to E$ is understood in the sense of t almost everywhere (a.e., $t \in [0,a]$) and $x_0, \ldots, x_{k-1} \in E$.

Observe that for $k = 1$ problem (4.1) reduces to the well-known Cauchy problem for differential inclusions. In what follows we shall denote by $\mathcal{S}(\phi, x_0, \ldots, x_{k-1})$ the set of all solutions of (4.1).

Our first application of Theorem 4.6 is the following:

Theorem 4.11. *Assume that φ is a mapping with compact values. Assume further that the following conditions hold:*

(4.11.1) *φ is bounded, i.e. there is an $M > 0$ such that $\|y\| \le M$ for every $t \in [0,a]$, $x \in E^k$ and $y \in \varphi(t,x)$,*
(4.11.2) *the map $\varphi(\,\cdot\,, x)$ is measurable for each $x \in E^k$,*
(4.11.3) *φ is a Lipschitz map with respect to the second variable, i.e. there exists an $L > 0$ such that for every $t \in [0,a]$ and for every $z = (z_1, \ldots, z_k)$, $y = (y_1, \ldots, y_k) \in E^k$ we have:*

$$d_H(\varphi(t,z), \varphi(t,y)) \le L \sum_{i=1}^{k} \|z_i - y_i\|.$$

Then the set $\mathcal{S}(\varphi, x_0, \ldots, x_{k-1})$ of all solutions of the problem (4.1) *is an* AR-*space.*

Sketch of proof. For the proof we define (single-valued) mappings[7]:
$h_j : M([0,a],E) \to AC^j$, $j = 0,\dots,k-1$, by putting

$$(h_j(z))(t) = x_0 + tx_1 + \dots + (t^j/j!)x_j + \int_0^t \int_0^{s_1} \dots \int_0^{s_j} z(s)\,ds\,ds_j \dots ds_1,$$

where $AC^j = \{u \in C^j([0,a],E) : u^{(j)}$ is absolutely continuous$\}$ and for $u \in AC^j$ we put:

$$\|u\| = \|u\|_{C^j} + \sup\mathrm{ess}_{t\in[0,a]}\{\|u^{(j+1)}(t)\|\}.$$

Now consider a multivalued mapping $\psi : M([0,a],E) \to M([0,a],E)$ defined as follows:

$$\psi(x) = \{z \in M([0,a],E) \mid z(t) \in \varphi(t, h_{k-1}(x)(t),\dots,h_0(x)(t)), \text{ for a.e. } t \in [0,a]\}.$$

It follows from the Kuratowski–Ryll–Nardzewski Selection Theorem and (4.11.1) that ψ is well defined (with closed decomposable values in $M([0,a],E)$. Moreover, it is easy to see that $h_{k-1}(\mathrm{Fix}(\psi)) = \mathcal{S}(\varphi, x_0,\dots,x_{k-1})$. Consequently, since h_{k-1} is a homeomorphism onto its image, in view of Theorem 4.6, it is sufficient to show that ψ is a contractive mapping. We shall do this by using the $M([0,a],E)$-version of Bielecki's method and the Kuratowski–Ryll–Nardzewski Theorem. In fact it is enough to see that for every $u, z \in M([0,a],E)$ and for every $y \in \psi(u)$ there is a $v \in \psi(z)$ such that

$$(\star) \qquad \|y - v\|_1 \le \alpha\|u - z\|_1,$$

where $\alpha \in [0,1)$ and $\|w\|_1 = \sup\mathrm{ess}_{t\in[0,a]}\{e^{-Lakt}\|w(t)\|\}$ is the Bielecki norm in $M([0,a],E)$. Observe that using Theorem 4.2 (in [66]) for ψ and z, we get a mapping $v \in \psi(z)$ and now $(\star)$ follows directly from 4.11.3. The proof of Theorem 4.11 is complete.

Remark 4.3. Note that if we impose more assumptions on φ then we are able to get better information on $= \mathcal{S}(\varphi, x_0,\dots,x_{k-1})$ (for details see [7], [66], [69]).

Now following [12], [61], [62] we would like to add that if we consider problem (4.1) for $k = 1$ and in Theorem 4.11 we assume moreover that $\varphi(t,x)$ is convex and $\dim \varphi(t,x) \ge n$ for some n and every $(t,x) \in [0,a] \times E$, then, in view of Theorem 4.9 we get that:

$$\dim \mathcal{S}(\varphi, x_0) \ge n.$$

Finally, let us remark that if we reject the assumption that φ has compact values, then still a characterization of $\mathcal{S}(\varphi, x_0)$ is possible (see Theorem 3.1 in [105]).

[7] here $M([0,a],E)$ is the Banach space of continuous essentialy bounded mappings.

5. The inverse limit method

The inverse limit method in differential equations and inclusions is quite new and it was indicated in 1999 by J. Andres, G. Gabor and L. Górniewicz (see: [5], [6] and [60]).

We shall start from the topological preparation. By an inverse system of topological spaces we mean a family $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$, where Σ is a set directed by the relation $\leq$, X_α is a topological (Hausdorff) space for every $\alpha \in \Sigma$ and $\pi_\alpha^\beta : X_\alpha \to X_\beta$ is a continuous mapping for every two elements $\alpha, \beta \in \Sigma$ such that $\alpha \leq \beta$. Moreover, for each $\alpha \leq \beta \leq \gamma$ the following conditions should hold: $\pi_\alpha^\alpha = \mathrm{id}_{X_\alpha}$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$.

A subspace of the product $\Pi_{\alpha\in\Sigma} X_\alpha$ is called a *limit of the inverse system* $\mathcal{IS}$ and it is denoted by $\lim_\leftarrow \mathcal{IS}$ or $\lim_\leftarrow \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ if

$$\lim_{\leftarrow} \mathcal{IS} = \{(x_\alpha) \in \Pi_{\alpha\in\Sigma} X_\alpha \mid \pi_\alpha^\beta(x_\beta) = x_\alpha \text{ for all } \alpha \leq \beta\}.$$

An element of $\lim_\leftarrow \mathcal{IS}$ is called a *thread* or a *fibre* of the system $\mathcal{IS}$. One can see that if we denote by $\pi_\alpha : \lim_\leftarrow \mathcal{IS} \to X_\alpha$ a restriction of the projection $p_\alpha : \Pi_{\alpha\in\Sigma} X_\alpha \to X_\alpha$ onto the α-th axis, then we obtain $\pi_\alpha = \pi_\alpha^\beta \pi_\beta$ for each $\alpha \leq \beta$.

Now we summarize some useful properties of limits of inverse systems which are well known (comp. [60]):

Proposition 5.1. *Let $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system.*

(5.1.1) *The limit $\lim_\leftarrow \mathcal{IS}$ is a closed subset of $\Pi_{\alpha\in\Sigma} X_\alpha$.*
(5.1.2) *If, for every $\alpha \in \Sigma$, X_α is*
- (i) *compact, then $\lim_\leftarrow \mathcal{IS}$ is compact;*
- (ii) *compact and nonempty, then $\lim_\leftarrow \mathcal{IS}$ is compact and nonempty;*
- (iii) *a continuum, then $\lim_\leftarrow \mathcal{IS}$ is a continuum;*
- (iv) *acyclic, and $\lim_\leftarrow \mathcal{IS}$ is nonempty, $\lim_\leftarrow \mathcal{IS}$ is acyclic;*
- (v) *metrizable, Σ is countable, and $\lim_\leftarrow \mathcal{IS}$ is nonempty, then $\lim_\leftarrow \mathcal{IS}$ is metrizable.*

The following further information is useful for applications.

Proposition 5.2 ([60]). *Let $\mathcal{IS} = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system. If each X_n is an R_δ-set, then so is $\lim_\leftarrow \mathcal{S}$.*

The following example shows that a limit of an inverse system of compact absolute retracts does not have to be an absolute retract.

Example 5.1. Consider a family $\{X_n\}_{n=1}^\infty$ of subsets of $\mathbb{R}^2$ defined as follows:

$$X_n = \left(\left[0, \frac{1}{n\pi}\right] \times [-1,1]\right) \cup \left\{(x,y) \,\middle|\, y = \sin\frac{1}{x} \text{ and } \frac{1}{n\pi} < x \leq 1\right\}.$$

One can see that for each $m, n \geq 1$ such that $m \geq n$ we have $X_m \subset X_n$.

Define the maps $\pi_n^m : X_m \to X_n$, $\pi_n^m(x) = x$. Therefore $\mathcal{IS} = \{X_n, \pi_n^m, \mathbb{N}\}$ is an inverse system of compact absolute retracts. It is evident that $\lim_{\leftarrow} \mathcal{IS}$ is homeomorphic to the intersection of all X_n. On the other hand

$$X = \bigcap_{n=1}^{\infty} X_n = \{(0,y) \mid y \in [-1,1]\} \cup \left\{(x,y) \,\middle|\, y = \sin\frac{1}{x} \text{ and } 0 < x \leq 1\right\}$$

and X is not an absolute retract since, for instance, X is not locally connected.

Note that in [60] the following information on a limit of an inverse system of absolute retracts has been formulated.

Proposition 5.3. *Let $\mathcal{IS} = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system of compact absolute retracts such that $X_n \subset X_p$ and π_n^p is a retraction for all $n \leq p$. Then* $\lim_{\leftarrow} \mathcal{IS}$ *has the fixed point property, i.e. every continuous map $f : \lim_{\leftarrow} \mathcal{IS} \to \lim_{\leftarrow} \mathcal{IS}$ has a fixed point.*

Example 5.2. Consider the inverse system $\mathcal{S} = \{X_n, \pi_n^p, \mathbb{N}\}$ such that $X_n = [n, \infty)$ and $\pi_n^p : X_p \hookrightarrow X_n$ are inclusion maps for $n \leq p$. It is obvious that $\lim_{\leftarrow} \mathcal{S}$ is homeomorphic to the intersection of all X_n which is an empty set.

Let us give important examples of inverse systems.

Example 5.3. Let, for every $m \in \mathbb{N}$, $C_m = C([0,m], \mathbb{R}^n)$ be a Banach space of all continuous functions of the closed interval $[0,m]$ into $\mathbb{R}$, and $C = C([0,\infty), \mathbb{R}^n)$ be an analogous Fréchet space of continuous functions.

Consider the maps $\pi_m^p : C_p \to C_m$, $\pi_m^p(x) = x|_{[0,m]}$. It is easy to see that C is isometrically homeomorphic to a limit of the inverse system $\{C_m, \pi_m^p, \mathbb{N}\}$. The maps $\pi_m : C \to C_m$, $\pi_m(x) = x|_{[0,m]}$ correspond to suitable projections.

Remark 5.1. In the same manner as above we can show that Fréchet spaces $C(J, \mathbb{R}^n)$, where J is an arbitrary interval, $L^1_{loc}(J, \mathbb{R}^n)$ of all locally integrable functions, $AC_{loc}(J, \mathbb{R}^n)$ of all locally absolutely continuous functions and $C^k(J, \mathbb{R}^n)$ of all continuously differentiable functions up to the order k can be considered as limits of suitable inverse systems.

More generally, every Fréchet space is a limit of some inverse system of Banach spaces.

Now we introduce the notion of multivalued maps of inverse systems. Suppose that two systems $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $\mathcal{IS}' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ are given.

Definition 5.1. By a *multivalued map of the system $\mathcal{IS}$ into the system $\mathcal{IS}'$* we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma : \Sigma' \to \Sigma$, that is $\sigma(\alpha') \leq \sigma(\beta')$, and of multivalued maps $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')} \multimap Y_{\alpha'}$ with nonempty values, defined for every $\alpha' \in \Sigma'$ and such that

$$\pi_{\alpha'}^{\beta'} \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}^{\sigma(\beta')}, \tag{5.1}$$

for each $\alpha' \le \beta'$.

A map of systems $\{\sigma, \varphi_{\sigma(\alpha')}\}$ induces a *limit map* $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}'$ defined as follows:

$$\varphi(x) = \Pi_{\alpha' \in \Sigma} \varphi_{\sigma(\alpha')}(x_{\sigma(\alpha')}) \cap \lim_{\leftarrow} \mathcal{IS}.$$

In other words, a limit map is a map such that

$$\pi_{\alpha'}\varphi = \varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')} \tag{5.2}$$

for every $\alpha' \in \Sigma'$.

Since a topology of a limit of an inverse system is the one generated by the base consisting of all sets of the form $\pi_\alpha(U_\alpha)$, where α runs over an arbitrary set cofinal in Σ and U_α are open subsets of the space X_α, it is easy to prove the following continuity property for limit maps:

Proposition 5.4 (see [5], Proposition 2.7). *Let $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $\mathcal{IS}' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$ be two inverse systems, and $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}'$ be a limit map induced by the map $\{\sigma, \varphi_{\sigma(\alpha')}\}$.*

If, for every $\alpha' \in \Sigma$, $\varphi_{\sigma(\alpha')}$ is

(i) *u.s.c., then φ is u.s.c.;*
(ii) *l.s.c., then φ is l.s.c.;*
(iii) *continuous, then φ is continuous (continuous means both u.s.c. and l.s.c.).*

The following crucial result allows us to study a topological structure of fixed point sets of limit maps.

Theorem 5.1 ([60]). *Let $\mathcal{IS} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ be an inverse system, and $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}$ be a limit map induced by a map $\{\mathrm{id}, \varphi_\alpha\}$, where $\varphi_\alpha : X_\alpha \multimap X_\alpha$. If fixed point sets of φ_α are acyclic, and the fixed point set of φ is nonempty, then it is acyclic, too.*

Theorem 5.2. *Let $\mathcal{IS} = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system, and $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}$ be a limit map induced by a map $\{\mathrm{id}, \varphi_n\}$, where $\varphi_n : X_n \multimap X_n$. If fixed point sets φ_n are compact R_δ, then the fixed point set of φ is R_δ, too.*

Corollary 5.1. *Let $\mathcal{IS} = \{X_n, \pi_n^p, \mathbb{N}\}$ be an inverse system, and $\varphi : \lim_{\leftarrow} \mathcal{IS} \multimap \lim_{\leftarrow} \mathcal{IS}$ be a limit map induced by a map $\{\mathrm{id}, \varphi_n\}$, where $\varphi_n : X_n \multimap X_n$. If all X_n are Fréchet spaces and all φ_n are contractions.*

Remark 5.2. Note that, following [70], we can prove Corollary 5.1 for a little larger class of multivalued maps (see [5], Corollary 2.9).

The inverse system approach described above gives us an easy way to study a topological structure of solution sets of differential problems on noncompact intervals. To illustrate it consider the following example:

Example 5.4. Let $F : [0,\infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a Carathéodory map, i.e.

(i) values of F are nonempty, compact and convex for all $(t,x) \in [0,\infty) \times \mathbb{R}^n$,
(ii) $F(t,\,\cdot\,)$ is u.s.c. for a.a. $t \in [0,\infty)$,
(iii) $F(\,\cdot\,,x)$ is measurable for all $x \in \mathbb{R}^n$,

with at most linear growth, i.e. there exists a locally integrable function $\alpha : [0,\infty) \to [0,\infty)$ such that, for every $x \in \mathbb{R}^n$ and for a.a. $t \in [0,\infty)$,

$$|F(t,x)| \le \alpha(t)(1 + \|x\|),$$

where $|F(t,x)| = \sup\{|y| \mid y \in F(t,x)\}$.

Consider the Cauchy problem

$$\left.\begin{aligned} &\dot{x}(t) \in F(t,x(t)) \quad \text{for a.a. } t \in [0,\infty), \\ &x(0) = x_0. \end{aligned}\right\} \tag{5.3}$$

We shall show, using the inverse systems technique, that the set of solutions $\mathcal{S}$ of problem (5.3) is R_δ. To do it, consider the family of Cauchy problems

$$\left.\begin{aligned} &\dot{x}(t) \in F(t,x(t)) \quad \text{for a.a. } t \in [0,m], \\ &x(0) = x_0, \end{aligned}\right\} \tag{5.4}$$

where $m \ge 1$. It is well known (see [46]) that, for every $m \ge 1$, the set $\mathcal{S}_m$ of the above problem is compact R_δ.

On the other hand, $\mathcal{S}_m$ is a fixed point set of the following map $\Psi_m : C_m = C([0,m],\mathbb{R}^n) \multimap C_m$,

$$\Psi_m(x) = \left\{ x_0 + \int_0^t u(s)\,ds \;\middle|\; u \in L^1([0,m],\mathbb{R}^n) \text{ and } u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,m] \right\}.$$

One can check that $\{\Psi_m\}$ is a map of the inverse system $\{C([0,m],\mathbb{R}^n), \pi_m^p, \mathbb{N}\}$, where $\pi_m^p(x) = x|_{[0,m]}$ for every $x \in C([0,p],\mathbb{R}^n)$. Moreover, it induces the limit map on $C([0,\infty),\mathbb{R}^n)$

$$\Psi(x) = \left\{ x_0 + \int_0^t u(s)\,ds \;\middle|\; u \in L^1_{loc}([0,\infty),\mathbb{R}^n) \text{ and } u(t) \in G(t,x(t)) \text{ for a.a. } t \in [0,\infty) \right\}$$

with the fixed point set $\mathcal{S}$. By Theorem 5.2 it follows that $\mathcal{S}$ is compact R_δ, as required.

Note that the above result on a topological structure of the solution set of the Cauchy problem on a halfline can be obtained by using different techniques (see e.g. [4] for the proof by using the Scorza–Dragoni type result).

Further applications of the inverse systems approach can be found in [5] and [60].

6. Concluding remarks and comments

Above we have presented different techniques of characterization of the set of fixed points and consequently solution sets for differential equations and inclusions. Now, we would like to show some consequences, which can be obtained by using the topological structure of solution sets.

We would like to show only results connected with multivalued Poincaré operator indicated by G. Dylawerski and L. Górniewicz in 1983 ([57]). We recommend also the following papers: [3], [8], [66], [69], [72], [91], [143], [45].

We shall formulate the simplest version. Most general results of this type are contained in [45].

Let $f : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and bounded map. We shall consider both the Cauchy problem:

$$\left.\begin{aligned} x'(t) &= f(t, x(t)),\\ x(t_0) &= x_0, \end{aligned}\right\} \tag{6.1}$$

and the following periodic problem:

$$\left.\begin{aligned} x'(t) &= f(t, x(t)),\\ x(0) &= x(a). \end{aligned}\right\} \tag{6.2}$$

We shall associate with (6.2) the multivalued Poincaré operator:

$$P_a : \mathbb{R}^n \multimap \mathbb{R}^n$$

defined as a composition of the following two maps:

$$\mathbb{R} \overset{P}{\mapsto} C([0,a], \mathbb{R}^n) \xrightarrow{e_a} \mathbb{R}^n,$$

where $P(x) = S(f; 0, x)$ and $e_a(x) = x(a)$. It follows from the Aronszajn Theorem that P has R_δ-values. On the other hand it is well known that P is u.s.c. (comp. [69] or [11]). Hence $P_a = e_a \circ P$ as a composition of u.s.c. R_δ-valued map P with a continuous map e_a is admissible in the sense of [67]. Therefore the topological degree of the field $(\mathrm{id}_{\mathbb{R}^n} - P)$ on any ball $B(0,r) \subset \mathbb{R}^n$ such that[8] $\mathrm{Fix}(P) \cap \partial B(0,r) = \emptyset$ is well defined (see: [67], [45] or [92]). In what follows P is called the Poincaré operator associated with (6.2).

The following proposition is straightforward.

Proposition 6.1. *If* $\mathrm{Fix}(P_a) \neq \emptyset$*, then problem* (6.2) *has a solution.*

In the terms of topological degree theory Proposition 6.3 can be expressed as follows:

Proposition 6.2. *Assume that for some* $r > 0$ *we have* $\mathrm{Fix}(P) \cap \partial B(0,r) = \emptyset$*. If the topological degree* $\deg(\mathrm{id}_{\mathbb{R}^n} - P_a, B(0,r))$ *of* $\mathrm{id}_{\mathbb{R}^n} - P_a$ *with respect to* $B(0,r)$ *is different from zero, then problem* (6.2) *has a solution.*

[8] $\partial B(0,r)$ denotes the boundary of $B(0,r)$ in $\mathbb{R}^n$.

In order to calculate the topological degree of Poincaré field $\mathrm{id}_{\mathbb{R}^n} - P_a$ we shall use the guiding function (or potential function) method (see: [91] or [45]).

A C^1-map $V : \mathbb{R}^n \to \mathbb{R}$ is called a guiding function (potential) for f provided that there exists $r_0 > 0$ such that:

$$\langle \operatorname{grad} V(x), f(t,x)\rangle > 0 \tag{6.3}$$

for every $t \in [0,a]$ and $x \in \mathbb{R}^n$ such that $\|x\| \geq r_0$, where $\operatorname{grad} V(x) = \Big(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\Big)$ is the gradient of V at the point x and $\langle\ ,\ \rangle$ stands for the inner product in $\mathbb{R}^n$.

It follows from (6.3) that for every $r \geq r_0$ and $x \in \mathbb{R}^n$ such that $\|x\| \geq r_0$ we have $\operatorname{grad} V(x) \neq 0$ so from the localization property of the topological degree it follows that for every $r \geq r_0$ we have $\deg(\operatorname{grad} V, B(0,r)) = \deg(\operatorname{grad} V(x), B(0, r_0))$. We let:

$$\operatorname{Ind}(V) = \deg(\operatorname{grad} V(x), B(0,r)). \tag{6.4}$$

Finally we obtain:

Theorem 6.1. *If f has a potential V such that* $\operatorname{Ind}(V) \neq 0$, *then* $\deg(P_a, B(0, r)) \neq 0$ *for some $r \geq r_0$.*

References

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